Lecture 10: Price discrimination Part II

EC 105. Industrial Organization.

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Price discrimination is endemic!

- In these lecture notes we examine the mathematical structure of the second-degree price discrimination model in more detail
- compare and contrast with the "usual" monopolist who only sets uniform price.

The Basic Model

- A firm produces a single good at marginal cost c.
- Consumers receive utility $\theta V(q) - T(q)$ if they purchase a quantity q and utility 0 otherwise.
- Two cases:
  1. $\theta \in \{\theta_1, \theta_2\}$
  2. $\theta \in [\underline{\theta}, \overline{\theta}]$

The Two Type Case

- Monopolist offers two bundles (we assume the monopolist serves both types; λ is sufficiently large):
  - (q_1, T_1), directed at type θ_1 consumers (in proportion λ), and
  - (q_2, T_2), directed at type θ_2 consumers (in proportion $1-\lambda$).

- The monopolist's profit is

$$\Pi^m = \lambda(T_1 - cq_1) + (1-\lambda)(T_2 - cq_2)$$

- Monopolist faces two types of constraints. The **individual rationality constraint** for type θ (IR(θ)) requires that consumers of type θ are willing to buy.
  - Since θ_2 consumers can always buy the θ_1 bundle, the relevant IR constraint is IR(θ_1):

$$\theta_1 V(q_1) - T_1 \geq 0 \quad (1)$$

- The **incentive compatibility constraint** for type θ (IC(θ)) requires that consumers of type θ prefer the bundle designed for them rather than that designed for type θ'
- The relevant IC constraint is that of the high-valuation consumers, IC(θ_2):

$$\theta_2 V(q_2) - T_2 \geq \theta_2 V(q_1) - T_1 \tag{2}$$

- In fact, we will proceed ignoring IC(θ_1) and then show that the solution of the subconstrained problem satisfies it.
- We thus solve the problem: max Π^m s.t. (1) and (3)

- The relevant IC constraint is that of the high-valuation consumers, IC(θ_2):

$$\theta_2 V(q_2) - T_2 \geq \theta_2 V(q_1) - T_1 \tag{3}$$

This suggests that T_2 must leave some net surplus to the type θ_2 consumers, because they can always buy the bundle (q_1, T_1) and have net surplus

$$\theta_2 V(q_1) - T_1$$

- To reduce this net surplus, set T_1 as high as possible:
  - Set $T_1 = \theta_1 V(q_1)$, so IR(θ_1) holds with equality.
  - Extract all surplus of the low types
- Now IC(θ_2) can be written as

$$T_2 \leq \theta_2 V(q_2) - [\theta_2 V(q_1) - T_1] = \theta_2 V(q_2) - (\theta_2 - \theta_1) V(q_1)$$

- Monopolist maximizes by setting T_2 as high as possible, ie.

$$T_2 = \theta_2 V(q_2) - (\theta_2 - \theta_1) V(q_1)$$

- Substituting this into the objective function, the monopolist solves the following *unconstrained* problem

$$max_{q_1,q_2} \lambda(\theta_1 V(q_1) - cq_1) + (1-\lambda)[\theta_2 V(q_2) - cq_2 - (\theta_2 - \theta_1)V(q_1)]$$

First Order Conditions are:

$$\theta_1 V'(q_1) = \frac{c}{\left[1 - \frac{1-\lambda}{\lambda}\frac{\theta_2-\theta_1}{\theta_1}\right]} \quad (4)$$

$$\theta_2 V'(q_2) = c \quad (5)$$

  - It follows from (5) that the quantity purchased by the high value consumers is socially optimal (marginal utility equal marginal cost). "No distortion at the top"
  - and from (4) that the quantity consumed by the low-demand consumers is socially suboptimal: $\theta_1 V'(q_1) > c$ and their consumption is distorted downwards. "quantity degradation"

- It remains to check that the low demand consumers do not want to choose the high demand consumers' bundle. Because they have zero surplus, we require that $0 \geq \theta_1 V(q_2) - T_2$. But this condition is equivalent to

$$0 \geq -(\theta_2 - \theta_1)[V(q_2) - V(q_1)],$$

which is satisfied

- Bundle 2, although offering higher quantity, is too expensive for the low types.

Intuition

- Monopolist attempt to extract the high demand consumers' large surplus faces threat of personal arbitrage:
  - High demand consumer can consume the low-demand consumers' bundle if his own bundle does not generate enough surplus.
- To relax this personal arbitrage constraint, the monopolist offers a relatively low consumption to the low demand consumers.
  - OK because typically high demand consumers suffer more from a reduction in consumption than low demand ones (single crossing property).
  - Since low demand consumers are not tempted to exercise personal arbitrage, no distortion at the top (recall welfare gains can be captured by the monopolist through an increase in T_2.

Nonlinear pricing and "quality degradation"

- As noted above, one feature of optimal nonlinear pricing by a firm with market power, typically they will *degrade* the lower tier products on its product line.
  - Since they can't *force* people to choose the higher-quality, more expensive products
  - ... they make lower-tier products so poor that people *willingly* choose higher-tier products
- But in real-world markets, do firms really degrade quality?
  - Look at cable TV

Evidence of quality *overprovision*

TABLE 7—WELFARE EFFECTS OF MARKET POWER OVER QUALITY (AND PRICE)

	Market power over quality (p^{Obs}, q^{Obs}) vs. (p^{Obs}, q^{SP})		Market power over price (p^{Obs}, q^{SP}) vs. (p^{SP}, q^{SP})		Total welfare effect (p^{Obs}, q^{Obs}) vs. (p^{SP}, q^{SP})	
	Mean	SD	Mean	SD	Mean	SD
	(1)		(2)		(3)	
Prices						
Low-quality products	–	–	−0.330	0.180	−0.330	0.180
Medium-quality products	–	–	−0.590	0.220	−0.590	0.220
High-quality products	–	–	−0.740	0.130	−0.740	0.130
Qualities						
Low-quality products	0.550	0.720	–	–	−0.230	0.910
Medium-quality products	0.070	0.110	–	–	−0.370	0.410
High-quality products	0.070	0.040	–	–	−0.550	0.260

- Firms actually offer consumers *too much quality* (and charge a lot)
- Consumers would prefer *lower quality* (and appropriately lower price)
  - Starbucks ("Venti"), Broadband internet (940Mbps)

The Continuum of Types Case

- Now we consider a mathematical generalization.
- Let θ be distributed with density $f(\theta)$ (and CDF F) on an interval $[\underline{\theta}, \overline{\theta}]$.
- Monopolist offers a nonlinear tariff $T(q)$. A consumer with type θ purchases $q(\theta)$ and pays $T(q(\theta))$.
- Monopolist's aggregate profit (across all consumer types) is:

$$\Pi^m = \int_{\underline{\theta}}^{\overline{\theta}} [T(q(\theta)) - cq(\theta)] f(\theta) d\theta$$

- The monopolist maximizes his profit subject to two types of constraints

IR constraints

- For all θ,
$$\theta V(q(\theta)) - T(q(\theta)) \geq 0$$
- As before, it suffices that IR ($\underline{\theta}$) holds:
$$\underline{\theta} V(q(\underline{\theta})) - T(q(\underline{\theta})) \geq 0 \tag{6}$$
- If (6) holds, any type θ can realize a nonegative surplus consuming $\underline{\theta}$'s bundle:
$$\theta V(q(\underline{\theta})) - T(q(\underline{\theta})) \geq (\theta - \underline{\theta}) V(q(\underline{\theta})) \geq 0$$

IC constraints

- θ should not consume the bundle designed for $\tilde{\theta}$ ($\tilde{\theta} \neq \theta$)
- IC(θ): for all $\theta, \tilde{\theta}$:

$$\mathbf{U}(\theta) \equiv \mathbf{U}(\theta, \theta) = \theta V(q(\theta)) - T(q(\theta)) \geq \theta V(q(\tilde{\theta})) - T(q(\tilde{\theta})) \equiv \mathbf{U}(\theta, \tilde{\theta}) \quad (7)$$

- These constraints are not very tractable in this form. However, we can show that it suffices to require that the ICs are satisfied "locally"; i.e., a necessary and sufficient condition for

$$\theta = argmax_{\tilde{\theta}} \mathbf{U}(\theta, \tilde{\theta}) = \theta V(q(\tilde{\theta})) - T(q(\tilde{\theta}))$$

is given by the FOC (evaluated at the true type θ):

$$\theta V'(q(\theta)) = T'(q(\theta)) \quad (8)$$

- This says that a small increase in the quantity consumed by they type θ consumer generates a marginal surplus $\theta V'(q(\theta))$ equal to the marginal payment $T'(q(\theta))$. Thus, the consumer does not want to modify the quantity at the margin.

- Now from the IC

$$\mathbf{U}(\theta) = max_{\tilde{\theta}} \theta V(q(\tilde{\theta})) - T(q(\tilde{\theta}))$$

- Using the envelope theorem (red part =0):

$$\frac{\partial \mathbf{U}(\theta)}{\partial \theta} = V(q(\theta)) + q'(\theta)\left[\theta V'(q(\theta)) - T'(q(\theta))\right]$$

- Thus we can write (use $\mathbf{U}(\underline{\theta}) = 0$)

$$\mathbf{U}(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\partial \mathbf{U}(t)}{\partial t} dt + \mathbf{U}(\underline{\theta}) = \int_{\underline{\theta}}^{\theta} V(q(t)) dt$$

- Note that consumer's utility grows with θ at a rate that increases with $q(\theta)$.
- This is important for the derivation of the optimal quantity function, as it implies that higher quantities "differentiate" different types more, in that the utility differentials are higher.
- Since leaving a surplus to the consumer is costly to the monopolist (recall $T(q(\theta)) = \theta V(q(\theta)) - \mathbf{U}(\theta)$), the monopolist will tend to reduce $\mathbf{U}$ and to do so, will induce (most) consumers to consume a suboptimal quantity.
- Intuition from the previous equations is that optimality will imply a bigger distortion for low-θ consumers.

- Since $T(q(\theta)) = \theta V(q(\theta)) - \mathbf{U}(\theta) = \theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(t))dt$, we can write

$$\Pi^m = \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(t))dt - cq(\theta) \right) f(\theta)d\theta \qquad (9)$$

Recall: Integration by Parts

- If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b), and if u and v are integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx = u(b)v(b) - u(a)v(a)$$

  - Let $g = uv$. Then g'=uv' + vu'. By the fundamental theorem of calculus, $\int_a^b g' = g(b) - g(a)$. Then

$$\int_a^b g'(x)dx = u(b)v(b) - u(a)v(a),$$

    and the result follows.

- Now, integrating by parts, with $f(\theta) = F'(\theta)$ and $\int_{\underline{\theta}}^{\theta} V(q(t))dt = G(\theta)$,

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[\int_{\underline{\theta}}^{\theta} V(q(t))dt \right] f(\theta)d\theta$$

is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} V(q(\theta))d\theta - 0 - \int_{\underline{\theta}}^{\overline{\theta}} V(q(\theta))F(\theta)d\theta = \int_{\underline{\theta}}^{\overline{\theta}} V(q(\theta))[1 - F(\theta)]d\theta$$

- Then going back to (9)

- We can write

$$\Pi^m = \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(t))dt - cq(\theta) \right) f(\theta)d\theta$$

- as

$$\Pi^m = \int_{\underline{\theta}}^{\overline{\theta}} \left([\theta V(q(\theta)) - cq(\theta)]f(\theta) - V(q(\theta))[1 - F(\theta)] \right) d\theta$$

- Now max Π^m w.r.t. $q(\cdot)$ requires that the term under the integral be maximized w.r.t. $q(\theta)$ for all θ, yielding:

$$[\theta V'(q(\theta)) - c]f(\theta) - V'(q(\theta))[1 - F(\theta)] = 0$$

or equivalently

$$\theta V'(q(\theta)) = c + \frac{[1 - F(\theta)]}{f(\theta)} V'(q(\theta)) \tag{10}$$

$$\theta V'(q(\theta)) = c + \frac{[1 - F(\theta)]}{f(\theta)} V'(q(\theta)) > c$$

- Thus marginal willingness to pay for the good $\theta V'(q(\theta))$ exceeds the marginal cost c for all but the highest value consumer $\theta = \overline{\theta}$
- So again, we have:
  - No distortion at the top ($MU = MC$) for $\bar{\theta}$
  - Quantity degradation (for $\theta < \bar{\theta}$)

- Moreover, for this specification of preferences, we can get a simple expression for the price-cost margin.
- Let $T'(q) \equiv p(q)$ denote the marginal price when the consumer is consuming q units.
- From consumer optimization

$$T'(q(\theta)) = \theta V'(q(\theta)) \Rightarrow V'(q(\theta)) = \frac{T'(q(\theta))}{\theta} = \frac{p(q(\theta))}{\theta}$$

- Substituting in (10), which I write again here

$$\theta V'(q(\theta)) = c + \frac{[1 - F(\theta)]}{f(\theta)} V'(q(\theta)),$$

we have:

$$\frac{p(q(\theta)) - c}{p(q(\theta))} = \frac{[1 - F(\theta)]}{\theta f(\theta)} \tag{11}$$

Price-cost margins

- We will assume that the "hazard rate" of the distribution of types, $\frac{f(\theta)}{[1-F(\theta)]}$, is increasing in θ (common assumption, satisfied by a variety of distributions).
- Recall

$$\frac{p(q(\theta)) - c}{p(q(\theta))} = \frac{[1 - F(\theta)]}{\theta f(\theta)} = \frac{1}{\theta \Gamma(\theta)}$$

- The derivative of the RHS with respect to θ is:

$$-1 \frac{1}{[\cdot]^2} [\Gamma(\theta) + \theta \Gamma'(\theta)] < 0$$

- Thus $\frac{p-c}{p}$ **decreases with consumer type,and therefore with output**
  - This is the margin for the "marginal unit". Note that the total profit obtained by the monopolist is always increasing in type though.

Quantities

- The (pointwise in θ) first-order condition (10) as:

$$\left(\theta - \frac{[1 - F(\theta)]}{f(\theta)}\right) V'(q(\theta)) = c$$

- Or letting $\Gamma(\theta) \equiv \left(\theta - \frac{[1-F(\theta)]}{f(\theta)}\right)$, simply as:

$$\Gamma(\theta) V'(q(\theta)) = c,$$

  where $\Gamma'(\theta) > 0$ by our increasing-hazard-rate assumption
- Totally differentiating (with variables $q(\theta)$ and θ), we obtain:

$$\frac{dq(\theta)}{d\theta} = -\frac{\Gamma'(\theta)}{\Gamma(\theta)} \frac{V'(q(\theta))}{V''(q(\theta))} > 0,$$

  using V concave and $\Gamma'(\theta) > 0$

- Thus $q'(\theta) > 0$: $q(\theta)$ increases with θ. (Higher types get higher quantity)

Quantity discounts

- Finally, recall $T'(q) = \theta V'(q)$. Hence

$$T''(q) = \theta V''(q) \quad \text{for } V \text{ concave}$$

- Thus $T(q)$ is concave . As a result:
  - Average price per unit $T(q)/q$ decreases with q (Maskin and Riley's **quantity discount** result).
  - Because a concave function is the lower envelope of its tangents, the optimal nonlinear payment schedule can also be implemented by offering a **menu of two part tariffs** (where the monopolist lets the consumer choose among the continuum of two-part tariffs)

Key takeaways

In this set of lecture notes, we studied the nonlinear pricing (2nd-degree price discrimination) model in more mathematical rigor and established key properties:

- *Quality degradation* ($\theta V'(q(\theta)) > c$) for all types $\theta < \bar{\theta}$
- .. except for highest type $\bar{\theta}$: *no distortion at the top* ($\bar{\theta} V'(q(\bar{\theta})) = c$)
- (empirical evidence from cable TV shows the opposite though..)
- $q'(\theta) > 0$: higher types get more quality
- $T''(\theta) < 0$: quantity discounts

IC constraints

- We argued that it was enough to require that the ICs are satisfied "locally"; i.e., that a necessary and sufficient condition for

$$\theta = argmax_{\tilde{\theta}} \mathbf{U}(\theta, \tilde{\theta}) = V(q(\tilde{\theta}), \theta) - T(q(\tilde{\theta}))$$

  is given by the FOC (evaluated at the true type θ):

$$\frac{\partial V(q(\theta), \theta)}{\partial q} = T'(q(\theta))$$

- This is because of the single crossing property, $\frac{\partial^2 V(q(\theta),\theta)}{\partial q \partial \theta} > 0$

The Inverse Elasticity Rule Again [skip]

- Decompose the aggregate demand function into independent demands for marginal units of consumption. Fix a quantity q and consider the demand for the q^{th} unit of consumption. By definition, the unit has price p. The proportion of consumers willing to buy the unit is

$$D_q(p) \equiv 1 - F(\theta_q^*(p))$$

, where $\theta_q^*(p)$ denotes the type of consumer who is indifferent between buying and not buying the q^{th} unit at price p:

$$\theta_q^*(p)V'(q) = p \tag{12}$$

- The demand for the q-th unit is independent of the demand for the $\tilde{q}th$ unit for $\tilde{q} \neq q$ (due to no income effects). We can thus apply the inverse elasticity rule.

The Inverse Elasticity Rule Again [skip]

- The optimal price for the *qth* unit is given by

$$\frac{p-c}{p} = -\frac{dp}{dD_q}\frac{D_q}{p}$$

- However

$$\frac{dD_q}{dp} = -f(\theta_q^*(p))\frac{d\theta_q^*(p)}{dp}$$

  and from (12)

$$\frac{d\theta_q^*(p)}{\theta_q^*(p)} = \frac{dp}{p}$$

- We thus obtain

$$\frac{p-c}{p} = \frac{1-F(\theta_q^*(p))}{\theta_q^*(p)f(\theta_q^*(p))}$$

- which is equation (11), thus unifying the theories of second degree and third degree price discrimination