**XY model**

\[
E(\{\phi_i\}) = -J \sum_{ij} \cos(\phi_i - \phi_j)
\]

Global \(U(1)\) symmetry: \(\phi_i \rightarrow \phi_i + \delta\)

\(\phi_i \in (0, 2\pi)\)

* can be viewed as a model of spins with easy-plane or more generally of interacting bosons

* can be viewed as a lattice realization of two-component \(\psi^\dagger\) theory \(\bar{\psi} = (\bar{\phi}_1, \bar{\phi}_2) \iff \psi = \phi_1 + i\phi_2 \in \mathbb{C}\)

\[
\int d^d x \left[ \frac{1}{2} \left| \nabla \psi \right|^2 + \frac{\mu}{2} |\psi|^2 + u|\psi|^4 + \ldots \right]
\]

Complex \(\psi\) - superfluid order parameter; this is Ginzburg-Landau theory for the normal to superfluid transition. (Very loosely, \(\psi\) \~ complex boson wavefunction, superfluidity being intrinsically quantum phenomenon.) Smoother more precisely, \(\bar{\psi}^\dagger \sim \) boson creation operator for the state into which bosons condense.

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**High-temperature phase**

\(\langle \psi \rangle = 0\) in the field theory

In the lattice model, \(\phi_i\)'s fluctuate wildly. Can proceed systematically e.g. constructing high-temperature series expansions:

\[
Z = \sum_{\text{closed graphs with each site covered by even # of bonds}} \prod_i (1 + t \cos(\phi_i - \phi_j))
\]

Slightly different lattice model (taken for simplicity), roughly:

\[
t \sim \frac{J}{t}
\]
The "closed graph" condition is similar to Ising series:

\[ \sum_{\Omega} \Pi_{i} \cos(\phi_{i} - \phi_{i'}) \] - this is what leads to the "closed graph" condition since \( \int_{0}^{2\pi} e^{im\phi} \, d\phi = 0 \) for any \( m \neq 0 \). These weights will of course be different from Ising.

**Correlation function**

\[ \langle e^{ip(r) - ip(r')} \rangle = \frac{2\pi}{\sum_{\Omega} \Pi_{i} e^{i\phi_{i}} e^{-i\phi_{i'}} (1 + \cos(\phi_{i} - \phi_{i'}))} \]

\[ \sum_{\Omega} \Pi_{i} \cos(\phi_{i} - \phi_{i'}) (1 + \cos(\phi_{i} - \phi_{i'})) \]

- Shortest path

Thus, in addition to \( t \) factor for each intermediate spin, there is a \( \frac{1}{2} \) factor for each intermediate spin.

\[ \sum_{0}^{2\pi} \cos(\phi_{1} - \phi_{2}) \cos(\phi_{2} - \phi_{3}) \]

\[ = \frac{1}{2} \cos(\phi_{1} - \phi_{3}) \]

\[ \Rightarrow \langle \cos(\phi(r) - \phi(r')) \rangle \sim \left( \frac{1}{2} \right)^{r-r'} = A e^{-r-r'/\xi} \]

\[ \xi = \frac{1}{\ln \left( \frac{2T}{J} \right)} - \frac{1}{\ln \left( \frac{2T}{J} \right)} \]

In short, correlations in spin models at high-\( T \) are short-ranged — decaying exponentially with distance. (Correlations can develop only through intervening interactions, and each step on the way brings a suppression factor.)
Spontaneous symmetry breaking and Goldstone modes

Low-temperature phase — spins order (angles $\phi_i$ lock to each other, $\phi_i = \phi_0 = \text{const}$ at $T \to 0$), breaking the global $U(1)$ symmetry. [In the field theory, $\langle \phi \rangle \neq 0$]

Since global rotation $\phi_i \rightarrow \phi_i + \pi$ does not change the energy, slow variation of $\phi_i$ in space will cost only small energy — the ordered system is not "gapped" but has low-energy modes — "Goldstone modes" (or "spin waves" in the spin context). These are seen and analyzed as follows:

Treating $\phi_i = \phi_0 + \delta\phi_i$, small slowly varying fluctuations expand the cosine

$$\frac{E}{T} \left[ \{ \delta\phi^2 \} \right] = -\frac{J}{T} \sum_{i,j} \cos (\delta\phi_i - \delta\phi_j) \propto \text{const} + \frac{J}{2T} \sum_{i,j} (\delta\phi_i - \delta\phi_j)^2$$

$$Z \propto \text{(const)} \int \prod_i \delta \phi_i \ e^{-\frac{J}{2T} \sum_{i,j} (\delta\phi_i - \delta\phi_j)^2}$$

Gaussian action if $\delta\phi_i$ are treated as $\mathcal{O}(1)$, can calculate everything using Gaussian integrals

$$\frac{J}{2T} \sum_{i,\mu \nu} (\delta\phi_i - \delta\phi_{\mu \nu})^2 = \frac{J}{2T} \sum_{i} \delta\phi_i^2 (1 - e^{-\Delta q \cdot \mathbf{r}^*}) e^{-\gamma r}$$

$$\delta\phi_i = \frac{1}{\sqrt{N}} \sum_q \delta\phi_q \ e^{i\mathbf{q} \cdot \mathbf{r}}$$

$$\delta\phi_q = \frac{1}{\sqrt{N}} \sum_i \delta\phi_i \ e^{-i\mathbf{q} \cdot \mathbf{r}}$$

$$\delta\phi_{-q} = \delta\phi^*_q$$

Spin wave theory
\[ K = \frac{\mathcal{J}}{\mathcal{P}} \]

\[ \mathcal{E}_c \left( \Phi_{q',3} \right) = \frac{K}{2} \sum_q \left[ \sum_\mu (2-2 \cos q \mu) \right] |8 \Phi_q|^2 = \]

\[ = K \sum_q' \left[ \sum_\mu (2-2 \cos q \mu) \right] \left( \Re \Phi_q^2 + (\Im \Phi_q)^2 \right) \]

one half of all \( q \)'s, e.g. specified by \( q_x > 0 \)

Can calculate arbitrary expectation values by using elementary

\[ \langle (\Re \Phi_q)^2 \rangle = \langle (\Im \Phi_q)^2 \rangle = \frac{1}{2K} \left[ \sum_\mu (2-2 \cos q \mu) \right] \]

and all other combinations (from the "half of all \( q \)-s" set) vanish. This can be concisely summarized as

\[ \langle \Phi_q \Phi_{q'} \rangle = \frac{S_{q+q'=0}}{K \left[ \sum_\mu (2-2 \cos q \mu) \right]} \]

with no restrictions on \( q \) & \( q' \).

Indeed,

\[ \langle \Phi_q \Phi_q \rangle = \langle (\Re \Phi_q)^2 \rangle + 2 \langle \Re \Phi_q \Im \Phi_q \rangle = 0 \]

\[ \langle (\Im \Phi_q)^2 \rangle = 0 \]

\[ \langle \Phi_q \Phi_{-q} \rangle = \langle |8 \Phi_q|^2 \rangle = \langle (\Re \Phi_q)^2 \rangle + \langle (\Im \Phi_q)^2 \rangle \]

Let's calculate

\[ \langle \Phi_{q(\mathbf{r})} \Phi_{q'\mathbf{r}} \rangle = \frac{1}{N} \sum_q \frac{e^{i q \cdot (\mathbf{r} - \mathbf{r}')}}{K \left[ \sum_\mu (2-2 \cos q \mu) \right]} \]

\[ = \alpha^d \int \frac{d^d q}{(2\pi)^d} \frac{e^{i q \cdot (\mathbf{r} - \mathbf{r}')}}{K \left[ \sum_\mu (2-2 \cos q \mu) \right]} \]

1st BZ

\[ 1q_n \leq \frac{\pi}{a} \]

\[ N \rightarrow \infty \text{ (large volume limit)} \]
For $r' = r$

$$\langle 8q(r)^2 \rangle = \frac{\Lambda}{K} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\Sigma(2 - 2aq, a)}$$

$$\sim \int \frac{q^{d-1} dq}{q^2} = \int \text{converges for } d > 2$$

$$\frac{sdq}{q^2} - \text{logarithmically divergent for } d = 2$$

$$\frac{sdq}{q^2} - \text{strongly divergent for } d = 1$$

So for $d > 2$ the assumption that $8q$ fluctuates only little is indeed justified at low $T$ ($K \to \infty$), while for $d = 2$ this is no longer the case. This is an example of Mermin-Wagner theorem that there can be no spontaneous breaking of continuous symmetry in (classical) $\dim \leq 2$.

Let's calculate

$$\langle (8q(r) - 8q(r'))^2 \rangle = 2 \langle 8q(r)^2 \rangle - 2 \langle 8q(r) 8q(r') \rangle =$$

$$= \frac{2}{K} \int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos(q^2 - 2rz)}{\Sigma(2 - 2aq, a)}$$

- now the integral converges at small $q$ for any $\dim$.

so in particular $\cos(8q(r) - 8q(r')) \approx 1 - \frac{1}{2}(8q(r) - 8q(r'))$ is justified at low $T$ ($K \to \infty$),

(but still need to check topological defects - vortices)

Want to examine long-distance behaviour of

$$\langle (8q(r) - 8q(0))^2 \rangle = \frac{2}{K} \int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos q^2 r^2}{\Sigma(2 - 2aq, a)} = \frac{2a^{-2} \Lambda}{K} \int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos q^2 r^2}{q^2}$$
\[ d=3 : \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} 1 - \text{co}(\theta) = \int_{0}^{\Lambda} \frac{1}{(2\pi)^3} 2\pi \int_{-1}^{1} \frac{d\cos(\theta)}{q^2} (1 - \text{co}(q\cdot r) \cdot \text{co}(\theta)) \]

\[ = \frac{1}{(2\pi)^3} \int_{0}^{\Lambda} (2 - 2\sin(q\cdot r)) \approx \frac{1}{2\pi^2} \left( \Lambda - \frac{1}{r} \right) \]

\[ = \frac{\Lambda}{2\pi^2} - \frac{1}{4\pi r} \]

\[ d=2 : \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2} (1 - \text{co}(\theta^2)) = \int_{0}^{\Lambda} \frac{dg}{2\pi^2} \int_{0}^{2\pi} \frac{d\phi}{2\pi} \frac{1}{q^2} \text{co}(g \cdot r) \]

\[ = \int_{0}^{\Lambda} \frac{dg}{2\pi^2} (1 - J_0(qr)) \approx \int_{0}^{\Lambda} \frac{dg}{2\pi^2} = \frac{1}{2\pi} \ln(\Lambda r) \]

provides cutoff at small \( q \approx \frac{1}{r} \)

More accurate treatment:

\[ \int \frac{dg}{g} (1 - J_0(qr)) = \int \frac{du}{u} (1 - J_0(u)) = \int_{0}^{\Lambda r} \frac{1 - J_0(u)}{u} du - \int_{1}^{\Lambda r} \frac{J_0(u)}{u} du = \int \frac{du}{u} J_0(u) + \int \frac{du}{u} \cdot u \]

\[ = \ln(\Lambda r) + \zeta + O\left( \frac{1}{(\Lambda r)^{3/2}} \right) \]

Cutoff-dependent properties

\[ d=1 : \int \frac{d^1q}{(2\pi)^1} \frac{1}{q^2} = \int_{0}^{\Lambda} \frac{du}{2\pi} \frac{1}{u^2} \text{co}(u) \approx \frac{\pi}{2} 1 + O\left( \frac{1}{\Lambda^2} \right) \]
Summary:

\[ \langle (\delta \psi(r) - \delta \psi(0))^2 \rangle \approx \begin{cases} 
\frac{(a \cdot A}{\pi^2} - \frac{a}{2\pi K^2}) \cdot \frac{1}{K} & d = 3 \\
\frac{1}{K} \ln \frac{|r|^d}{a} \cdot \frac{1}{K} & d = 2 \\
\frac{\pi x}{a} \cdot \frac{1}{K} & d = 1 
\end{cases} \]

⇒ Correlation function of spins:

\[ \langle e^{i \psi(r) - i \psi(0)} \rangle = \exp \left[ -\frac{1}{2} \left( \langle \psi(r) - \psi(0) \rangle^2 \right) \right] = e^{\frac{1}{2} \langle \psi^2 \rangle} \]

(Use \langle e^A \rangle \text{ linear in } x \Rightarrow \text{ Gaussian weights in } x)

\[ \exp \left( - \left( \text{const} - \frac{\text{const}}{r^{d-2}} \right) \right) = \text{(#)} e^{- \frac{\text{const}}{r^{d-2}}} \rightarrow \text{const} + \frac{\text{const}}{r^{d-2}} \]

\[ \begin{align*} 
\text{const} & \text{ for } d \geq 3; \\
\frac{1}{|r|^{1/(d \pi K)}} & \text{ for } d = 2; \\
\exp \left( - \frac{\pi x}{K a} \right) & \text{ for } d = 1; 
\end{align*} \]

Remarks:

* In the spin wave theory, we have long range order for \( d \geq 3 \). The slowly decaying power law correction \( \sim \frac{1}{r^{d-2}} \) is due to gapless spin wave modes depending on temperature!

* In \( d = 2 \), spin waves destroy long range order, but there remains quasi-long-range order - power law spin correlations.

* In \( d = 1 \), spin waves completely destroy any order.
Remarks on the lecture:

Continuum limit of spin wave theory

\[ E = \frac{1}{2} \sum_{r, k} (\phi_{r+k} - \phi_r)^2 \approx \frac{1}{2} \sum_{r, k} (\phi_{r+k} - \phi_r - \frac{e^2}{\kappa} \cdot \phi_r) \cdot a^d \cdot a^{2-d} \approx \frac{1}{2} \frac{Ja^{2-d}}{\omega} \int d^d r \ (\nabla^2 \phi)^2 \]

k-space analysis: \( \phi_r = \int \frac{d^d k}{(2\pi)^d} \hat{\phi}_k e^{i \mathbf{k} \cdot \mathbf{r}} \)

\( \hat{\phi}_k = \int d^d r \ \phi(r) e^{-i \mathbf{k} \cdot \mathbf{r}} \)

\[ E = \frac{\omega}{2} \int \frac{d^d k}{(2\pi)^d} \mathbf{k}^2 |\hat{\phi}_k|^2 \text{ where we used } \int d^d r e^{i \mathbf{q} \cdot \mathbf{r}} = (2\pi)^d \delta(\mathbf{0}) \]

\[ \langle \phi(\mathbf{k}) \phi(\mathbf{k'}) \rangle = \frac{(2\pi)^d \delta^{(d)}(\mathbf{k} + \mathbf{k'})}{\omega} \]

"Spin wave theory" for the superfluid:

\[ \int d^d x \left[ \frac{\omega}{2} |\nabla \psi|^2 + \frac{m}{2} |\psi|^2 + u |\psi|^4 \right] \]

\( m < 0 \Rightarrow \psi \) develops an expectation value. Write \( \psi = \sqrt{\frac{\omega}{2}} e^{i \phi} \).

Ignoring fluctuations of \( \psi \)

\[ \Rightarrow \int d^d x \left[ \frac{\omega}{2} |\nabla \psi|^2 \right] \]

Superfluid "stiffness"

\( \int \frac{d^d k}{(2\pi)^d} \frac{1-\cos q \cdot r}{q^2} \) can be done neatly by noting that \( \frac{1}{q^2} \) is the Fourier transformation of the Coulomb potential - see Kardar's lecture on Goldstone modes L3

Spin waves in the quantum language: \( \omega_k \approx c |\mathbf{k}| - \text{see next addendum} \)