Representations (variants) of the XY model

General discussion in d-dim; say, working on a cubic lattice

Phase representation - usual "cosine" interaction

\[ Z = \prod_{i} e^{\beta \sum_{a} \cos (\phi_{i} + \phi_{i+1} - \phi_{i+2})} \]

Villain variant: the idea is that what is important is the \( 2 \pi \)-periodicity of the interaction potential but not the detailed form

\[ e^{\beta \cos \alpha} \approx e^{-\frac{1}{2} (\alpha - 2\pi p)^2} \]

(rentually, \( \alpha = \nabla \phi \))

"Justification" of the Villain model (e.g., if we want to connect the "cosine" and the Villain models more quantitatively)

* Cosine model with large \( \beta \): For \( \alpha = 2\pi p + 2\pi \)

\[ e^{\beta \cos \alpha} \approx \text{const} \times e^{-\frac{1}{2} (\alpha - 2\pi p)^2} \]

- precisely reproduced by the Villain form with \( \beta \approx \beta_v \)

* Cosine model with small \( \beta \) - will correspond to the Villain model with small \( \beta_v \). To establish the correspondence between couplings \( \beta \) and \( \beta_v \), use Poisson-resummed form

\[ \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\beta_v}{2} (\alpha - 2\pi p)^2} = (\text{const}), \sum_{J=-\infty}^{\infty} e^{-\frac{\beta}{2\beta_v} + iJ \alpha} \]
For small $\beta v$, 
\[ \sum_{J=-\infty}^{+\infty} e^{-J^2/2\beta v} + i J\alpha = 1 + e^{-1/2\beta v} \cdot 2 \cos \alpha \approx \begin{cases} \exp \left( e^{-\beta v} \cdot 2 \cos \alpha \right) \\ \beta \approx 2 e^{-1/2\beta v} \end{cases} \text{ for small } \beta \ll 1 \]

\[ \beta v \ll 1 \]

* In general, the Villain model will have qualitatively similar physics to the original XY model.

Villain model

\[ Z_v = \int_{-\pi}^{\pi} d\phi_i \sum_{i=1}^{+\infty} e^{-\beta \sum_{i=1}^{+\infty} (\phi_{i+1} - \phi_i - 2\pi \phi_{i+1})^2} \]

Current loops representation: use the Poisson-resummed form of the Villain potential:

\[ Z_v = \int_{-\pi}^{\pi} d\phi_i \sum_{i=1}^{+\infty} e^{-\sum_{i=1}^{+\infty} \frac{J_{i+1}}{2\beta^2} + i \sum_{i=1}^{+\infty} J_{i+1} (\phi_{i+1} - \phi_i)} \]

Recall derivation of the XY model as path integral for the quantum rotor model. As an intermediate step, we had

\[ \langle \phi(t+\delta t) \mid e^{-\delta t \mathbf{H}^2} \mid \phi(t) \rangle = \sum_{n=-\infty}^{+\infty} e^{-\delta t \frac{1}{2} n^2 + i n (\phi(t+\delta t) - \phi(t))} \]

Thus the above form can be viewed as a path integral where we keep both $\phi$-variables and $n$-variables, with the identification $J_x = n$. The meaning of
the "spatial" components \( J_x, J_y \) will become clear shortly as "currents".

The form with both \( \mathbf{J} \)'s and \( \phi \)'s present is not convenient by itself because of the "complex weights"

\[ e^{i \Sigma \mathbf{J} \cdot \mathbf{\phi}} - "\text{Berry phases}" \text{ for quantum bosons.} \]

We can obtain representation in terms of \( \mathbf{J} \)'s only by integrating out \( \phi \)'s:

\[ e^{i \Sigma \mathbf{J}_\mu (\phi_i + \phi_j - \phi_i)} = e^{i \Sigma \mathbf{J} \cdot \mathbf{\phi}} = e^{-i \Sigma \mathbf{\phi}_c (\mathbf{J} \cdot \mathbf{J})_c} \]

\[ \int_{-\pi}^{\pi} \frac{d\phi_i}{2\pi} e^{-i\phi_i (\mathbf{J} \cdot \mathbf{J})_c} = \delta_{\mathbf{J} \cdot \mathbf{J} = 0} \]

\[ Z \_v = Z \_v \left( \sum \frac{1}{J_{i\mu} = -\infty} \prod \delta (\mathbf{J} \cdot \mathbf{J})_c = 0 \right) e^{-\frac{1}{2} \sum \frac{J_{i\mu}^2}{2 \beta}} \]

\[ \mathbf{J}_c \] are integer-valued conserved currents with steric (or more generally short-range) interactions.

If we interpret one direction as imaginary time and the rest as spatial directions, we have

\[ \nabla_n J_n + \nabla_{\text{spat}} \cdot J_{\text{spat}} = 0 \] - "continuity" equation

\[ \nabla_n \]

boson number is the temporal component of the current.

Remarks: * Even though the \( \mathbf{J} \)-formulation looks "gaussian" the discreteness of \( \mathbf{J} \)'s is crucial, making the model nontrivial (e.g., has two phases)

* Current loop models can be efficiently simulated in Monte Carlo
Formulation in terms of vortices

Formal construction

\[ Z_V = \sum_{\pi=1}^{\infty} e^{i\pi p_\mu - \pi^2 \sum_{i=1}^{\infty} (\phi(x) - \phi(x) - 2\pi p_\mu x)^2} \]

Divide all \( p_\mu \) into classes equivalent under integer-valued transformations

\( p_\mu \rightarrow p_\mu + N_i \mu - N_i = p_\mu + \nabla_\mu N \)

\( \mathcal{E} p_\mu \) and \( \mathcal{E} p'_\mu \) are equivalent if there exists an integer-valued \( \{N_i\} \) s.t.

\( p'_\mu = p_\mu + \nabla_\mu N \).

If we do not worry about boundaries, which is OK if we study bulk properties, the classes are completely characterized by

\[ q_{\mu\nu} = \nabla_\mu p_\nu - \nabla_\nu p_\mu = p_{i\mu} \delta_{i\nu} - p_{i\nu} \delta_{i\mu} + p_{i\mu} \]

objects are "oriented plaquettes"

Indeed, if \( \mathcal{E} p_3 \) and \( \mathcal{E} p'_3 \) are equivalent, they will give the same \( q \). On the other hand, \( \mathcal{E} p_3 \) and \( \mathcal{E} p'_3 \) are equivalent.

(More formally, using differential forms:

\( \mathcal{E} p \) is a 1-form: \( \omega_\mathcal{E} = \sum_{\mu} \epsilon_{\mu\nu} dx^\nu \)

\( q \) is a 2-form: \( \omega_q = \sum_{\mu\nu} q_{\mu\nu} dx^\mu \wedge dx^\nu = d(\omega_\mathcal{E}) \)

If \( d\omega_\mathcal{E} = d\omega_q, \) then \( d\omega_{\mathcal{E} - \mathcal{E}} = 0 \Rightarrow \omega_q \Rightarrow p - p' = \nabla N. \)
Note that $q$’s are not arbitrary but satisfy

$$d\omega_q^{(a)} = dd\omega_q^{(b)} = 0.$$ 

Conversely, if $d\omega_q^{(a)} = 0$, then there exists $\hat{p}$ such that $\omega_q^{(a)} = d\omega_{\hat{p}}^{(b)}$. Thus, \text{one-to-one}

Thus, \text{classes of } \{\hat{p}_{i\mu}\} \leftrightarrow \text{arbitrary } q_{\mu\nu} \text{ such that } d\omega_q^{(a)} = 0

\textbf{Examples:}

\[2D\]

\[\hat{q} \mapsto [x, y] \mapsto q_{xy} = \nabla_x p_y - \nabla_y p_x\]

If \[\nabla_x p_y - \nabla_y p_x = \nabla_x p'_y - \nabla_y p'_x \Rightarrow \nabla_x (p_y - p'_y) - \nabla_y (p_x - p'_x) = 0\]

\[\Rightarrow \Delta \hat{p} = (p_x - p'_x, p_y - p'_y) \text{ has zero circulation around any plaquette and can therefore be written as a gradient.}\]

(Note also that in 2d for any 2-form $d\omega^{(a)} = 0$).

Conversely, for any $q_{\hat{q}}$’s on the dual plaquettes, one can always find $p$’s to give $q = \nabla_x p_y - \nabla_y p_x$

\[\{\text{classes of } \hat{p}_{i\mu}\} \leftrightarrow \{\text{arbitrary integer-valued } q_{\mu\nu} \text{ on the dual lattice}\}\]

\[3D\] can use (dual) vector notation to connect with familiar 3D formulae:

\[\vec{q} = \vec{\nabla} \times \vec{p} \Rightarrow \vec{\nabla} \cdot \vec{q} = 0\]

* If $\vec{\nabla} \times \vec{p} = \vec{\nabla} \times \vec{p'} \Rightarrow \vec{\nabla} \times (\vec{p} - \vec{p'}) = 0 \Rightarrow \vec{p} - \vec{p'} = \vec{\nabla} N$

* If $\vec{\nabla} \cdot \vec{q} = 0 \Rightarrow$ there exists $\vec{p}$ such that $\vec{q} = \vec{\nabla} \times \vec{p}$

\[\{\text{classes of } \vec{p}\} \leftrightarrow \text{integer-valued } \vec{q}$’s such that $\vec{\nabla} \cdot \vec{q} = 0\]
Physical meaning of \( q_{\mu\nu} = \nabla_\mu p_\nu - \nabla_\nu p_\mu \) - vorticity.

**2D example:** States of the system are described by \( \{\phi_i, \pi\} \). Consider configuration with a vortex at the origin and \( \phi \)'s accumulating \( 2\pi \) as we go around it:

![Vortex configuration diagram](image)

these \( \phi \)'s are \( \approx 0+ \)
these \( \phi \)'s are \( \approx 2\pi-0 \)

For this to be a low-energy configuration of \( \{\phi_i, \pi\} \), the variables \( p_\mu \) will be zero everywhere except for the cut where they will be \( p_\mu = 1 \)

\[
(\phi_i - \phi_i + \pi - 2\pi p_\mu)^2 \approx 0 \\
\approx 2\pi-0 \approx 0+ \\
\text{small energy}
\]

For this configuration of \( p_\mu \), \( \nabla \times p \) is \( +1 \) on the vortex plaquette and \( 0 \) everywhere else, i.e. \( \nabla \times p \) is vortex "density" (vortex number) in this 2D case.

Villain model formally working with \( \{\phi, p_\mu\} \) ensemble allows to formally define vorticities by monitoring the configurations of \( \{p_\mu\} \).

In 2D, we have vortex number on each plaquette, with no additional conditions on \( q \)'s.

**3D example:** Stack the above picture in the \( z \)-dir.

We have \( \nabla \times p \) nonzero on \( x-y \) plaquettes at the origin at any \( z \) (in each layer), and zero everywhere else.
The corresponding is \( \tilde{q} = +\hat{2} @ (\text{dual}) \)

\[(x,\tilde{y}) = (0,0)\]

and \( \tilde{q} \) everywhere else

- vortex line.

In 3D, \( \tilde{q} \)'s satisfy \( \nabla \cdot \tilde{q} = 0 \)

- vortex lines have no sinks or sources and must form closed loops.

Viewing this in 3D = (2+1)D vortex lines can be viewed as worldlines in the space-time. In the 2dim quantum problem, vortices are quantum particles.

(Will derive this interactions later).

In contrast, in 2D = (1+1)D vortices are points in the space-time. In the 1dim quantum problem, vortices are instantons — tunneling events (phase slip events).

4D Objects \( q_{\mu\nu} = \nabla_\mu \rho_\nu - \nabla_\nu \rho_\mu \) satisfying \( d\omega^{(3)} = 0 \)

describe closed surfaces, which can be viewed as world sheets in (3+1)D of vortices which are quantum strings in the 3dim quantum problem.

Summary on vortices in different dimensionalities:

- 1+1d: instantons (space-time points)
- 2+1d: particles (lines in space-time)
- 3+1d: strings (surfaces in space-time)

Next: "dynamics" of vortices — their interactions and statistical mechanics.
Formulation focusing on vortices:

In the Villain model

\[ \sum_{\ell p^{(e)} \in C_p} = \sum_{\text{classes of } \ell p^{(e)}} \sum_{C_p} \sum_{\ell p^{(e)} \in C_p} \quad \text{e.g. } p^{(e)}_{\mu} - \text{one member,} \]

then \( p^{(e)}_{\mu} = p^{(e)}_{\mu} + \nabla \mu N \) with arbitrary integer-valued \( N \).

same as sum over allowed vortex configurations

\[ Z_V = \sum_{C_p} \sum_{N_i = -\infty}^{+\infty} \prod_{i \mu} e^{-\frac{1}{2} \sum_{i \mu} \left( \phi_{i \mu} - \phi_i - 2\pi p^{(e)}_{\mu} - 2\pi \times N_i \right)^2} \]

Denoting \( \tilde{\phi}_i = \phi_i - 2\pi N_i \), then \( \int_{-\infty}^{+\infty} d\phi_i \sum_{N_i = -\infty}^{+\infty} \) is effectively the same as \( \int_{-\infty}^{+\infty} d\tilde{\phi}_i \)

\[ Z_V = \sum_{C_p} \int_{-\infty}^{+\infty} \prod_{i \mu} e^{-\frac{1}{2} \sum_{i \mu} \left( \nabla \mu \tilde{\phi}_i - 2\pi p^{(e)}_{\mu} \right)^2} \]

one member of \( C_p \) does not matter which

Now the integration variables are non-compact \( \phi_i \) and we can do the gaussian integrals completely. The final result will depend only on \( \nabla \times p^{(e)} \), i.e., only on the vortex configuration \( q = \nabla \times p^{(e)} \) \( \Rightarrow \) problem of interacting vortices.
The integral over $\tilde{\phi}$ can be done in many different ways (any Gaussian integral is doable with enough perseverance!). Here we will use an intermediate step (which will be also useful later), writing:

\[
    e^{-\frac{B}{2} (\nabla \tilde{\phi}^2 - 2 \pi \rho_{\mu})^2} = (\text{const}) \int_{-\infty}^{+\infty} e^{-\frac{B_{\mu}}{2B} \rho_{\mu}^2 + i \sum_{\mu} \left( \nabla \tilde{\phi}^2 - 2 \pi \rho_{\mu} \right)}
\]

Plugging this into $Z_V$ and reorganizing

\[
    \Sigma_{i\mu} (\nabla \tilde{\phi}^2 - 2 \pi \rho_{i\mu}) = -\Sigma_{i\mu} \tilde{\phi} (\nabla \tilde{\phi}) - \Sigma_{i\mu} 2 \pi \rho_{i\mu},
\]

now $\int_{-\infty}^{+\infty} d\tilde{\phi}$ can be done trivially and gives $S(\tilde{\phi}, \tilde{j} = 0)$

\[
    Z_V = \sum_{c_p} \int_{-\infty}^{+\infty} (\prod_{i\mu} d\rho_{i\mu}) \prod_{i\mu} \delta(\tilde{\rho}_{i\mu} - 0) e^{-\Sigma_{i\mu} \rho_{i\mu}^2 - i \Sigma_{i\mu} 2 \pi \rho_{i\mu}}
\]

This reminds the "current loop model" formulation

\[
    Z_V = \sum_{J_{i\mu} = -\infty}^{+\infty} \prod_{i\mu} \delta(\tilde{\rho}_{i\mu} - 0) e^{-\Sigma_{i\mu} \rho_{i\mu}^2 - \frac{J_{i\mu}^2}{4B}}
\]

but $J_{i\mu}$ are integer-valued, which is how the original compactness of $\rho_{i\mu}$ is encoded, while $\rho_{i\mu}$ are real-valued and the compactness of $\phi_{i\mu}$ is encoded in the vorticity variables $\nabla \phi_{i\mu}$ explicitly present.

Very loosely speaking, $\tilde{j}$ can be viewed as "coarse-grained real-valued versions" of the original boson currents $j$.

It is easier (and safer) to work with the real-valued $\tilde{j}$'s.
2D

\[ Z_v = \sum_{j} \int_{-\infty}^{+\infty} d j_{+j} \prod_{\mu} e^{-i \frac{j_{+j}^2}{2\beta}} - i \sum_{j} j_{+j} \cdot 2\pi \rho^{(0)} \]

\[ \mathbf{j} = (j_x, j_y) ; \quad \mathbf{\nabla} \cdot \mathbf{j} = \mathcal{V}_x j_x + \mathcal{V}_y j_y = 0 \]

can be solved by

\[ j_x = \mathcal{V}_y \chi \quad \int \chi \text{ - real-valued field} \]

\[ j_y = -\mathcal{V}_x \chi \quad \text{with no conditions} \]

\[ \sum \mathbf{j} \cdot \mathbf{\beta} = \sum (p_x \mathcal{V}_y - p_y \mathcal{V}_x) = \sum \chi (\mathcal{V}_x p_y - \mathcal{V}_y p_x) = \sum \chi \mathbf{q} \quad \text{explicitly depends only on vorticities} \]

\[ Z_v = \sum_{\text{vortex configs}} \int_{-\infty}^{+\infty} \mathcal{D} \chi e^{-\frac{\mathcal{V}(\chi)^2}{2\beta}} - i \cdot 2\pi \sum_{\text{lattice site}} \chi_{I} \cdot q_{I} \]

From here can go two separate ways:

\[ \begin{array}{c}
\text{\textcircled{X}} \quad \text{Integrate out } \chi \text{'s completely } \Rightarrow \text{Coulomb gas of vorticities}
\end{array} \]

\[ \int_{-\infty}^{+\infty} \mathcal{D} \chi e^{-i \alpha \sum \chi_{I} \cdot q_{I}} e^{-\frac{\mathcal{V}(\chi)^2}{2\beta}} = \frac{\beta^{-1} \mathcal{L}_{II}'}{\text{inverse of the lattice Laplacian } \Sigma (x^2 = \Sigma x \mathcal{A} x + \mathcal{C})} \]

\[ \begin{align*}
\langle \chi(R) \chi(R') \rangle &= \int \frac{d^2 k}{BZ (2\pi)^2} \cdot \frac{\beta \cdot e^{i \mathbf{k} \cdot (R - R')}}{4 - 2 \cos k_x - 2 \cos k_y} \\
\langle \chi(R) \chi(R') \rangle - \langle \chi^2 \rangle &= \beta \int \frac{d^2 k}{BZ (2\pi)^2} \frac{\cos k(R - R') - 1}{4 - 2 \cos k_x - 2 \cos k_y}
\end{align*} \]
\[ Z = \sum \exp \left[ -\beta \cdot \frac{e^{2}}{8\pi^{2}} \sum_{\vec{R},\vec{R}'} q_{\vec{R}} q_{\vec{R}'} C(\vec{R} - \vec{R}') \right]; \]

\[ C(\vec{R} - \vec{R}') = \int \frac{d^{2}k}{(2\pi)^{2}} \frac{\cos k \cdot (\vec{R} - \vec{R}') - 1}{4 - 2c_{x}k_{x} - 2c_{y}k_{y}} - \text{ lattice version of the Coulomb interaction between vortices.} \]

At large \( R - R' \), can focus on small \( k \)

\[ C(\vec{R} - \vec{R}') \approx \int \frac{d^{2}k}{(2\pi)^{2}} \frac{\cos k \cdot (\vec{R} - \vec{R}') - 1}{R^{2}} = -\frac{1}{2\pi} \ln \frac{|R - R'|}{a} \]

\[ Z \approx \sum \exp \left[ \beta \cdot \frac{1}{2\pi} \sum_{\vec{R},\vec{R}'} q_{\vec{R}} q_{\vec{R}'} \ln \frac{|R - R'|}{a} \right] - \frac{1}{\beta} \]

- essentially identical to the vortex system derived thinking in continuum; here the cutoff is explicitly the lattice spacing \( a \).

Can now analyze the Coulomb gas using KT RG.

\[ \text{Alternatively, can sum out the vortex number degrees of freedom obtaining description in terms of the dual field } \mathcal{X} \Rightarrow \text{ Sine-Gordon field theory.} \]

To proceed formally, can say that modifying the vortex interactions at short distances should not change the universal properties of the phases (and phase transitions). Consider therefore adding

\[ \sum_{I} \frac{q_{I}^{2}}{2\lambda} \text{ to the action. Now can do the summation} \]

\[ \sum_{q_{I}=-\infty}^{+\infty} e^{-\frac{q_{I}^{2}}{2\lambda} - i 2\pi \chi_{I} q_{I}} = (\text{const}) \sum_{\rho=-\infty}^{+\infty} e^{-\frac{\lambda}{2}(2\pi \chi_{I} - 2\pi \rho)^{2}} \sim e^{\lambda \chi_{I}} \text{ in the villain sense.} \]
In the continuum limit
\[ S[\chi] = \int d^2r \left( \frac{1}{2\beta} (\nabla \chi)^2 - \lambda \int d^2r \cos(2\pi \chi) \right) \]

Remarks: \( \times \) The original Villain model in terms of \( \chi_I \) and \( q_I \) corresponds to \( \lambda \to \infty \) limit. In this limit, \( \chi_I \) must be integer-valued.

\[
\text{orig. } Z_V = \left. Z'[\chi \to \infty] \right|_{\text{orig.}} = \exp \sum \frac{(\delta \chi)^2}{2\beta} \text{- integer-valued } \]

integer-valued "height model"

- Model of surface constructed with discrete steps, with the energy function penalizing large differences between neighbouring sites.

The height model can be obtained directly from the integer-valued loop model \( \sum \frac{\nabla^2 \chi^2}{2\beta} \) by solving the condition \( \nabla \cdot \mathbb{J} = 0 \) via
\[ J_x = \nabla_y \chi, \quad J_y = -\nabla_x \chi, \]

\( \chi \)-integer-valued field residing on the dual lattice.

A more direct derivation of the Sine-Gordon model is to say that the universal properties should not change if we modify the "hard" integer-valued ness...
condition by a "soft" version

"hard" integer $X \rightarrow$ real-valued $X$

plus energy term $-\lambda c a(2\pi X)$
causing $X$ to lie preferentially near integer values.

We want to make this approximation in order to be able to work in the continuum limit where it makes sense to work only with real-valued fields.

* We took a longer route to the Sine–Gordon model to see explicitly that "vector potential" "seen" by
* continuum $X$ variable is the vortex number $q$ ($X$'s encode phases "accumulated" by vortices in the presence of bosons)

* $\lambda = \infty$ and $\lambda = \text{finite}$ models really differ only by the "vortex core energy", i.e. bare vortex fugacities, $(1/\lambda = 0$ or $1/\lambda = \text{finite})$. This can shift the system in the phase diagram, but cannot modify qualitative physics.

* To summarize:

- 2D $XY$ model
- Conserved integer-valued current loop model
- Height model
- Sine–Gordon field theory
- Coulomb gas model

(can be all viewed as different formulations of the same physical system)
Correspondence of phases:

<table>
<thead>
<tr>
<th>Model</th>
<th>High-T phase</th>
<th>Low-T phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D XY model</td>
<td>high-T phase; e^Φ scrambled</td>
<td>low-T phase e^Φ have ALRO</td>
</tr>
<tr>
<td>Coulomb gas of vortices</td>
<td>unbound charges (plasma)</td>
<td>tightly bound charges (dielectric)</td>
</tr>
<tr>
<td>Current loop model</td>
<td>J-s pinned to ∅, with only small loops present (Mott insulator of bosons)</td>
<td>loops proliferate - arbitrary large loops in the system</td>
</tr>
<tr>
<td>Height model</td>
<td>X pinned to const. int. - &quot;flat surface&quot;</td>
<td>X fluctuates (within allowable) &quot;rough surface&quot;</td>
</tr>
</tbody>
</table>

Sine-Gordon model
- λ-relevant perturbation and "explodes" (flows to strong coupling), pinning X integer = constant
- λ is irrelevant; continuum $\Sigma^2(x)^2$
- $j_x \sim \nabla_x^2 \chi$
- $j_y \sim -\nabla_x \chi$
- X describes fluct. of the coarse-grained current densities.

The mapping to the Sine-Gordon model allows us to use local field theory techniques to address dynamical questions about vortices. In the original XY model, vortices are non-local (topological defect) objects, and we had to use non-field-theory technique - analysis of the Coulomb gas.
Characterization of classes $C_p$ in arbitrary dimension

Classes $C_p$ are in one-to-one correspondence with integer-valued fields on plaquettes

"vorticity fluxes":

$$ q_{\mu \nu} = \partial_\mu p_\nu - \partial_\nu p_\mu \quad \text{i.e.} \quad \omega^{(i)}_{q, \mu \nu} = d \omega^{(i)}_{p, \mu} $$

such that

$$ d \omega^{(o)}_q = d \left( \sum_{\mu \nu} q_{\mu \nu} dx^\mu \wedge dx^\nu \right) = 0 $$

$$ \partial_\lambda q_{\mu \nu} + \partial_\mu q_{\lambda \nu} + \partial_\nu q_{\lambda \mu} = 0 $$

Indeed,

$$ p \rightarrow q \quad \omega^{(i)}_p \rightarrow \omega^{(o)}_q = d \omega^{(i)}_p $$

- If $d \omega^{(i)}_p = d \omega^{(i)}_{p'} \Rightarrow d \omega^{(o)}_{p-p'} = 0 \Rightarrow \omega^{(i)}_{p-p'} = d \omega^{(o)}_N \Rightarrow p \rightarrow p' = \nabla N \quad C_p = C_{p'}$

- If $d \omega^{(o)}_q = 0 \Rightarrow \omega^{(o)}_q = d \omega^{(i)}_{p'}$ for some $p'$.

In 2d and 3d do not really need this formalism, but let's see how things look in this language:

\[ \underline{2d} \]

$$ \overline{p} = (p_x, p_y) \quad q_{xy} = \partial_x p_y - \partial_y p_x $$

2-form in 2d is just a scalar

(more precisely, dual to 2-form in 2d is 0-form, i.e. scalar)

Since in 2d $d \omega^{(o)}_q = 0$ for any 2-form, there exists $p'$ s.t. $\omega^{(o)}_q = d \omega^{(i)}_{p'}$ without any restrictions on $q$.

\[ \underline{3d} \]

2-form $q_{\mu \nu}$ can be also viewed as a (dual) 1-form - vector on the dual lattice

$$ \omega^{(o)}_{q, \mu \nu} \leftrightarrow \omega^{(i)}_{\delta, \mu \nu} \quad q_{\mu \nu} = \delta_{\mu \nu} \delta \lambda $$

$$ dw^{(i)}_q = 0 \leftrightarrow \nabla \cdot \delta = 0 $$

$C_p \leftrightarrow \delta$ satisfying $\nabla \cdot \delta = 0$