Papers to be covered:

- Laffont, Ossard, and Vuong (1995)
- Guerre, Perrigne, and Vuong (2000)
- Haile, Hong, and Shum (2003)

In this lecture, focus on empirical of auction models. Auctions are models of asymmetric information which have generated the most interest empirically. We begin by summarizing some relevant theory.

1 Theoretical background

An auction is a game of incomplete information. Assume that there are N players, or bidders, indexed by $i = 1, \ldots, N$. There are two fundamental random elements in any auction model.

- Bidders' private signals $X_1, \ldots, X_N$. We assume that the signals are scalar random variables, although there has been recent interest in models where each signal is multi-dimensional.
- Bidders' utilities: $u_i(X_i, X_{-i})$, where $X_{-i} \equiv \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N\}$, the vector of signals excluding bidder i's signal. Since signals are private, $V_i \equiv u_i(X_i, X_{-i})$ is a random variable from *all* bidders' point of view. In what follows, we will also refer to bidder i's (random) utility as her *valuation*.

Differing assumptions on the form of bidders' utility function lead to the important distinction between **common value** and **private value** models. In the private value case, $V_i = X_i$, $\forall i$: each bidder knows his own valuation, but not that of his rivals.[1] In the (pure) common value case, $V_i = V$, $\forall i$, where V is in turn a random variable from all bidders' point of view, and bidders' signals are to be interpreted as their noisy estimates of the true but known common value V. Therefore, signals will generally not be independent when common values are involved. More generally a common value model arises when $u_i(X_i, X_{-i})$ is functionally dependent on X_{-i}.

[1] More generally, in a private value model, $u_i(X_i, X_{-i})$ is restricted to be a function *only* of X_i.

Before proceeding, we give some examples to illustrate the auction formats discussed above.

- Symmetric independent private values (IPV) model: $X_i \sim F$, *i.i.d.* across all bidders i, and $V_i = X_i$. Therefore, $F(X_1, \ldots, X_N) = F(X_1) * F(X_2) \cdots * F(X_N)$, and $F(V_1, X_1, \ldots, V_N, X_N) = \prod_i [F(X_i)]^2$.
- Conditional independent model: signals are independent, conditional on a common component V. $V_i = V, \forall i$, but $F(V, X_1, \ldots, X_N) = F(V) \prod_i F(X_i|V)$.

Models also differ depending on the auction rules. In a **first-price** auction, the object is awarded to the highest bidder, at her bid. A **second-price** auction also awards the object to the highest bidder, but she pays a price equal to the bid of the second-highest bidder. (Sometimes second-price auctions are also called "Vickrey" auctions, after the late Nobel laureate William Vickrey.) In an **English** or **ascending** auction, the price the raised continuously by the auctioneer, and the winner is the last bidder to remain, and he pays an amount equal to the price at which all of his rivals have dropped out of the auction. In a **Dutch** auction, the price is *lowered* continuously by the auctioneer, and the winner is the first bidder to agree to pay any price.

There is a large amount of theory and empirical work. In this lecture, we focus on first-price auction models. We also discuss a few theoretical concepts that will come up in the empirical papers discussed later.

1.1 Equilibrium bidding

In discussing equilibrium bidding in the different auction models, we will focus on the general symmetric affiliated model, used in the seminal paper of Milgrom and Weber (1982). The assumptions made in this model are:

- $V_i = u_i(X_i, X_{-i})$
- Symmetry: $F(V_1, X_1, \ldots, V_N, X_N)$ is symmetric (i.e., exchangeable) in the indices i so that, for example, $F(V_N, X_N, \ldots, V_1, X_1) = F(V_1, X_1, \ldots, V_N, X_N)$.
- The random variables $V_1, \ldots, V_N, X_1, \ldots, X_N$ are affiliated. Let $Z_1, \ldots, Z_M$ and $Z_1^*, \ldots, Z_M^*$ denote two realizations of a random vector process, and $\bar{Z}$ and $\underline{Z}$ denote, respectively, the component-wise maximum and minimum. Then we say that

$Z_1, \ldots, Z_M$ are affiliated if $F(\bar{Z})F(\underline{Z}) \geq F(Z_1, \ldots, Z_M)F(Z_1^*, \ldots, Z_M^*)$. In other words, large values for some of the variables make large values for the other variables most likely. Affiliation implies useful stochastic orderings on the conditional distributions of bidders' signals and valuations, which is necessary in deriving monotonic equilibrium bidding strategies.

Let $Y_i \equiv \max_{j \neq i} X_j$, the highest of the signals observed by bidder i's rivals. Given affiliation, the conditional expectation $E[V_i|X_i, Y_i]$ is increasing in both X_i and Y_i.

Winner's curse Another consequence of affiliation is the *winner's curse*, which is just the fact that

$$E[V_i|X_i] \geq E[V_i|X_i > Y_i]$$

where the conditioning event in the second expectation ($X_i > Y_i$) is the event of winning the auction.

To see this, note that

$$\begin{aligned}
E[V_i|X_i] = E_{X_{-i}} E\left[V_i|X_i; X_{-i}\right] &= \underbrace{\int \cdots \int}_{N-1} E\left[V_i|X_i; X_{-i}\right] F\left(dX_1, \ldots, dX_N\right) \\
&\geq \underbrace{\int^{X_i} \cdots \int^{X_i}}_{N-1} E\left[V_i|X_i; X_{-i}\right] F\left(dX_1, \ldots, dX_N\right) \\
&= E\left[V_i|X_i > X_j, j \neq i\right] = E\left[V_i|X_i > Y_i\right].
\end{aligned}$$

In other words, if bidder i "naively" bids $E\left[V_i|X_i\right]$, her expected payoff from a first-price auction is negative for every X_i. In equilibrium, therefore, rational bidders should "shade down" their bids by a factor to account for the winner's curse.

This winner's curse intuition arises in many non-auction settings also. For example, in two-sided markets where traders have private signals about unknown fundamental value of the asset, the ability to consummate a trade is "good news" for sellers, but "bad news" for buyers, implying that, without ex-ante gains from trade, traders may not be able to settle on a market-clearing price. The result is the famous "lemons" result by Akerlof (1970), as well as a version of the "no-trade" theorem in Milgrom and Stokey (1982). Glosten and Milgrom (1985) apply the same intuition to explain bid-ask spreads in financial markets.

Next, we cover some specific auction results in some detail, in order to understand methodology in the empirical papers.

1.2 First-price auctions

We derive the symmetric monotonic equilibrium bidding strategy $b^*(\cdot)$ for first-price auctions. If bidder i wins the auction, he pays his bid $b^*(X_i)$. His expected profit is

$$\begin{aligned}
&= E\left[(V_i - b)\mathbf{1}\left(b^*(Y_i) < b\right)|X_i = x\right] \\
&= E_{Y_i} E\left[(V_i - b)\mathbf{1}\left(Y_i < b^{*-1}(b)\right)|X_i = x, Y_i\right] \\
&= E_{Y_i}\left[(V(x, Y_i) - b)\mathbf{1}\left(Y_i < b^{*-1}(b)\right)|X_i = x\right] \\
&= \int_{-\infty}^{b^{*-1}(b)} (V(x, Y_i) - b) f(Y_i|x) dY_i.
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
0 = & - \int_{-\infty}^{b^{*-1}(b)} f(Y_i|x)dY_i + \frac{1}{b^{*\prime}(x)}\left[(V(x,x) - b) * f_{Y_i|X_i}(x|x)\right] \Leftrightarrow \\
0 = & - F_{Y_i|x}(x|x) + \frac{1}{b^{*\prime}(x)}\left[(V(x,x) - b) * f_{Y_i|X_i}(x|x)\right] \Leftrightarrow \\
b^{*\prime}(x) = & \left(V(x,x) - b^*(x)\right)\left[\frac{f(x|x)}{F(x|x)}\right] \Rightarrow \\
b^*(x) = & \exp\left(-\int_{\underline{x}}^{x} \frac{f(s|s)}{F(s|s)} ds\right) b(\underline{x}) + \int_{\underline{x}}^{x} V(\alpha, \alpha) dL(\alpha|x)
\end{aligned}$$

where

$$L(\alpha|x) = \exp\left(-\int_{\alpha}^{x} \frac{f(s|s)}{F(s|s)}\right).$$

Initial condition: $b(\underline{x}) = V(\underline{x}, \underline{x})$.

For the IPV case:

$$\begin{aligned}
V(\alpha, \alpha) &= \alpha \\
F(s|s) &= F(s)^{N-1} \\
f(s|s) &= (n-1)F(s)^{N-2} f(s)
\end{aligned}$$

An example $X_i \sim \mathcal{U}[0,1]$, *i.i.d.* across bidders i. Then $F(s) = s$, $f(s) = 1$. Then

$$\begin{aligned} b^*(x) &= 0 + \int_0^x \alpha \exp\left(-\int_\alpha^x \frac{(n-1)f(s)}{sF(s)} ds\right) \frac{(n-1)f(\alpha)}{\alpha F(\alpha)} d\alpha \\ &= \int_0^x \exp\left(-(n-1)(\log \frac{x}{\alpha})\right)(n-1)d\alpha \\ &= \int_0^x \left(\frac{\alpha}{x}\right)^{N-1} (N-1)d\alpha \\ &= \alpha\left(\frac{N-1}{N}\right)\left(\frac{\alpha}{x}\right)^N\Big]_0^x \\ &= \left(\frac{N-1}{N}\right)x. \end{aligned}$$

1.2.1 Reserve prices

A reserve price just changes the initial condition of the equilibrium bid function. With reserve price r, initial condition is now $b(x^*(r)) = r$. Here $x^*(r)$ denotes the screening value, defined as

$$x^*(r) \equiv \inf\left\{x: \ E\left[V_i | X_i = x, Y_i < x\right] \geq r\right\}. \tag{1}$$

Conditional expectation in brackets is value of winning to bidder i, who has signal x. Screening value is lowest signal such that bidder i is willing to pay at least the reserve price r.

(Note: in PV case, $x^*(r) = r$. In CV case, with affiliation, generally $x^*(r) > r$, due to winners curse.)

Equilibrium bidding strategy is now:

$$b^*(x) \begin{cases} = \exp\left(-\int_{x^*(r)}^x \frac{f(s|s)}{F(s|s)} ds\right) r + \int_{x^*(r)}^x V(\alpha,\alpha) dL(\alpha|x) & \text{for } x \geq x^*(r) \\ < r & \text{for } x < x^*(r) \end{cases}$$

For IPV, uniform example above:

$$b^*(x) = \left(\frac{N-1}{N}\right)x + \frac{1}{N}\left(\frac{r}{x}\right)^{N-1} r.$$

1.3 Second-price auctions

Assume the existence of a monotonic equilibrium bidding strategy $b^*(x)$. Next we derive the functional form of this equilibrium strategy.

Given monotonicity, the price that bidder i will pay (if he wins) is $b^*(Y_i)$: the bid submitted by his closest rivals. He only wins when his bid $b < b^*(Y_i)$. Therefore, his expected profit from participating in the auction with a bid b and a signal $X_i = x$ is:

$$\begin{aligned}
& E_{Y_i}\left[(V_i - b^*(Y_i))\,\mathbf{1}\left(b^*(Y_i) < b\right)|X_i = x\right] \\
=& E_{Y_i}\left[(V_i - b^*(Y_i))\,\mathbf{1}\left(Y_i < X_i\right)|X_i = x\right] \\
=& E_{Y_i|X_i} E\left[(V_i - b^*(Y_i))\,\mathbf{1}\left(Y_i < X_i\right)|X_i = x, Y_i\right] \\
=& E_{Y_i|X_i}\left[(E(V_i|X_i, Y_i) - b^*(Y_i))\,\mathbf{1}\left(Y_i < X_i\right)\right] \\
\equiv& E_{Y_i|X_i}\left[(v(X_i, Y_i) - b^*(Y_i))\,\mathbf{1}\left(Y_i < X_i\right)\right] \\
=& \int_{-\infty}^{(b^*)^{-1}(b)} \left(v(x, Y_i) - b^*(Y_i)\right) f\left(Y_i|X_i = x\right).
\end{aligned} \tag{2}$$

Bidder i chooses his bid b to maximize his profits. The first-order conditions are (using Leibniz' rule):

$$\begin{aligned}
0 &= b^{*-1\prime}(b) * \left[v(x, b^{*-1}(b)) - b^*(b^{*-1}(b))\right] * f(b^{*-1}(b)|X_i) \Leftrightarrow \\
0 &= \frac{1}{b^{*\prime}(b)}\left[v(x,x) - b^*(x)\right] * f(b^{*-1}(b)|X_i) \Leftrightarrow \\
b^*(x) &= v(x,x) = E\left[V_i|X_i = x, Y_i = x\right].
\end{aligned}$$

In the PV case, the equilibrium bidding strategy simplifies to

$$b^*(x) = v(x,x) = x.$$

With reserve price, equilibrium strategy remains the same, except that bidders with signals less than the screening value $x^*(r)$ (defined in Eq. (1) above) do not bid.

2 Laffont-Ossard-Vuong (1995): "Econometrics of First-Price Auctions"

- Structural estimation of 1PA model, in IPV context.
- Example of a parametric approach to estimation.
- Goal of empirical work:
  - We observe bids $b_1, \ldots, b_n$, and we want to recover valuations $v_1, \ldots, v_n$.

  - Why? Analogously to demand estimation, we can evaluate the "market power" of bidders, as measured by the margin $v - p$.

    Could be interesting to examine: how fast does margin decrease as n (number of bidders) increases?
  - Useful for the optimal design of auctions:
    1. What is auction format which would maximize seller revenue?
    2. What value for reserve price would maximize seller revenue?

- Another exercise in simulation estimation

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MODEL

- I bidders

- Information structure is IPV: valuations $v^i,\ i = 1, \ldots, I$ are *i.i.d.* from $F(\cdot|z_l, \theta)$ where l indexes auctions, and z_l are characteristics of l-th auctions
- θ is parameter vector of interest, and goal of estimation
- p^0 denotes "reserve price": bid is rejected if $< p^0$.
- Dutch auction: strategically identical to first-price sealed bid auction.

Equilibrium bidding strategy is:

$$b^i = e\left(v^i, I, p^0, F\right) = \begin{cases} v^i - \frac{\int_{p^0}^{v^i} F(x)^{I-1} dx}{F(v^i)^{I-1}} & \text{if } v^i > p^0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Note: (1) $b^i(v^i = p^0) = p^0$; (2) strictly increasing in v^i.

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Dataset: only observe *winning bid* b_l^w for each auction l. Because bidders with lower bids never have a chance to bid in Dutch auction.

Given monotonicity, the winning bid $b^w = e\left(v_{(I)}, I, p^0, F\right)$, where $v_{(I)} \equiv \max_i v^i$ (the highest order statistic out of the I valuations).

Furthermore, the CDF of $v_{(I)}$ is $F(\cdot|z_l,\theta)^I$, with corresponding density $I \cdot F^{I-1} f$.

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Goal is to estimate θ by (roughly speaking) matching the winning bid in each auction l to its expectation.

Expected winning bid is (for simplicity, drop z_l and θ now)

$$\begin{aligned} E_{v_{(I)}>p^0}(b^w) &= \int_{p^0}^{\infty} e\left(v_{(I)}, I, p^0, F\right) I \cdot F(v|\theta)^{I-1} f(v|\theta) dv \\ &= I \int_{p^0}^{\infty} \left(v - \frac{\int_{p^0}^{v} F(x)^{I-1} dx}{F(v)^{I-1}} \right) F(v|\theta)^{I-1} f(v|\theta) dv \\ &= I \int_{p^0}^{\infty} \left(v \cdot F(v)^{I-1} - \int_{p^0}^{\infty} F(x)^{I-1} dx \right) f(v) dv. \quad (*). \end{aligned}$$

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If we were to estimate by simulated nonlinear least squares, we would proceed by finding θ to minimize the sum-of-squares between the observed winning bids and the predicted winning bid, given by expression (*) above. Since (*) involves complicated integrals, we would simulate (*), for each parameter vector θ.

How would this be done:

- Draw valuations v^s, $s = 1, \ldots, S$ i.i.d. according to $f(v|\theta)$. This can be done by drawint $u_1, \ldots, u_S$ i.i.d. from the $U[0,1]$ distribution, then transform each draw:
$$v_s = F^{-1}(u_s|\theta).$$
- For each simulated valuation v_s, compute integrand $\mathcal{V}_s = v_s F(v_s|\theta)^{I-1} - \int_{p^0}^{v_s} F(x|\theta)^{I-1} dx$. (Second term can also be simulated, but one-dimensional integral is that very hard to compute.)
- Approximate the expected winning bid as $\frac{1}{S} \sum_s \mathcal{V}_s$.

However, the authors do not do this— they propose a more elegant solution. In particular, they simplify the simulation procedure for the expected winning bid by appealing to the **Revenue-Equivalence Theorem**: an important result for auctions where bidders' signals are independent, and the model is symmetric. (This was first derived explicitly in Myerson (1981), and this statement is due to Klemperer (1999).)

Theorem 1 *(Revenue Equivalence) Assume each of N risk-neutral bidders has a privately-known signal X independently drawn from a common distribution F that is strictly increasing and atomless on its support $[\underline{X}, \bar{X}]$. Any auction mechanism which is (i)* efficient *in awarding the object to the bidder with the highest signal with probability one; and (ii) leaves any bidder with the lowest signal $\underline{X}$ with zero surplus yields the same expected revenue for the seller, and results in a bidder with signal x making the same expected payment.*

From a mechanism design point of view, auctions are complicated because they are multiple-agent problems, in which a given agent's payoff can depend on the reports of all the agents. However, in the independent signal case, there is no gain (in terms of stronger incentives) in making any given agent's payoff depend on her rivals' reports, so that a symmetric auction with independent signal essentially boils down to independent contracts offered to each of the agents individually.

Furthermore, in any efficient auction, the probability that a given agent with a signal x wins is the same (and, in fact, equals $F(x)^{N-1}$). This implies that each bidder's expected surplus function (as a function of his signal) is the same, and therefore that the expected payment schedule is the same.

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By RET:

- expected revenue in 1PA same as expected revenue in 2PA
- expected revenue in 2PA is $Ev^{(I-1)}$
- with reserve price, expected revenue in 2PA is $E\max(v^{(I-1)}, p^0)$. (Note: with IPV structure, reserve price r screens out same subset of valuations $v \leq r$ in both 1PA and 2PA.)

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In 1PA, expected revenue corresponds to expected winning bid. Hence, expected winning bid is:

$$E_{v_{(I)}} b^*(v_{(I)}) = E_{v_{(I)}} E\left[\max\left(v_{(I-1)}, p^0\right) | v_{(I)}\right] = E\left[\max\left(v_{(I-1)}, p^0\right)\right]$$

which is insanely easy to simulate:

For each parameter vector θ, and each auction l

- For each simulation draw $s = 1, \dots, S$:
  - Draw $v_1^s, \dots, v_{I_l}^s$: vector of simulated valuations for auction l (which had I_l participants)
  - Sort the draws in ascending order: $v_{1:I_l}^s < \cdots < v_{I_l:I_l}^s$
  - Set $b_l^{w,s} = v_{I-1_l:I_l}$ (ie. the second-highest valuation)
  - If $b_l^{w,s} < p_l^0$, set $b_l^{w,s} = p_l^0$. (ie. $b_l^{w,s} = \max\left(v_{I-1_l:I_l}^s, p_l^0\right)$)
- Approximate $E\left(b_l^w; \theta\right) = \frac{1}{S} \sum_s b_l^{w,s}$.

Estimate θ by simulated nonlinear least squares:

$$\min_\theta \frac{1}{L} \sum_{l=1}^{L} \left(b_l^w - E\left(b_l^w; \theta\right)\right)^2 .$$

Results.

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Remarks:

- Problem: bias when number of simulation draws S is fixed (as number of auctions $L \to \infty$). Propose bias correction estimator, which is consistent and asymptotic normal under these conditions.
- This clever methodology is useful for independent value models: works for all cases where revenue equivalence theorem holds.
- Does not work for affiliated value models (including common value models)

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3 Application: internet used car auctions

- Consider Lewis (2007) paper on used cars sold on eBay (simplified exposition)
- Question: does information revealed by sellers lead to high prices? (Question about the credibiltiy of information revealed by sellers.)

- Observe transsactions price in ascending auction. Assume that transaction price is equal to
$$v(X_{n-1:n}, X_{n-1:n})$$
(as in second-price auction).

- Consider pure common value setup with conditionally independent signals. Log-normality is assumed:
$$\begin{aligned}\tilde{v} \equiv \log v = \mu + \sigma\epsilon_v \;\sim\; N(\mu, \sigma^2)\\ x_i|\tilde{v} = \tilde{v} + r\epsilon_i \;\sim\; N(\tilde{v}, r^2)\end{aligned}$$
These are, respectively, the prior distribution of valuations, and the conditional distribution of signals.

- Allow seller information variables z to affect the mean and variance of the prior distribution:
$$\begin{aligned}\mu &= \alpha' z\\ \sigma &= \kappa(\beta' z).\end{aligned}$$
$\kappa(\cdot)$ is just a transformation of the index $\beta' z$ to ensure that the estimate of $\sigma > 0$.

- z includes variables such as: number of photos, how much text is on the website. (Larger z denotes better information.)

- Question: is $\alpha > 0$?

- Results.

4 Guerre-Perrigne-Vuong (2000): Nonparametric Identification and Estimation in IPV First-price Auction Model

The recent emphasis in the empirical literature is on *nonparametric* identification and estimation of auction models. Motivation is to estimate bidders' unobserved valuations, while avoiding parametric assumption (as in the LOV paper).

- Recall first-order condition for equilibrium bid (general affiliated values case):
$$b'(x) = (v(x,x) - b(x)) \cdot \frac{f_{y_i|x_i}(x|x)}{F_{y_i|x_i}(x|x)}; \quad y_i \equiv \max_{j\neq i} x_i. \tag{4}$$

- In IPV case:

$$\begin{aligned}
v(x, x) &= x \\
F_{y_i|x_i}(x|x) &= F(x)^{n-1} \\
f_{y_i|x_i}(x|x) &= \frac{\partial}{\partial x} F(x)^{n-1} = (n-1)F(x)^{n-2}f(x)
\end{aligned}$$

so that first-order condition becomes

$$\begin{aligned}
b'(x) &= (x - b(x)) \cdot (n-1)\frac{F(x)^{n-2}f(x)}{F(x)^{n-1}} \\
&= (x - b(x)) \cdot (n-1)\frac{f(x)}{F(x)}.
\end{aligned} \tag{5}$$

- Now, note that because equilibrium bidding function $b(x)$ is just a monotone increasing function of the valuation x, the change of variables formulas yield that (take $b_i \equiv b(x_i)$)

  -
  $$G(b_i) = F(x_i)$$

  -
  $$g(b_i) = f(x_i) \cdot 1/b'(x_i)$$

  .

Hence, substituting the above into Eq. (5):

$$\begin{aligned}
\frac{1}{g(b_i)} &= (n-1)\frac{x_i - b_i}{G(b_i)} \\
\Leftrightarrow x_i &= b_i + \frac{G(b_i)}{(n-1)g(b_i)}.
\end{aligned} \tag{6}$$

Everything on the RHS of the preceding equation is observed: the equilibrium bid CDF G and density g can be estimated directly from the data, in nonparametric fashion (assuming a dataset consisting of T n-bidder auctions):

$$\begin{aligned}
\hat{g}(b) &\approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{1}{h} \mathcal{K}\left(\frac{b - b_{it}}{h}\right) \\
\hat{G}(b) &\approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{\mathcal{K}\left(\frac{b - b_{it}}{h}\right) \cdot \mathbf{1}(b_{it} \leq b)}{\sum_{t=1}^{T} \sum_{i=1}^{n} \mathcal{K}\left(\frac{b - b_{i't'}}{h}\right)}.
\end{aligned} \tag{7}$$

The first is a kernel density estimate of bid density. The second is a kernel-smoothed empirical CDF. The bandwidth h is a design variable which should be chosen such that it $\rightarrow 0$ as $T \rightarrow \infty$.

- Hence, the IPV first-price auction model is nonparametrically identified. For each observed bid b_i, the corresponding valuation $x_i = b^{-1}(b_i)$ can be recovered as:

$$\hat{x}_i = b_i + \frac{\hat{G}(b_i)}{(n-1)\hat{g}(b_i)}. \tag{8}$$

Hence, GPV recommend a two-step approach to estimating the valuation distribution $f(x)$:

1. In first step, estimate $G(b)$ and $g(b)$ nonparametrically, using Eqs. (7).

2. In second step, estimate $f(x)$ by using kernel density estimator of recovered valuations:

$$\hat{f}(x) \approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{1}{h} \mathcal{K}\left(\frac{x - \hat{x}_{it}}{h}\right). \tag{9}$$

Deriving the asymptotics of the quantity (9) is complicated by the fact that the valuations $\hat{x}_{it}$ are estimated. In their paper, GPV derive an upper bound on the rate at which the function $f(x)$ can be estimated using their procedure, and show how this rate can be achieved (by the right choice of the bandwidth sequence).

Athey and Haile (2002) shows many nonparametric identification results for a variety of auction models (first-price, second-price) under a variety of assumption on the information structure (symmetry, asymmetry). They focus on situations when only a subset of the bids submitted in an auction are available to a researcher.

As an example of such a result, we see that identification continues to hold, even when only the highest-bid in each auction is observed. Specifically, if only $b_{n:n}$ is observed, we can estimate $G_{n:n}$, the CDF of the maximum bid, from the data. Note that the relationship between the CDF of the maximum bid and the marginal CDF of an equilibrium bid is

$$G_{n:n}(b) = G(b)^n$$

implying that $G(b)$ can be recovered from knowledge of $G_{n:n}(b)$. Once $G(b)$ is recovered, the corresponding density $g(b)$ can also be recovered, and we could solve Eq. (8) for every b to obtain the inverse bid function.

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5 Affiliated values models

Can this methodology be extended to affiliated values models (including common value models)?

However, Laffont and Vuong (1996) nonidentification result: from observation of bids in n-bidder auctions, the affiliated private value model (ie. a PV model where valuations are dependent across bidders) is indistinguishable from a CV model.

- Intuitively, all you identify from observed bid data is joint density of $b_1, \ldots, b_n$. In particular, can recover the correlation structure amongst the bids. But correlation of bids in an auction could be due to both affiliated PV, or to CV.

5.1 Affiliated private value models

Li, Perrigne, and Vuong (2002) proceed to consider nonparametric identification and estimation of the affiliated private values model. In this model, valuations $x_i, \ldots, x_n$ are drawn from some joint distribution (and there can be arbitrary correlation amongst them).

First order condition for this model is: for bidder i

$$b'(x_i) = (x_i - b(x)) \cdot \frac{f_{y_i|x_i}(x|x)}{F_{y_i|x_i}(x|x)}; \quad y_i \equiv \max_{j \neq i} x_i.$$

where $y_i \equiv \max_{j \neq i} \{x_1, \ldots, x_n\}$.

Procedure similar to GPV can be used here to recover, for each bid b_i, the corresponding valuation $x_i = b^{-1}(b_i)$. As before, exploit the following change of variable formulas:

- $$G_{b^*|b}(b|b) = F_{y|x}(x|x)$$
- $$g_{b^*|b}(b|b) = f_{y|x}(x|x) \cdot 1/b'(x)$$

where b^* denotes (for a given bidder), the highest bid submitted by this bidder's rivals: for a given bidder i, $b_i^* = \max_{j \neq i} b_j$. To prepare what follows, we introduce n subscript (so we index distributions according to the number of bidders in the auction).

Li, Perrigne, and Vuong (2000) suggest nonparametric estimates of the form

$$\begin{aligned}\hat{G}_n(b;b) &= \frac{1}{T_n \times h \times n}\sum_{t=1}^{T}\sum_{i=1}^{n} K\left(\frac{b-b_{it}}{h}\right)\mathbf{1}\left(b_{it}^{*} < b, n_t = n\right) \\ \hat{g}_n(b;b) &= \frac{1}{T_n \times h^2 \times n}\sum_{t=1}^{T}\sum_{i=1}^{n} \mathbf{1}(n_t = n) K\left(\frac{b-b_{it}}{h}\right) K\left(\frac{b-b_{it}^{*}}{h}\right).\end{aligned} \tag{10}$$

Here h and h are bandwidths and $K(\cdot)$ is a kernel. $\hat{G}_n(b;b)$ and $\hat{g}_n(b;b)$ are nonparametric estimates of

$$G_n(b;b) \equiv G_n(b|b) g_n\,(b) = \frac{\partial}{\partial b}\Pr(B_{it}^{*} \le m, B_{it} \le b)|_{m=b}$$

and

$$g_n(b;b) \equiv g_n(b|b) g_n\,(b) = \frac{\partial^2}{\partial m \partial b}\Pr(B_{it}^{*} \le m, B_{it} \le b)|_{m=b}$$

respectively, where $g_n(\cdot)$ is the marginal density of bids in equilibrium. Because

$$\frac{G_n(b;b)}{g_n(b;b)} = \frac{G_n(b|b)}{g_n(b|b)} \tag{11}$$

$\frac{\hat{G}_n(b;b)}{\hat{g}_n(b;b)}$ is a consistent estimator of $\frac{G_n(b|b)}{g_n(b|b)}$. Hence, by evaluating $\hat{G}_n(\cdot,\cdot)$ and $\hat{g}_n(\cdot,\cdot)$ at each observed bid, we can construct a pseudo-sample of consistent estimates of the realizations of each $x_{it} = b^{-1}(b_{it})$ using Eq. (4):

$$\hat{x}_{it} = \frac{\hat{G}_n(b_{it}; b_{it})}{\hat{g}_n(b_{it}; b_{it})}. \tag{12}$$

Subsequently, joint distribution of $x_1, \ldots, x_n$ can be recovered as sample joint distribution of $\hat{x}_1, \ldots, \hat{x}_n$.

5.2 Common value models: testing between CV and PV

Laffont-Vuong did not consider variation in n, the number of bidders.

In Haile, Hong, and Shum (2003), we explore how variation in n allows us to test for existence of CV.

Introduce notation:

$$v(x_i, x_i, n) = E[V_i | X_i = x_i, \max_{j \neq i} X_j = x_i, n].$$

Recall the winner's curse: it implies that $v(x, x, n)$ is invariant to n for all x in a PV model but strictly decreasing in n for all x in a CV model.

Consider the first-order condition in the common value case:

$$b'(x,n) = (v(x,x,n) - b(x,n)) \cdot \frac{f_{y_i|x_i,n}(x|x)}{F_{y_i|x_i,n}(x|x)}; \quad y_i \equiv \max_{j \neq i} x_i.$$

Hence, the Li, Perrigne, and Vuong (2002) procedure from the previous section can be used to recover the "pseudovalue" $v(x_i, x_i, n)$ corresponding to each observed bid b_i. Note that we cannot recover $x_i = b^{-1}(b_i)$ itself from the first-order condition, but can recover $v(x_i, x_i, n)$. (This insight was also articulated in Hendricks, Pinkse, and Porter (2003).)

In Haile, Hong, and Shum (2003), we use this intuition to develop a test for CV:

$$\begin{aligned} H_0 \text{ (PV)} :\ & E\left[v(X,X;\underline{n})\right] = E\left[v(X,X;\underline{n}+1)\right] = \cdots = E\left[v(X,X;\bar{n})\right] \\ H_1 \text{ (CV)} :\ & E\left[v(X,X;\underline{n})\right] > E\left[v(X,X;\underline{n}+1)\right] > \cdots > E\left[v(X,X;\bar{n})\right] \end{aligned}$$

6 Revenue Equivalence, Optimal Auctions, and Linkage Principle

6.1 Revenue Equivalence

Consider the auction as an example of a *mechanism*, which is a general scheme whereby agents with private types v give a report $\tilde{v}$ of their type (where possibly $\tilde{v} \neq v$). The auction mechanism consists of two functions:

- $T(\tilde{v})$: payment agent must make if his report is $\tilde{v}$
- $P(\tilde{v})$: probability of winning the object if his report is $\tilde{v}$.

The *surplus* of an agent with type v, who reports that his type is $\tilde{v}$, is "quasilinear":

$$S(\tilde{v}; v) \equiv vP(\tilde{v}) - T(\tilde{v}).$$

The seller's mechanism design problem is to choose the functions $P(\cdot)$ and $T(\cdot)$ to maximize his expected profits. By the "revelation principle" (for which the 2007 Nobel Laureate Roger Myerson is most well known for), we can restrict attention to mechanisms which satisfy an **incentive compatibility condition**, which requires that agents find it in their best interest to report the truth.

The revenue equivalence result falls out as an implication of quasilinearity and incentive compatibility. Assuming differentiability, incentive compatibility implies the following first-order condition:

$$\left. \frac{\partial S(\tilde{v}; v)}{\partial \tilde{v}} \right|_{\tilde{v}=v} = vP'(v) - T'(v) = 0. \tag{13}$$

Hence, under an IC mechanism, the surplus of an agent with type v is just equal to

$$\bar{S}(v) \equiv S(v; v) = vP(v) - T(v).$$

Given quasilinearity, instead of viewing the seller as setting P and T functions, can view him as setting P and $\bar{S}$ functions, and $T(v) = \bar{S}(v) - vP(v)$.

By Eq. (13), we see that IC places restrictions on the $\bar{S}$ function that seller can choose. Namely, the derivative of the $\bar{S}$ function must obey:

$$\begin{aligned} \bar{S}'(v) &= P(v) + vP'(v) - T'(v) \\ &= P(v) + 0 \end{aligned} \tag{14}$$

where the second line follows from Eq. (13). This operation is the "envelope condition". Hence, integrating up, the $\bar{S}$ function obeys

$$\bar{S}(v) = \bar{S}(\underline{v}) + \int_{\underline{v}}^{v} P(v)dv \tag{15}$$

where $\underline{v}$ denotes the lowest type. From the above, we see that any IC-auction mechanism with quasilinear preferences is completely characterized by (i) the quantity $\bar{S}(\underline{v})$, the surplus gained by the lowest type; and (ii) $P(\cdot)$, the probability of winning.

Now note that, under independence of types, the monotonic equilibrium in the standard auction forms (including first-price and second-price auctions) have the same $P(v)$ function. This is because the object is always awarded to the bidder with the highest type, so that for any v, $P(v)$ is just the probability that a bidder with type v has the highest type; that is, with N bidders, and each type v distributed i.i.d. from distribution $F(v)$, we have $P(v) = [F(v)]^{N-1}$. Furthermore, the standard auction forms also have the condition that $b(\underline{v}) = \underline{v}$, so that $\bar{S}(\underline{v}) = 0$.

Hence, by quasilinearity, the standard auction forms also have the same T function, given by $T(v) = S(v) - vP(v)$. Since the T function is the revenue function for the seller, for each type, we conclude that the standard auction forms give the seller the same expected revenue.

6.2 Optimal auctions

Now consider the seller's optimal choice of mechanism. As the discussion above showed, we can just focus on the seller's choice of $\bar{S}(\underline{v})$ and $P(\cdot)$. These are chosen to maximize expected revenue from selling to each bidder, which is equal to

$$\begin{aligned} ER &= \int_{\underline{v}}^{\bar{v}} T(v)f(v)dv \\ &= \int_{\underline{v}}^{\bar{v}} [vP(v) - \bar{S}(v)]f(v)dv \\ &= \int_{\underline{v}}^{\bar{v}} vP(v)f(v)dv - \int_{\underline{v}}^{\bar{v}} f(v)[\int_{\underline{v}}^{v} P(x)dx]dv - \int_{\underline{v}}^{\bar{v}} \bar{S}(\underline{v})f(v)dv \qquad (16) \\ &= \int_{\underline{v}}^{\bar{v}} P(v)f(v)\left[v - \frac{1-F(v)}{f(v)}\right]dv - \bar{S}(\underline{v}) \\ &\equiv \int_{\underline{v}}^{\bar{v}} P(v)f(v)v^{*}dv. \end{aligned}$$

The third equality follows by substituting in the IC constraint (15). The fourth equality uses integration by parts to simplify the second term, with the substitution $u = \int_{\underline{v}}^{\bar{v}} P(x)dx$ and $dv = f(v)dv$. In the above, V^* is called the "virtual type" corresponding to a type v. The interpretation is that the revenue that the seller makes from a bidder with type v is $v^* \leq v$, so that the deviation $\frac{1-F(v)}{f(v)}$ is the bidder's "information rent" which arises because only the bidder knows his type.

Under the additional assumption:

$$(*) \qquad \frac{1-F(v)}{f(v)} \text{ is decreasing in } v$$

then v^* is increasing in v.[2] In this case, given a set of N bidders, the seller just wants to award the object to the bidder with the highest report. To implement this, he will set $P(v) = Pr(v \text{ is highest out of } N \text{ signals}) = F(v)^{N-1}$. This is just like in the standard auction forms.

However, because of the information rent, he will not be willing to sell to all bidders, but only those with non-negative v^*. Thus he will set a *reserve price*, such that all bidders with $v^* \leq 0$ are excluded. Let r^* denote this optimal reserve price. Recall that, with a reserve price r and IPV, the lower bound on bids is given by the condition $b(r) = r$, that is, the reserve price r screens out bidders with valuations $v \leq r$. Hence, the optimal reserve price r^* satisfies

$$r^* - \frac{1-F(r^*)}{f(r^*)} = 0.$$

r^* depends on the distribution of types $f(v)$. For example, if $v \sim U[0,1]$, then

$$0 = r^* - \frac{1-F(r^*)}{f(r^*)} = r^* - \frac{1-r^*}{1} = 2r^* - 1 \Rightarrow r^* = \frac{1}{2}.$$

6.3 Linkage principle

See handout from Krishna (2002).

6.4 Important results under non-independence

The crucial assumption for the RET is independence of bidder types. This turns a complicated multi-agent mechanism design problem to the single-agent problem, which was how

[2]In statistics, the quantity $f(v)/[1 = F(v)]$ is called the hazard function, so that assumption (*) is equivalently that the hazard function of v is increasing.

we presented it in the previous sections.

When types are no longer independent (as in the general affiliated case of Milgrom and Weber (1982)), revenue equivalence of the standard auction forms no longer holds. However, a remarkable series of papers (Cremer and McLean (1985), Cremer and McLean (1988), McAfee and Reny (1992)) argued that the seller can potentially exploit dependence among bidder types to extract all the surplus in the auction.

Note that in the independent case, if a bidder's valuation is x, but he only needs to pay $x^* = x - \frac{1-F(x)}{f(x)}$, implying that he gets an "information rent" equal to the inverse hazard rate, $\frac{1-F(x)}{f(x)}$ (the highest type $\bar{x}$ gets zero information rent; this is the familiar "no distortion at the top" result from optimal mechanism design). Thus it is indeed remarkable that relaxing the independence assumption allows the designer to extract all the surplus.

We consider this insight for a two-bidder correlated private values auction, as considered in McAfee and Reny (1992) (pp. 397–398). Bidders 1 and 2 have values v^1 and v^2, which are both discrete-valued from the finite set $\{v_1, \ldots, v_n\}$. Let P denote the $n \times n$ matrix of conditional probabilities: the (i, j)-th element of P is the probability that (from bidder 1's point of view) bidder 2 has value v_j, given that bidder 1 has value v_i. Clearly, the rows of P should sum to one.

Cremer and McLean focus on simple two-part mechanisms. First, bidders find out their values, and then choose from among a menu of participation fee schedules offered by the seller. These schedules specify the fixed fee that each bidder must pay, as a function of *both bidders'* reports. Second, the two bidders participate in a Vickrey (second-price) auction. Truth-telling is assured in the Vickrey auction, so we focus on the optimal design of the participation fee schedules assuming that both bidders report the truth, i.e., report v^1 and v^2.

In what follows, we focus on bidder 1 (argument for bidder 2 is symmetric). Intuitively, the seller wants to design the fee schedules so that bidder 1 pays an amount arbitrarily close to his expected profit π_1 from participating which, given his value v^1, is $v^1 - E[v^2|v^1]$ (due to second-stage Vickrey auction). Given the discreteness in values, without loss of generality we can consider fee schedules of the form where, for the m-th schedule, bidder 1 must pay an amount z_{mij} if he announces $v^1 = v_i$, but bidder 2 announces $v^2 = v_j$. Therefore bidder 1's expected profit from schedule m, given his type $v^1 = v_i$, is $v_i - E[v^2|v^1 = v_i] - \sum_{j=1}^{n} p_{ij} z_{mij}$.

Assume that the seller offers n different schedules, indexed $m = 1, \ldots n$. Also assume that

$z_{mij} = v_m - E[v^2|v^1 = v_m] + \alpha_m * x_{mj}$. Hence, the net expected payoff to bidder 1, who has valuation v_i, chooses schedule m, and then the bidders announce (v_i, v_j), are:

$$v_1 - E[v^2|v^1 = v_i] - (v_m - E[v^2|v^1 = v_m]) - \alpha_m \sum_{j=1}^{n} p_{ij} x_{mj}.$$

It turns out that the parameter α will be the same across all schedules, and so we eliminate the m subscript from these quantities.

In order for the seller to design schedules which extract all of bidder 1's surplus, we require that:

1. For each $i = 1, \ldots, n$, there exists a vector $\vec{x}_i \equiv (x_{i1}, \ldots, x_{in})'$ such that

$$\sum_{j=1}^{n} p_{ij} x_{ij} = 0, \quad \sum_{j=1}^{n} p_{i'j} x_{ij} \leq 0, \quad \forall i' \neq i :$$

   that is, the vector $\vec{x}_i$ "separates" the i-th row of P from the other rows, at 0.

2. α must be set sufficiently large so that, for all $i = 1, \ldots, n$, if bidder 1 has value $v^1 = v_i$ he will choose schedule i. This implies that his expected profit from choosing schedule i exceeds that from choosing any schedule $i' \neq i$. These incentive compatibility conditions pin down α:

$$-\alpha * \sum_{j=1}^{n} p_{ij} x_{ij} \geq v_i - E[v^2|v^1 = v_i] - (v_{i'} - E[v^2|v^1 = v_{i'}]) - \alpha * \sum_{j=1}^{n} p_{ij} x_{i'j}, \ \forall i' \neq i$$

$$\Leftrightarrow 0 \geq v_i - E[v^2|v^1 = v_i] - (v_{i'} - E[v^2|v^1 = v_{i'}]) - \alpha_i * \sum_{j=1}^{n} p_{ij} x_{i'j}, \ \forall i' \neq i$$

$$\Leftrightarrow \alpha > \max_{i' \neq i} \frac{\left(v_i - E[v^2|v^1 = v_i]\right) - \left(v_{i'} - E[v^2|v^1 = v_{i'}]\right)}{\sum_{j=1}^{n} p_{ij} x_{i'j}}.$$

A sufficient condition for the first point above is that, for all $i = 1, \ldots, n$, the i-th row of P is not in the (n-dimensional) convex set spanned by all the other rows of P.[3] Then we can apply the separating hyperplane theorem to say that there always exists a vector $(x_{i1}, \ldots, x_{in})'$ which separates the i-th row of P from the convex set composed of convex combinations of the other $n-1$ rows of P.

[3] Or, using terminology in McAfee and Reny (1992), the i-th row of P is not in the convex hull of the other $n-1$ rows of P.

In order for the i-th row of P is not in the (n-dimensional) convex set spanned by all the other rows of P, we require that the i-th row of P cannot be written as a linear combination of the other $n-1$ rows, i.e., that all the rows of P are linear independent. As Cremer and McLean (1988) note, this is a *full rank* condition on P.

As an example, consider the case where the P matrix is diagonal (that is, $p_{ii} = 1$ for all $i = 1, \ldots, n$). This is the case of "Stalin's food taster". In this case, the vector $\vec{x}_i$ with element $x_{ii} = 0$ and $x_{ij} = -1$ for $j \neq i$ satisfies point 1. The numerator of the expression for α is equal to 0. Thus any arbitrarily small participation fee will lead to full surplus extraction!

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