Ising gauge theory - thinking directly in the Hamiltonian language

\[ \hat{H} = -K \sum_p \hat{G}^x_{12} \hat{G}^x_{23} \hat{G}^x_{34} \hat{G}^x_{41} - \Gamma \sum_{l \in \text{links}} \hat{G}^x_{rr'} = 1 \quad \forall r \]

"Strong coupling" phase \( \Gamma \gg K \) (confined phase)

- \( K = 0 \): \( \hat{H} \) is diagonal in the \( \hat{S}^x_e \) basis

\[ E[\sum S^x_e] = -\Gamma \sum S^x_e \]

But \{\( S^x_e \}\} cannot be arbitrary - must satisfy \( \prod_{r \in \text{links}} S^x_{rr'} = 1 \)

Ground state: \( S^x_{rr'} = 1 \)

First excited state: \(-1 \) - small loop of the '\( \mathbb{Z}_2 \) electric field'

Further excited states:

Confinement of charges: \( r_0 \rightarrow r_1 \) - minimal energy cost is \( 2\Gamma \cdot |r_0 - r_1| \)

- \( K \neq 0 \): provides the loops with quantum dynamics, e.g.,

"closed electric field lines"

(creates/annihilates elementary loops)
deconfined phase

\[ \Gamma = 0 \]

\[ \text{Weak coupling phase} \]

\[ \text{Confinement phase} \]

\[ \text{Changes shape of loops} \]

\[ \text{Largest energy excitation at } \Gamma \ll K \text{ is the elementary loop} \]

\[ \text{One can do e.g. perturbative analysis} \]

\[ \text{As } K \text{ gets large, we have more virtual states to lower their energy} \]

\[ \text{At some point, the loops proliferate and one} \]

\[ \text{deconfined phase} \]
Thus, the ground state is an equal weight superposition of all allowed configurations in the $g^2$ basis — an equal weight superposition of all pictures of closed "$Z_2$ electric field" lines.

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| 1 \rangle = \frac{1}{\sqrt{3}} | 3 \rangle
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"Condensate" of such lines.

Inserting test charges then costs only finite energy, since the connecting string has no tension.

In the ground state, $G_{12} G_{23} G_{34} G_{41} |14\rangle = |14\rangle$, and the energy is indeed minimized $E = -K \cdot Np$.

(can be checked either by direct action on $|14\rangle = \sum_{f, s} |f_{12}, f_{23}, f_{34}, f_{41}, s\rangle$

and noting that $G_{12} G_{23} G_{34} G_{41}$ is a one-to-one mapping of allowed $\{f_{12}, f_{23}, f_{34}, f_{41}\}$, or by noting that the projection commutes with $G_{12} G_{23} G_{34} G_{41}$ and evaluate it before the projection).

The "gauge-invariant" definition of the ground state is that all placket terms are "happy" ($\text{"unfrustrated"}$, or zero flux if we interpret the $+1 = e^{i\phi}$).

Excited states — can be constructed and analyzed similarly. For example, starting with $S_z = +1$ before projection, changing one $S_z$ to $-1$ frustrates two plackets → energy cost $= 2 \times 2K$. 

But same energy cost will result also from changing $S_i^z \rightarrow -S_i^z$ on a string of bonds, since the flux on the "intermediate" plackets remains unchanged.

A more "elementary" excitation is then one frustrated placket (although frustrated plackets must come in pairs, the two can be separated far apart and one can focus on single one at a time).

In the "gauge-invariant" procedure, we can insert such "-1" plackets (or "J* fluxes") by acting with a string operator

$$\prod_{\text{string}} \hat{\sigma}_x$$

Since this operator commutes with all "plackets" other than end-points and anticommutes with the end-point plackets, it indeed inserts the J* fluxes only at the end-points. Note that because of the constraint $\prod_{r^*} \hat{\sigma}_x = 1$, the result depends only on the end-points and not on the path itself.

Equivalent paths for the string operator because of the constraints on all enclosed sides.
(deconfined phase contd)

- \( \Gamma \neq 0 \) — imparts quantum dynamics on the fluxes.

\[
\begin{array}{c}
\Phi_p \\
\Phi_{p'}
\end{array}
\]

acting with \( \hat{g}^x \) changes
\[
\Phi_p \rightarrow -\Phi_p
\]
\[
\Phi_{p'} \rightarrow -\Phi_{p'}
\]

Acting on the g.s., this would create two frustrated plackets.

Acting in the first excited states sector with one frustrated placket, the \( \Gamma \)-terms can now hop this placket around

\[
\begin{array}{ccc|ccc}
+ & + & + & + & + & + \\
+ & - & + & + & + & + \\
+ & + & + & + & + & + \\
\end{array}
\rightarrow
\begin{array}{ccc|ccc}
+ & + & + & + & + & + \\
+ & + & + & + & + & + \\
+ & + & + & + & + & + \\
\end{array}
\]

- quantum hopping with amplitude \( \Gamma \)

\[\Rightarrow E(k) = 2K - 2\Gamma (\cos k_x + \cos k_y)\]

\( \pi \)-flux excitations are quantum particles moving around. When these particles \text{proliferate}, the gap closes and the deconfined phase is destroyed and the system goes into strong coupling confined phase.
Summary: Ising gauge theory in (2+1)D

Confined phase: excitations are closed loops = closed electric field lines = closed worldsheet in space-time

Deconfined phase: excitations are T-fluxes – quantum particles = worldlines in space-time

Compare with global Ising model in (2+1)D

Paramagnetic phase: excitations are \( x^T = -1 \) particles = worldlines in space-time

Ferromagnetic phase: excitations are closed domain walls = closed worldsheet in space-time

Note qualitative similarity between confined \( \leftrightarrow \) ferro and deconfined \( \leftrightarrow \) paramagnetic phases in the IST \( \leftrightarrow \) global Ising – can be formalized by duality mapping.
Formal duality mapping for the quantum IGT

Define \( \hat{\mu}_p = G_{12} G_{23} G_{34} G_{41} \)
\( \hat{\nu}_p = \prod_{p \in P} \hat{\sigma}_e \)

(any path from infinity or boundary that terminates at \( P \))

\( \hat{\mu}_p \) commute among themselves & \( (\hat{\mu}_p)^2 = 1 \)

\( \hat{\mu}_p \) -1 - & \( (\hat{\mu}_p)^2 = 1 \)
\( \hat{\mu}_p \) & \( \hat{\mu}_p \) commute when \( P \neq P' \) (since then \( P \) is crossed even number of times by the string defining \( \hat{\mu}_p \))

\( \hat{\mu}_p \) & \( \hat{\mu}_p \) anticommute; \( \hat{\mu}_p \hat{\mu}_p = -\hat{\mu}_p \hat{\mu}_p \)

\( \Rightarrow \) \( \hat{\mu}_p \), \( \hat{\mu}_p \) can be viewed as Pauli matrices

\[
\hat{H} = -k \sum_{p} \hat{\mu}_p \hat{\mu}_p - \Gamma \sum_{p,p'} \hat{\mu}_p \hat{\mu}_{p'}
\]

Quantum Ising gauge theory is dual to the quantum global Ising model, with the notions of strong coupling (dominant "x" terms) and weak coupling (dominant "z" terms) interchanged.

Correspondence between phases and excitations (\( g.s. \)) is now transparent.
Ising gauge theory \[ \hat{\mathcal{g}}_{12,23}. \hat{\mathcal{g}}_{3,41,1} = \hat{x} \]
\[ \prod_{P} \hat{\mathcal{g}} \rightarrow \hat{\mu}_{P} \]
\[ \hat{\mu} = \hat{\mu}_{P} \hat{\mu}_{P}^{1} \]
\[ \hat{H} = -K \sum_{P} \hat{\mathcal{g}}_{12,23} \hat{\mathcal{g}}_{3,41} \hat{\mathcal{g}}^{1} - K \sum_{P} \hat{\mu}_{P} \hat{\mu}_{P}^{1} \]

\* Confined phase \[ \iff \rightarrow \] Ferro-ordered phase

g.s. \[ \hat{\mathcal{g}} = 1 \]
excitations:

loops of \( \hat{\mathcal{g}}^{x} = -1 \)

\* Deconfined phase \[ \iff \rightarrow \] Paramagnetic phase

g.s. \[ \hat{\mathcal{g}}^{x} \hat{\mathcal{g}}^{z} \hat{\mathcal{g}}^{z} = 1 \]
excitations:

frustrated plackets

\[ \hat{\mathcal{g}}^{2} \hat{\mathcal{g}}^{2} \hat{\mathcal{g}}^{2} = -1 \]

\* Deconfined \( \rightarrow \) confined transition
condensation of \( \pi \)-fluxes
(frustr. plackets)

\[ \langle \prod_{P} \hat{\mathcal{g}} \rangle \neq 0 \]
nonlocal order param.

\* Confined \( \rightarrow \) deconfined transition
proliferation of electric field loops
Ferro \( \rightarrow \) para
proliferation of domain walls.
Parting remarks on the Ising systems (global Ising model & Ising gauge theory (2+1)D)

* Throughout, emphasized simple pictures of the phases and excitations above the ground state. The phase transition is viewed as a closing of the gap and proliferation of these excitations.

- Coming from the ferro-Ising, the excitations are domain walls — defects in the order — and the order is destroyed when such defects proliferate.

  (More general theme of the course — thinking in terms of topological defects and disordering transitions as defect proliferations)

- Coming from the deconfined IGT, the defects are \( \gamma \)-fluxes and their proliferation leads to confinement.

Duality mappings — way to formalize such thinking in terms of relevant defects

Duality mappings by themselves do not give dynamical information about the phase transitions — one still needs to formulate field theory for the dual degrees of freedom and study this. But clearly this can be a very different field theory compared with thinking in terms of original variables and can provide valuable different perspective.

Other applications of such thinking:

- Vortices in global XY model (neutral superfluid)
- Vortices in charged superfluids
- Monopoles in quantum electrodynamics