Problem 1: Current-current correlations in the low-temperature phase of the XY model

Consider the spin-wave theory for the XY model

$$ S = \frac{J}{2} \int d^d r (\nabla \phi)^2. \quad (1) $$

a) Show that the “currents” $\nabla \phi$ have correlations given by

$$ \langle \nabla_\mu \phi(r) \nabla_\nu \phi(0) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{J k^2} e^{i k \cdot r}. \quad (2) $$

b) In $d = 2$, calculate

$$ \langle \nabla_\mu \phi(r) \nabla_\nu \phi(0) \rangle \sim \frac{1}{r^2} \left( \delta_{\mu\nu} - \frac{2 r_\mu r_\nu}{r^2} \right), \quad (3) $$

which has a characteristic “dipole” shape.

Suggestion: The long-distance power laws arise from singularities near $k = 0$, while the treatment of the upper cutoff is not important; try, e.g., doing the integral over $k_y \in (-\infty, +\infty)$ first, followed by the integral over $k_x$.

c) What power law is expected in general $d$? You will get full credit for an argument by power-counting or by simple examples, and extra credit if you brave to calculate the general form in $d = 3$ similar to what was done in part b) in $d = 2$.

Remark: $\nabla \phi$ are conserved currents. Identical dipole-shaped correlations arise in a statistical mechanics model $S = \frac{K}{2} \int d^d r B^2$ with constraints $\nabla \cdot B = 0$.

Problem 2: Sine-Gordon and neutral plasma in $d$-dimensions

Show that the Sine-Gordon theory

$$ S = \int dr \left[ \frac{T}{2J} (\nabla \chi)^2 - 2y/a^2 \cos(2\pi \chi) \right] \quad (4) $$

in general $d$ dimensions describes a system of $\pm 1$ charges interacting via $d$-dimensional Coulomb interaction $V(r - r')$. Show that $V(r)$ obeys the Poisson equation $-\nabla^2 V(r) = 4\pi^2 \delta(r)$ and discuss the $r$-dependence.

Suggestion: One way to show the correspondence is by working on the lattice like we do in the lecture notes for the 2d case, but there are other ways also.