Application: Luttinger liquid theory in 1+1d

["Harmonic liquid theory" or (Harmonic) hydrodynamics].

* Classical XY model $\leftrightarrow$ quantum rotor model

$$\hat{H} = \sum_r \frac{1}{2} \hat{n}_r^2 - \sum_{\langle r, r' \rangle} J \cos(\phi_r - \phi_{r'})$$

- models boson fluctuation about some integer value
- can immediately apply everything we learned to say that for sufficiently weak $U \ll J$ ($K' \equiv \sqrt{\frac{J}{U}}$) we are in the phase with power law correlations

$$\langle e^{i\phi(r)} e^{-i\phi(r')} \rangle \sim \frac{1}{|r-r'|^{1/(\pi K')}}.$$ 

As we increase $U$, the system undergoes transition to Mott insulator (vortices proliferate and pin boson density). The threshold is $K = \frac{2}{\sqrt{J}}$ and the power law decay at this threshold is $\sim \frac{1}{r^{1/4}}$

(everywhere in the QLRO phase the power law decay is more slow than $\frac{1}{r^{1/4}}$).

* What if the boson density per site is noninteger?

Model with:

$$\hat{H} = \sum_r \frac{1}{2} (\hat{n}_r - \bar{n})^2 - J \sum_{\langle r, r' \rangle} \cos(\phi_r - \phi_{r'})$$

average boson number per site

IF we try to do path integral in the $\phi$-variables:

$$\langle \phi(t+\delta t) | e^{-\frac{\delta t}{2} \frac{1}{2} (n-\bar{n})^2} | \phi(t) \rangle = \sum_{n=\infty}^{+\infty} e^{-\frac{\delta t}{2} \frac{1}{2} (n-\bar{n})^2} e^{i n (\phi(t+\delta t) - \phi(t))} e^{-\delta t \frac{1}{2} n^2}$$

$$= \sum_{n=-\infty}^{+\infty} e^{-\delta t \frac{1}{2} n^2} e^{i n (\phi(t+\delta t) - \phi(t))} e^{-\delta t \frac{1}{2} n^2}$$
Poisson resum. on $n$

\[
\sum_{\rho=-\infty}^{+\infty} e^{\frac{1}{2 u s t} \left( \phi(r+s \tau) - \phi(r) - i \bar{n} u s t - 2 \pi \rho \right)^2} e^{s \bar{n}^2}
\]

\[
\sum_{\rho=-\infty}^{+\infty} e^{\frac{1}{2 u s t} \left( \phi(r+s \tau) - \phi(r) - 2 \pi \rho \right)^2 + i \bar{n} \left( \phi(r+s \tau) - \phi(r) \right) - 2 \pi \rho}
\]

Berry phase!

\[
e^{-\frac{1}{2 u} (\partial \bar{n})^2 / \bar{s} \tau + i \bar{n} (\partial r) / \bar{s} \tau}
\]

schematic writing. Cannot drop $e^{-\frac{1}{2 \bar{s} \tau} (\partial \bar{n})^2}$ because $\bar{n}$ may wind!

We have complex contributions to the $Z$ in the $\bar{\phi}$-representation which are very difficult to work with. The only way we know how to deal with them is by doing exact "resummations" (reformulations) where the final theory has no Berry phases. The original Berry phases produce new features in the dual theory.

Current loop model

\[
Z = \sum_{\tilde{j}, \tilde{\phi}, \tilde{j} = 0} e^{-\frac{1}{2 u s t} \left( \tilde{j} \tilde{\phi} - \tilde{n} \right)^2} \frac{\tilde{j}}{2 \beta_{\text{spot}}}
\]

- has no sign problem, so in principle can be simulated in Monte Carlo (QMC)

Exact reformulations

\[
Z = \sum_{\tilde{j}} e^{i \Sigma \bar{j} \cdot \bar{S} \left( \bar{j}, \phi = 0 \right) e^{-\frac{1}{2 u s t} \left( \tilde{j} \tilde{\phi} - \tilde{n} \right)^2} - \frac{\tilde{j}}{2 \beta_{\text{spot}}}} - i \Sigma j \cdot 2 \pi \rho
\]

Here $\bar{j}$ are real-valued, so can change vars $\tilde{j} = j - \bar{n}$

\[
Z = \sum_{\tilde{j}} e^{-i \Sigma \bar{j} \cdot 2 \pi \rho \left( \bar{n} \right)} \sum_{\tilde{j}} e^{i \Sigma \bar{j} \cdot \bar{S} \left( \bar{j}, \phi = 0 \right) e^{-\frac{1}{2 u s t} \left( \tilde{j} \tilde{\phi} - \tilde{n} \right)^2} - \frac{\tilde{j}}{2 \beta_{\text{spot}}}} - i \Sigma j \cdot 2 \pi \rho
\]

Same as at zero density
\[ \langle \phi(\tau+i\delta) \mid e^{-\delta \tau \frac{1}{2} (\pi^{-1}\cdot \nabla)^2} \mid \phi(\tau) \rangle = \]
\[ = \sum_{\mathbf{p} \equiv \mathbf{0}} e^{-\frac{1}{2\delta \tau} (\phi(\tau+i\delta) - \phi(\tau) - 2\pi \mathbf{p})^2} + i\nabla (\phi(\tau+i\delta) - \phi(\tau) - 2\pi \mathbf{p}) \cdot \nabla \]

In the path integral:
\[ e^{i \sum_{\mathbf{p} \equiv \mathbf{0}} \nabla (\phi(\tau+i\delta) - \phi(\tau) - 2\pi \mathbf{p})} = e^{i \sum_{\mathbf{p} \equiv \mathbf{0}} 2\pi \mathbf{p} \cdot \nabla} \]

Everything else is the same as for zero average density, and all manipulations are the same as long as we keep \( \mathbf{p} \equiv \mathbf{0} \) (or \( \mathbf{C} \equiv \mathbf{vortex \ configs} \)).

Berry phase factor associated with each configuration:
\[ e^{-i \sum_{\mathbf{p} \equiv \mathbf{0}} 2\pi \mathbf{p} \cdot \nabla} \]

One can show that this depends only on vorticity \( \Omega_{xy} = \partial_x \mathbf{p}_y - \partial_y \mathbf{p}_x \)
(will see explicitly in \( d=2,3 \)).

(Original static boson density produces static "sources" of complex phases for the vortices.)

Specialize to 1+1d: Write
\[ \begin{align*}
\bar{\chi} &= -\bar{\nabla} \cdot \chi \\
\bar{\nabla} &= \bar{\nabla} = \nabla = -\nabla \cdot \bar{\chi}
\end{align*} \]

\[ \bar{\chi} = -\mathbf{p} \cdot \mathbf{x} \] indeed gives

\[ \sum \mathbf{p} \cdot \bar{\chi} = \sum (\mathbf{p}_y \bar{j}_y + \mathbf{p}_x \bar{j}_x) = \sum (\mathbf{p}_y (-\nabla_x \bar{\chi}) + \mathbf{p}_x \nabla_y \bar{\chi}) \]

\[ = \sum \bar{\chi} \cdot \nabla (\mathbf{p}_y \bar{\chi} - \mathbf{p}_x \nabla \mathbf{p}_x) = \sum \bar{\chi} \cdot \mathbf{q} \]

Dual lattice sites
Berry phase associated with vortex config $\chi_I$ is

$$e^{i S_{\text{Berry}}[\chi]} e^{-i \pi \sum_{I} \chi_I}$$

$$Z = \sum_{\text{vortex configs}} e^{i S_{\text{Berry}}[\chi]} - E_{\text{vortex cos}}[\chi]$$

Can map directly to sine-Gordon model modified

$$S' [\chi] = \sum_{I} \left( \frac{\partial \chi_I}{\partial x} \right)^2 - \lambda \sum \cos \left( 2 \pi \chi_I + 2 \pi \chi_I \right)$$

Example: Lattice bosons at half-filling, $\overline{\eta} = \frac{1}{2}$

$$\chi_I = \frac{1}{2} \chi_I$$

$$\sum \cos (2 \pi \chi_I + 2 \pi \chi_I) = \sum \cos (2 \pi \chi_I + \pi \chi_I) = \sum (-1) \cos (2 \pi \chi_I)$$

- averages to zero for slowly varying $\chi_I$!

Physically, single vortex @ $x_I$ and single vortex @ $x_{I+1}$ have Berry phases differing by $\pi$, and such configurations interfere destructively in the path integral! Because of the Berry phase interference effects, single vortices do not destroy the phase coherence of the original $\phi$'s.

On the other hand, double vortex @ $x_I$ and double vortex @ $x_{I+1}$ have Berry phases differing by $2\pi = 0$, and there is no such destructive interference.
In the field theory, besides \( \cos(2\pi x_i + 2\pi \overline{x}_i) \), we should allow also any powers, e.g.,

\[
\cos\left(2 \cdot (2\pi x_i + 2\pi \overline{x}_i)\right) \quad \text{interpreted as double vortex insertion}
\]

For \( \overline{n} = \frac{1}{2} \)

\[
\sum \cos\left(2 \cdot (2\pi x_i + 2\pi \overline{x}_i)\right) = \sum \cos(4\pi x_i + 4\pi \overline{x}_i)
\]

\[
= \sum \cos(4\pi x_i) = \int d^2r \cos(4\pi x_i)
\]

Continuum limit.

Thus, for \( \overline{n} = \frac{1}{2} \), the first term allowed in the nonlinear potential Sine-Gordon theory is \( \cos(4\pi x_i) \to \lim_{J} = 4\pi K \)

This perturbation is irrelevant for \( 4\pi K > 2 \);

in particular, the power-law at threshold is

\[
\langle e^{i\varphi(r)} e^{-i\varphi(r')} \rangle \sim \frac{1}{|r-r'|^{1/\omega_{mk}}} \sim \frac{1}{|r-r'|}
\]

Application: XY Heisenberg spin chain

\[
\hat{H} = \sum_i \left[ J_x (s_i^x s_{i+1}^x + s_i^y s_{i+1}^y) + J_z s_i^z s_{i+1}^z \right]
\]

for \( J_z < J_x \) is in the "Luttinger liquid" phase with continuously varying power law. At \( J_z = J_x \) the power law is \( \frac{1}{|r-r'|} \), and for larger \( J_z > J_x \) gap opens up and the system has Ising antiferromagnetic order \( \uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow \).
Bosons @ half-filling - what is the state when \( \cos(\pi x) \) becomes relevant? Expect this to be a Mott insulator, but can we describe it in more detail? - Yes, but need to work a bit harder: Keeping the oscillatory term that we dropped in the analysis of Luttinger liquid phase:

\[
S[x] = \sum \frac{(\nabla x)^2}{2K} - \lambda_1 \sum (-1)^x \cos(\pi x) - \lambda_2 \sum \cos(4\pi x)
\]

\( \nabla x = x_{i+1} - x_i \)

In the phase where \( \lambda_2 \) flows to large value, we need to simply minimize \( x \) to determine the ground state:

\[
\frac{1}{2K} \cdot 2(\chi_i - \chi_{i+1} + \chi_i - \chi_{i-1}) + \lambda_1 (-1)^x \sin(2\pi \chi_i) \cdot 2\pi + \lambda_2 \sin(4\pi \chi_i) = 0
\]

\( \lambda_2 > 0 \) \( \Rightarrow \) the solution minimizing \( S \) is \( \chi_i = 0 \). (Any that gives \( \cos(\pi x) = 1 \))

\( n \approx \nabla x \) is uniform, but there is staggered bond energy \( E \approx (-1)^x \cos(\pi x) \approx (-1)^x \)

\( \chi_i \approx \text{dual lattice sites} \)

\( \lambda_2 < 0 \) if \( \lambda_1 = 0 \), we would minimize by \( \chi_i = \frac{1}{4} \)

But now this does not solve (\( \ast \)) if \( \lambda_1 \neq 0 \) since \( \sin(\pi x) \approx 1 \)

One can solve (\( \ast \)) by \( \chi_i = \frac{1}{4} + (-1)^x \frac{1}{2} \approx n - \nabla x \approx (-1)^x \)

- charge density
- wave of bosons!
More details on the quantum rotor \( \leftrightarrow \) bosons \( \leftrightarrow \) at half-filling

\( \leftrightarrow \) XY spin model \( \leftrightarrow \) XXZ antiferromagnet (easy-plane)

Quantum rotors at half-filling

\[
H = \sum_i \left( \frac{\Delta}{2} (n_i - \frac{1}{2})^2 - J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) \right) - J \sum_{\langle i,j \rangle} \left( e^{i\phi_i} e^{i\phi_j} + e^{i\phi_j} e^{i\phi_i} \right)
\]

\([\hat{\phi}, \hat{n}] = i \Rightarrow e^{i\hat{\phi}} \) is raising operator \( e^{i\phi} |n_i\rangle = |n_i + 1\rangle \)

\( e^{-i\hat{\phi}} \) is lowering operator \( e^{-i\phi} |n_i\rangle = |n_i - 1\rangle \)

\( e^{i\phi_i} e^{-i\phi_j} |n_i; n_j\rangle = |n_i + 1; n_j - 1\rangle \)

\( \leftrightarrow \) moves boson from \( j \) to \( i \).

Suppose \( u \) is quite large \( \Rightarrow \) main configurations that have lowest \( u \)-energy are \( n_i = 0 \) or \( 1 \) \( \Rightarrow \) "hard-core bosons". \( b_i \) restricted to have \( b_i^+ b_i = 0 \) or \( 1 \)

\( e^{\phi_i} e^{-\phi_j} - b_i^+ b_j \) - boson hopping

For qualitative physics, quantum rotors at half-filling are similar to hard-core boson model at half-filling

\[
H = -t \sum_{\langle i,j \rangle} (b_i^+ b_j + h.c.) + \text{perhaps some additional interactions, e.g.}
\]

\( V_{ij} [b_i^+ b_i - \frac{1}{2}] [b_j^+ b_j - \frac{1}{2}] \)
Hard-core bosons ↔ spin-$\frac{1}{2}$ system

\[
\begin{align*}
\hat{n} = 0 & \leftrightarrow \text{spin-\downarrow, } S^z = -\frac{1}{2} \\
1 & \leftrightarrow \text{spin-\uparrow, } S^z = +\frac{1}{2}
\end{align*}
\]

\[\hat{n} - \frac{1}{2} = \hat{S}^z\]

\[b_i^+ = S_i^+ \quad \text{spin-raising operator}\]

\[b_i^- = S_i^- \quad \text{spin-lowering operator}\]

Boson hopping \[= -t(b_i^+ b_j + h.c.) = -t(S_i^+ S_j^- + S_i^- S_j^+)\]

\[= -2t(S_i^x S_j^x + S_i^y S_j^y)\]

"XY" spin interaction

Boson interaction \[V_{ij}(\hat{n}_i - \frac{1}{2})(\hat{n}_j - \frac{1}{2}) = V_{ij}S_i^z S_j^z\]

"Z" spin interaction

\[
H_{XXZ} = -\sum_{\langle ij \rangle} J^+(S_i^z S_j^z + S_i^x S_j^x) + J^z \sum_{\langle ij \rangle} S_i^z S_j^z
\]

Spin-$\frac{1}{2}$ model

1d chain - the model is exactly soluble and correlations are known exactly:

\[J^z < J^+ \quad \text{- easy-plane limit}\]

\[
\langle S_i^z S_j^z \rangle \sim \frac{1}{|i-j|^2}
\]

- can understand this using our "hydrodynamic" description

\[
S_i^2 \sim \hat{n}_i - \frac{1}{2} \sim \nabla_\alpha x
\]

and

\[
\langle \nabla_\alpha x(x) \nabla_\alpha x(x') \rangle \sim \frac{1}{|x-x'|^2}
\]
\[ \langle S^x_i S^x_j + S^y_i S^y_j \rangle \sim \frac{1}{|i-j|^{1/2 \kappa_0 K_{\text{eff}}}} \]

- can understand correlations qualitatively as

\[ \langle e^{i \phi_i} e^{-i \phi_j} \rangle \sim \frac{1}{|i-j|^{1/2 \kappa_0 K_{\text{eff}}}} \]

but cannot calculate \( K_{\text{eff}} \) accurately in the hydrodynamic treatment. Can match with one exact result

\[ K_{\text{eff}} = \frac{1}{2 \pi} \left( 1 - \frac{1}{\pi} \arccos \frac{J_2}{J_1} \right)^2 \]

and can then describe any other property (everything is contained in one hydrodynamic parameter \( K_{\text{eff}} \))

\[ J_2 = J_1 \quad \langle S^x_i S^x_j + S^y_i S^y_j \rangle \sim \frac{1}{|i-j|} \]

\[ K_{\text{eff}} = \frac{1}{2 \pi} \quad \text{threshold for stability} \]

\[ J_2 > J_1 \quad \text{antiferromagnetic order} \]

schematically: \[ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \]

n = 1 0 1 0 1 0 — charge order of bosons

\[ K_{\text{eff}} = \frac{1}{2(\pi - \arccos \frac{J_2}{J_1})} \]
∗ If bosons are at incommensurate density, then any multiple vortex insertions will still interfere destructively:

\[ \lambda \sum_{\mathbf{a}} \exp \left[ i \left( \mathbf{p} \cdot \mathbf{x} + 2\pi \mathbf{a} \cdot \mathbf{x} \right) \right] \rightarrow \text{averages to } \overline{0}. \]

⇒ at incommensurate density the bosons are always in liquid phase (described by the harmonic hydrodynamic theory).