Oligopoly
Market demand is $p = a - bQ$
$Q = q_1 + q_2$.

1. Cournot

General description of this game:

Players: firm 1 and firm 2.
Firm 1 and firm 2 are identical.
Firm 1's strategies: q_1
Firm 2's strategies: q_2

- Solve for Cournot Nash Equilibrium

  Payoff (or profit) function

  Firm 1: $\Pi_1(q_1;q_2) = q_1(a - b(q_1 + q_2)) - cq_1$.
  Firm 2: $\Pi_2(q_1;q_2) = q_2(a - b(q_1 + q_2)) - cq_2$.

Firm 1's problem: Choose q_1 as to maximize its profit by <u>taking q_2 as given</u>.

$$\max_{q_1} \Pi_1 = q_1(a - b(q_1 + q_2)) - cq_1$$

FOC: $a-b(q_1 + q_2) - c - bq_1 = 0$

$$a-bq_2 - c = 2bq_1$$

$$q_1 = \frac{a-c}{2b} - \frac{q_2}{2}$$

$$= BR_1(q_2)$$

The solution q_1^* is the best response of firm 1 to the quantity chosen by firm 2.

Firm 2's problem:

$$\max_{q_2} \Pi_2 = q_2(a - b(q_1 + q_2)) - cq_2$$

$$BR_2(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}$$

Since both firms are identical (i.e. same demand and cost functions) in NE

$$q_1 = q_2 = q^*.$$

Substitute $BR_2(q_1)$ into $BR_1(q_2)$

$$q_1 = \frac{a-c}{2b} - \frac{\left(\frac{a-c}{2b} - \frac{q_1}{2}\right)}{2}$$

$$\frac{3}{4}q_1 = \frac{a-c}{4b}$$

$$\boxed{q^* = \frac{a-c}{3b}}$$

$$p^* = a-bQ^*$$
$$= a-b(q^* + q^*)$$
$$= a-b2(\frac{a-c}{3b})$$

$$\boxed{p^* = \frac{a+2c}{3}}$$

$$\Pi^* = \Pi_1 = \Pi_2 = q^*(p^*\text{-}c)$$
$$= \left(\frac{a\text{-}c}{3b}\right)\left(\frac{a+2c}{3} - c\right)$$
$$\boxed{\Pi^* = \frac{(a-c)^2}{9b}}$$

- <u>Cartel</u> (this is similar to the monopoly case)

Let q^j be the quantity produced by firm1 and firm2 under cartel.

Joint profit $\Pi^j = \Pi_1(q^j) + \Pi_2(q^j)$

$= 2\{q^j(a - b(2q^j)) - c\}$

Cartel chooses q^j so as to maximize joint profit.

$$\underset{q^j}{\text{Max}}\,\Pi^j = 2(q^j(a - b(2q^j)) - c)$$

FOC: $2(a - b(2q^j) - c) - 4bq^j = 0$

$2(a\text{-}c) = 8bq^j$

$$\boxed{q^j = \frac{a\text{-}c}{4b}}$$

$< q^*$

$$p^j = a\text{-}b(2q^j)$$
$$= a\text{-}2b\left(\frac{a\text{-}c}{4b}\right)$$
$$\boxed{= \frac{a+c}{2}}$$

$> p^*$

$$\Pi^j = 2q^j(p^j\text{-}c)$$
$$= 2\left(\frac{a\text{-}c}{4b}\right)\left(\frac{a+c}{2} - c\right)$$
$$= \frac{(a-c)^2}{4b}$$
$$= \frac{1}{2}\Pi^j = \Pi_1^j = \Pi_2^j$$

$$\boxed{\Pi^j = \frac{(a-c)^2}{8b}} > \Pi^*$$

- <u>Firm 2 cheat</u> when firm 1 sets $q_1 = q^j$.

How will firm 2 cheat?

Set q_2 that maximize firm 2's profit given $q_1 = q^j$. That is, firm 2 produce at $BR_2(q^j)$.

$$BR_2(q^j) = \frac{a\text{-}c}{2b} - \frac{1}{2}(\frac{a\text{-}c}{4b})$$

$$= (1-\frac{1}{4})(\frac{a\text{-}c}{2b})$$

$$\boxed{q_2 = \frac{3}{8}\frac{(a-c)}{b}} \quad > q^j$$

$$Q = q_1+q_2$$
$$= q^j+q_2$$
$$p = a\text{-}b(q^j+q_2)$$
$$= a\text{-}b(\frac{a\text{-}c}{4b} + \frac{3}{8}\frac{(a-c)}{b})$$

$$\boxed{p = \frac{3a+5c}{8}} \quad < p^j$$

$$\Pi_1 = q^j(p\text{-}c)$$
$$= \frac{(a-c)}{4b}(\frac{3a+5c}{8}\text{-}c)$$
$$= \frac{3(a-c)^2}{32b} \quad < \Pi^*$$
$$\Pi_2 = q_2(p\text{-}c)$$
$$= \frac{3(a-c)}{8b}(\frac{3a+5c}{8}\text{-}c)$$
$$= \frac{9(a-c)^2}{64b} \quad > \Pi_2^j$$

- We can summarize the results so far by using payoff matrix

Firm1, Firm2	Don't Coop	Coop
Don't Coop (Cournot)	$\frac{(a-c)^2}{9b}, \frac{(a-c)^2}{9b}$	$\frac{9(a-c)^2}{64b}, \frac{3(a-c)^2}{32b}$
Coop (Cartel)	$\frac{3(a-c)^2}{32b}, \frac{9(a-c)^2}{64b}$	$\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{8b}$

NE of this game is that both firms choose "Don't coop", set $q = \frac{a-c}{3b}$. This is a unique equilibrium, which is inferior to what would have been if they cooperate. But both firms decide to cheat because of the higher profit. This is similar to Prisoner's Dilemma case.

- To solve multi-period game, we use "Backward Induction" method.

In the 2-period game, we solve 2[nd] period first. The best response of both firms is "Don't coop". Knowing that they will play Cournot strategy in the last period, they will also cheat in the first period. Thus, cartel will not survive. Analogically, the 10-period game will be played in the same way. Both firms will cheat in 10[th], 9[th],…..through the first period.

As long as the game has finite horizon, firms are always better off cheating in the last period. However, if they play forever, a cartel may survive. To see this, suppose firm 1 cheats in period t then, in period t+1, firm 2 can retaliate by increasing its output. As a result, payoffs of both firms will be lowered. The threat of retaliation is likely to prevent firms from cheating. Survival of cartel also depends on firms' time preference. That is how much firms care about future punishments.

2. Bertrand Game

Players: Firm 1 and firm 2 , both are identical.

Strategies: p_1 and p_2

Inverse demand curve: $Q = \frac{a}{b} - \frac{1}{b}p$

Consider profit of firm 1.

If $p_1 < p_2$, all consumers will buy form firm 1. Firm 1's demand is the market demand.

If $p_1 > p_2$, all consumers will buy form firm 2. Firm 1's demand is zero.

If $p_1 = p_2$, firm 1 share the market with firm 2 by 1/2 : 1/2 .

Thus,

$$\Pi_1 = \begin{cases} (p_1-c)(\frac{a}{b} - \frac{1}{b}p_1) & \text{if } p_1 < p_2 \\ \frac{1}{2}(p_1-c)(\frac{a}{b} - \frac{1}{b}p_1) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2. \end{cases}$$

$$\Pi_2 = \begin{cases} (p_2-c)(\frac{a}{b} - \frac{1}{b}p_2) & \text{if } p_1 > p_2 \\ \frac{1}{2}(p_2-c)(\frac{a}{b} - \frac{1}{b}p_2) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 < p_2. \end{cases}$$

The best response for firm 1

As long as p_2 is above marginal cost, firm 1 can maximize profit by set price slightly lower than p_2 and capture the entire market.

If p_2 is equal to marginal cost, the best that firm 1 can do is to set price equal to marginal cost as well. It will not set price below marginal cost since it will incur a loss.

Thus,

$$BR_1(p_2) = \begin{cases} p_2 - \varepsilon & \text{if } p_2 - \varepsilon > c \\ c & \text{otherwise} \end{cases}$$

$$BR_2(p_1) = \begin{cases} p_1 - \varepsilon & \text{if } p_1 - \varepsilon > c \\ c & \text{otherwise.} \end{cases}$$

Nash Equilibrium

Suppose firm 1 set price at p_1 where $p_1 > c$. Firm 2's best response is to set $p_2 = p_1 - \varepsilon$. But this cannot be NE since , given $p_2 = p_1 - \varepsilon$, the best response of firm 1 is to set $p_1 = p_2 - \varepsilon$ not p_1.

As long as $p > c$, both firms will undercut each other. They will not do so if $p = c$. Thus, NE occurs when both firms choose price equal to MC.

$$p^b = c$$
$$q^b = \frac{a-c}{2b}$$
$$\Pi^b = 0$$

3. Stakelberg Game

Players : Firm 1 (leader) , firm 2 (follower)
Strategy: q_1 and q_2
Period 1: firm 1 chooses q_1
Period 2: given q_1, firm 2 chooses q_2 to maximize its profit.

Use "Backward Induction." Solve for period 2 first then solve for period 1.

Period 2

Firm 2 uses $BR_2(q_1)$ as we derived in question 1.

$$BR_2(q_1) = \frac{a\text{-}c}{2b} - \frac{q_1}{2}$$

Period 1

Firm 1 knows how much firm 2 will produce through BR_2.

$$\underset{q_1}{Max}\ \Pi_1 = q_1(a - b(q_1 + BR_2(q_1))) - cq_1$$

$$q_1^s = \frac{a-c}{2b}$$

Substitute into $BR_2(q_1)$

$$BR_2(q_1^s) = \frac{a-c}{2b} - \frac{a-c}{4b}$$

$$q_2^s = \frac{a-c}{4b}$$

$$p^s = a\text{-}b(q_1^s + q_2^s)$$
$$= a\text{-}b\left(\frac{a-c}{2b} + \frac{a-c}{4b}\right)$$

$$p^s = \frac{a+3c}{4}$$

$$\Pi_1^s = q_1^s(p^s - c)$$
$$= \frac{(a-c)}{2b}\left(\frac{a+3c}{4} - c\right)$$

$$\boxed{\Pi_1^s = \frac{(a-c)^2}{8b}}$$

$$\Pi_2^s = q_2^s(p^s - c)$$
$$= \frac{a-c}{4b}\left(\frac{a+3c}{4} - c\right)$$

$$\boxed{\Pi_2^s = \frac{(a-c)^2}{16b}}$$

4. <u>Rankings</u>

At firms level

$q^b = q_1^s > q^* > q^j = q_2^s$

$\Pi^j = \Pi_1^s > \Pi^* > \Pi_2^s > \Pi^b$

At industry level

$Q = q_1 + q_2$

$Q^b > Q^s > Q^* > Q^j$

$p^j > p^* > p_1^s > p^b$

$\Pi^j > \Pi^* > \Pi^s > \Pi^b$

5.

(a) Player 1's strategies are A, B, and C

(b) Player 2's strategies are (L,L,L), (L,L,R), (L,R,R), (R,L,R), (R,R,R), (R,R,L), (L,R,L), (R,L,L)

(c) To find NE, write down payoff matrix

Player 1, Player 2	A	B	C
LLL	+5,9*	4,2	4,2
LLR	5,9*	4,2	+7,3*
LRR	5,9*	6,3	+7,3*
RLR	2,2	4,2	+7,3*
RRR	2,2	6,3*	+7,3*
RRL	2,2	+6,3*	4,2
LRL	5,9*	+6,3*	4,2
RLL	2,2	+4,2	+4,2

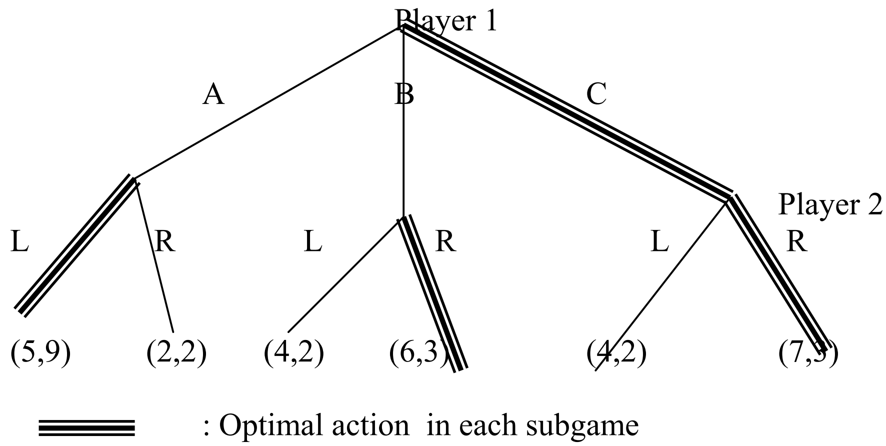
+: BR for player 1

*: BR for player 2

shaded area: NE

NE are (A;LLL), (B;RRL), (B;LRL), (C;LLR), (C;LRR), (C;RLR), (C;RRR).

(d) To solve for SPE, use backward induction.



SPE is (C;LRR)

6. Single period NE in Bertrand game

NE: $p_1 = p_2 = c$

$\Pi_1 = \Pi_2 = 0$

If firms cooperate, firms play $p_1^j = p_2^j = p^j$ such that p^j maximize cartel's profit,

$$\Pi = (p-c)\left(\frac{a}{b} - \frac{1}{b}p\right)$$

$$\Pi_1 = \Pi_2 = \frac{1}{2}\Pi$$

FOC: $\left(\frac{a}{b} - \frac{1}{b}p\right) - \frac{1}{b}(p-c) = 0$

$a-p-p+c=0$

$p^j = \frac{a+c}{2}$ (notice that this is the same as question 1 part 3)

$$\Pi^j = \frac{(a-c)^2}{8b}$$

If firm 1 plays p^j, firm 2 can cheat by playing

$$BR_2(p^j) = p^j - \varepsilon = \frac{a+c}{2} - \varepsilon$$

In the situation where one firm cheat wile the other cooperate, profit of the coop firm = 0.

Profit of the cheating firm,

$$\Pi^x = \left(\frac{a-c}{2} - \varepsilon\right)\left(\frac{a}{b} - \frac{1}{b}\left(\frac{a+c}{2} - \varepsilon\right)\right) = \left(\frac{a-c-2\varepsilon}{2}\right)\left(\frac{a-c+2\varepsilon}{2b}\right)$$

$$= \frac{(a-c)^2 - 4\varepsilon^2}{4b}$$

This is also a prisoner's dilemma problem.

Payoff Matrix

Firm1, Firm 2	Cheat	Cartel
Cheat	0,0	Π^x,0
Cartel	$0, \Pi^x$	Π^j, Π^j

Nash reversion strategy for this game is

Firm 1 plays $p_1 = p^j$ if $p_{2,t-1} = p^j$

plays $p_1 = c$ otherwise.

Similarly for firm 2.

To show that these strategy constitute an SPE, we have to find conditions that prescribe best response behavior for firm1 given firm 2 is following this strategy in each subgame.

I. Consider the period after which cheating has occurred. Suppose firm 1 cheated. If firm 2 follows Nash reversion , it should play p = c. Firm1's best response is also to play p=c. Thus, in these subgames, Nash reversion prescribes best response behavior.

II. Consider period after which no cheating has occurred. Will firm 1 play p^j given firm 2 plays Nash reversion?

Recall $\boxed{\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}}$

Let δ be discount factor where $0 \le \delta \le 1$.

PDV (present discounted value) of profit from playing $p^j = \frac{\Pi^j}{1-\delta}$

PDV of profit form cheating $= \Pi^x + \frac{\delta}{1-\delta} 0 = \Pi^x$

Nash reversion prescribes best response if $\frac{\Pi^j}{1-\delta} > \Pi^x$

$$\frac{(a-c)^2}{(1-\delta)8b} > \frac{(a-c)^2 - 4\varepsilon^2}{4b}$$

$$\delta > 1- \frac{(a-c)^2}{2((a-c)^2 - 4\varepsilon^2)}$$

$$\lim_{\varepsilon \to 0} 1- \frac{(a-c)^2}{2((a-c)^2 - 4\varepsilon^2)} = \frac{1}{2}$$

As $\varepsilon \to 0$, if $\delta > \frac{1}{2}$, then Nash reversion is SPE.