Problem 1: Ising matter coupled to Ising gauge theory

In the class we studied pure Ising gauge theory without dynamical matter fields. For many Condensed Matter applications (see e.g. next problem), we also have matter fields, and we will introduce this topic here.

Consider the following Hamiltonian

\[
H = -K \sum_P \sigma^z_P \sigma^z_P - \Gamma \sum_\ell \sigma^x_\ell - J \sum_{\ell \equiv (rr')} \sigma^x_r \sigma^x_{r'}, - h \sum_r \hat{\tau}^x_r .
\]  

(1)

The model is defined on a \(d\)-dimensional cubic lattice whose sites are labelled by \(r\) and whose links are labelled by \(\ell\), and it is often convenient to specify a link by its end-points \(r\) and \(r'\): \(\ell \equiv (rr')\). We have spin-1/2 degrees of freedom residing on the links, and Pauli matrices \(\hat{\sigma}^x, \hat{\sigma}^z\) operate on these spins; the first two terms represent the Ising gauge theory part. We also have spin-1/2 degrees of freedom residing on the sites and operated upon by Pauli matrices \(\hat{\tau}^x, \hat{\tau}^z\); the last two terms represent dynamics of this “Ising matter” part coupled to the gauge field. The Hamiltonian is supplemented by the Hilbert space constraints

\[
\prod_{r' \in r} \hat{\sigma}^x_{rr'} = \hat{\tau}^x_r \quad \text{for every } r ,
\]  

(2)

where \(r'\) goes over all nearest neighbors of \(r\). We can interpret the constraints as Gauss’ law for the Ising gauge theory in the presence of dynamical charges.

a) Show that operators \(\hat{G}(r) = \hat{\tau}^x_r \prod_{r' \in r} \hat{\sigma}^x_{rr'}\) commute with the Hamiltonian, and hence it is consistent to restrict the Hilbert space as above. Describe how these operators act in the \(\hat{\sigma}^z\) basis and explain why these operators are often called as generators of gauge transformations.

b) Develop Euclidean path integral for this model in the \(\hat{\sigma}^x, \hat{\tau}^z\) basis. \(\text{Suggestion:}\) combine the derivation of the Euclidean path integral in the gauge theory and in the global Ising model, dealing with the constraint by introducing Ising Lagrange multipliers which are then interpreted as temporal gauge fields. Show that the final action has the form

\[
S = - \sum_{\text{spatial and temporal plackets } P} K_P S_{12} S_{23} S_{34} S_{41} - \sum_{\text{spatial and temporal links } (ij)} J_{ij} T_i T_j ,
\]  

(3)

and determine the couplings \(K_P\) and \(J_{ij}\) on the spatial and temporal plackets and links.


Problem 2: Kitaev’s toric code

\(\text{Remark: Kitaev’s toric code is a simple yet remarkably influential model exhibiting emergence of a topological phase from local interactions. It first appeared in the paper arXiv:quant-ph/970702, which also introduced much more general and very important idea of fault-tolerant quantum computation using topological phases. A pedagogical account of the toric code can be found also in lecture notes arXiv:0904.2771. This problem is reasonably self-contained to guide you through the main physics, but you can also consult these references.}\)
Consider the following Hamiltonian

$$H = -\alpha \sum_r \hat{A}_r - \beta \sum_P \hat{B}_P ,$$  \hspace{1cm} (4)

$$\hat{A}_r = \prod_{r' \in r} \hat{\sigma}^x_{rr'}, \quad \hat{B}_P = \hat{\sigma}^z_{12} \hat{\sigma}^z_{23} \hat{\sigma}^z_{34} \hat{\sigma}^z_{41} .$$  \hspace{1cm} (5)

Here we have spin-1/2 degrees of freedom residing on links of a two-dimensional square lattice. We defined a “star operator” $\hat{A}_r$ as a product of $\hat{\sigma}^x$'s on the links connected to a site $r$, and a placket operator $\hat{B}_P$ as a product of $\hat{\sigma}^z$'s on the links belonging to a placket $P$. Note that there are no constraints, and this model can be viewed as a true local Hamiltonian for some spin system. We will assume that $\alpha$ and $\beta$ are both positive.

a) Show that all operators $\hat{A}_r$ and $\hat{B}_P$ commute. Hence, they can be simultaneously diagonalized. The ground state(s) will have the corresponding eigenvalues equal to +1. Show that the following state written in the $\hat{\sigma}^x$ basis is a ground state:

$$|\Psi_0\rangle = \sum \{S_{r,r'}^x\} \text{ satisfying } \prod_{r' \in r} S_{r,r'}^x = 1 \text{ for every } r .$$  \hspace{1cm} (6)

b) Consider now such a two-dimensional spin system on a torus, i.e., the system has periodic boundary conditions in both directions. Show that there are actually four degenerate ground states. That is, specifying $A_r = +1$ and $B_P = +1$ does not fix the state uniquely.

**Suggestion:** There are many ways to proceed (and you are encouraged to devise your own proof). One way to construct the four states is to start with the above $|\Psi_0\rangle$ and act upon it with the following operators

$$\hat{X}_1 \equiv \prod_{x=0}^{L_x-1} \hat{\sigma}^z_{(x,0),(x+1,0)} (i.e., \text{ product of } \hat{\sigma}^z\text{'s on vertical links taken along one horizontal line}) ,$$  \hspace{1cm} (7)

$$\hat{X}_2 \equiv \prod_{y=0}^{L_y-1} \hat{\sigma}^z_{(0,y),(0,y+1)} (i.e., \text{ product of } \hat{\sigma}^z\text{'s on horizontal links taken along one vertical line}) .$$  \hspace{1cm} (8)

Here we took finite $L_x \times L_y$ system with sites labelled as $r = (x,y)$, $0 \leq x \leq L_x - 1$, $0 \leq y \leq L_y - 1$. The precise positions of the lines are actually not important, we just took some concrete choice. What is important is that one line wraps around the torus in one direction and the other wraps in the other direction; the two lines are not contractible and cannot be deformed into each other. Show that $|\Psi_0\rangle$, $\hat{X}_1|\Psi_0\rangle$, $\hat{X}_2|\Psi_0\rangle$, $\hat{X}_1\hat{X}_2|\Psi_0\rangle$ are all ground states of the Hamiltonian (i.e., lowest energy eigenstates) and are orthogonal to each other. To show the latter, examine how the following two operators act on the four states:

$$\hat{Z}_1 \equiv \prod_{x=0}^{L_x-1} \hat{\sigma}^z_{(x,0),(x+1,0)} (i.e., \text{ product of } \hat{\sigma}^z\text{'s on horizontal links along one horizontal line}) ,$$  \hspace{1cm} (9)

$$\hat{Z}_2 \equiv \prod_{y=0}^{L_y-1} \hat{\sigma}^z_{(0,y),(0,y+1)} (i.e., \text{ product of } \hat{\sigma}^z\text{'s on vertical links along one vertical line}) .$$  \hspace{1cm} (10)

One can also show that there are no other ground states.

Any local perturbations to the Hamiltonian lead to splitting of the degeneracy which is exponentially small in the system size and can be neglected for large enough systems, and Kitaev proposed to use
the degenerate ground states to form a topologically protected qubit (e.g., for quantum memory). The above model is called Kitaev’s toric code.

c) Let us now consider excitations in the toric code model. Start with one of the ground states, say $|\Psi_0\rangle$, and show that by acting on it with $\hat{\sigma}_{rr'}^z$ on one link $\langle rr' \rangle$ we create an excitation with $A_r = -1$ and $A_{r'} = -1$ while all other $A$-stars and all $B$-plackets remain unchanged. These are called “electric” (“charge”) excitations. Show that by acting with $\hat{\sigma}_z$’s along a path on the lattice we get such “defect stars” only at the ends of the path (“string”), hence we can view each defect star as an elementary excitation (but the “strings”, even though they are locally not detectable, are important to keep track of for some properties, see below).

Similarly, show that by acting on the ground state with $\hat{\sigma}_{rr'}^x$ on one link, we create an excitation with $B_P = -1$ and $B_{P'} = -1$ on the two neighboring plackets $P$ and $P'$ containing this link. These are called “magnetic” (“flux”) excitations. Show that by acting with $\hat{\sigma}_x$’s along a path on the dual lattice we can separate such “placket defects” and can view an individual placket defect as an elementary excitation.

Consider now a charge excitation and a flux excitation, with the corresponding “strings” creating these excitations running off to some fixed far location. When we move the charge excitation around the flux excitation by “extending” the corresponding string for the charge, it necessarily crosses the string for the flux and the corresponding operators anticommute. This corresponds to the charge and flux having mutual $\pi$ statistics: when one particle encircles another, there is an extra phase of $\pi$. The appearance of such excitations with unusual statistics and the topological degeneracy on the torus are intimately related: Indeed, e.g., the $\hat{X}_1$ operator in part b) can be viewed as creating a pair of fluxes, moving one flux around the horizontal direction going around the torus and then annihilating with its partner, while the $\hat{Z}_2$ operator does the same with the charge excitations and the vertical direction, and the crucial aspect in the argument in part b) is the anticommutation of $\hat{X}_1$ and $\hat{Z}_2$.

d) In the above exactly-solvable Kitaev’s toric code, the electric and magnetic excitations are localized on the lattice. Consider perturbing the model by external fields acting on the spins:

$$\delta H = -\sum_\ell \left[ \gamma_z \hat{\sigma}_\ell^z + \gamma_x \hat{\sigma}_\ell^x \right].$$  \hspace{1cm} (11)$$

Argue that the $\gamma_z$ term allows hopping of the electric excitations, while the $\gamma_x$ allows hopping of the magnetic excitations.

e) For sufficiently large $\gamma_x$ or $\gamma_z$, the corresponding gap closes and the topological phase is destroyed. The model with $\gamma_x$ and $\gamma_z$ is no longer exactly solvable, but one can study their effects and such transitions using Monte Carlo method. A convenient formulation is provided by the Ising gauge theory with Ising matter fields considered in the previous problem. To this end, show that the toric code with the above external field is mathematically equivalent to the model Eq. (1).

**Suggestion:** By working in the $\hat{\sigma}_x, \hat{\tau}_x$ basis, first show that there is one-to-one mapping between the states in the constrained Hilbert space in the gauge theory Eq. (2) and the unconstrained Hilbert space in the toric code. Then examine the action of all operators in Eq. (1) in terms of the toric code variables.

This connection was used e.g. in paper I. S. Tupitsyn, A. Kitaev, N. V. Prokofev, and P. C. E. Stamp, Phys. Rev. B 82, 085114, to study detailed aspects of the phase transitions in the toric code.