Quantum – classical $\overset{\text{statmech}}{}$ mapping (Euclidean path integral or imaginary time path integral)

d-dimensional quantum system at $T=0$ $\longleftrightarrow$ $(d+1)$ - dimensional statistical mechanics system

General structure: $\hat{H}$ – quantum Hamiltonian; want to calculate partition fnctn at temperature T (inverse temperature $\beta = 1/T$)

$$\boxed{Z = \mathrm{Tr}\, e^{-\beta H} = \sum_{m} \langle m | e^{-\beta \hat{H}} | m \rangle}$$

↑ all states (some basis)

Generally, problem is that $\hat{H}$ has non-commuting parts and it is difficult to calculate $e^{-\beta\hat{H}}$. On the other hand, if we consider small $\delta\tau$

Trotter decomposition

$$\boxed{e^{-\delta\tau \hat{H}} = e^{-\delta\tau \sum_j \hat{h}_j} \approx e^{-\delta\tau \hat{h}_1} e^{-\delta\tau \hat{h}_2} \dots}$$ with accuracy $O(\delta\tau^2)$

↑ different parts

$$\approx 1 - \delta\tau \sum_j h_j + \frac{1}{2}\delta\tau^2 (\Sigma)^2 \dots \quad \Big\| \quad \left(1 - \delta\tau h_1 + \frac{1}{2}\delta\tau^2 h_1^2 + \dots\right)\left(1 - \delta\tau h_2 + O(\delta\tau^2)\right)$$

$$\approx 1 - \delta\tau (h_1 + h_2 + \dots) + O(\delta\tau^2)$$

Write $e^{-\beta\hat{H}} = \underbrace{e^{-\delta\tau\hat{H}} e^{-\delta\tau\hat{H}} \dots e^{-\delta\tau\hat{H}}}_{N \text{ times}}$; insert $\mathbb{1} = \sum |m\rangle\langle m|$ between each pair

$$\boxed{Z = \sum_{m_0, m_1 \dots, m_{N-1}} \langle m_0 | e^{-\delta\tau\hat{H}} | m_{N-1} \rangle \langle m_{N-1} | e^{-\delta\tau\hat{H}} | m_{N-2} \rangle \langle m_{N-2} | \dots \dots | m_1 \rangle \langle m_1 | e^{-\delta\tau\hat{H}} | m_0 \rangle}$$

$\delta\tau = \frac{\beta}{N}$; can introduce $m_N \equiv m_0$ – "periodic boundary conditions" in the imaginary time direction

Euclidean path integral for the transverse field Ising model (quantum Ising model)

$$\boxed{\hat{H} = \underbrace{-J\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z}_{H_z} \underbrace{- \Gamma \sum_i \hat{\sigma}_i^x}_{H_x}}$$

H_z, H_x do not commute

Pauli matrices

$$\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Want to write path integral in the basis of $\hat{\sigma}^z$

one spin: $\hat{\sigma}^z |S_z\rangle = S_z |S_z\rangle$, $S_z = \pm 1$

many spins: $|m\rangle = |\{S_z(i)\}\rangle$

One time step:

$$\langle m_{\tau+\delta\tau}| e^{-\delta\tau \hat{H}} |m_\tau\rangle \approx \langle m_{\tau+\delta\tau}| e^{-\delta\tau H_x} e^{-\delta\tau H_z} |m_\tau\rangle =$$

$$= e^{-\delta\tau H_z(\{S_z(i;\tau)\})} \langle m_{\tau+\delta\tau}| e^{-\delta\tau H_x} |m_\tau\rangle .$$

Single spin: need $\langle S_z(\tau+\delta\tau)| e^{\delta\tau\Gamma\hat{\sigma}^x} |S_z(\tau)\rangle =$

$$= \sum_{S_x=\pm1} \langle S_z(\tau+\delta\tau)| e^{\delta\tau\Gamma\hat{\sigma}^x} |S_x\hat{x}\rangle\langle S_x\hat{x}| S_z(\tau)\rangle =$$

$$= \sum_{S_x=\pm1} e^{\delta\tau\Gamma S_x} \langle S_z(\tau+\delta\tau)\hat{z}| S_x\hat{x}\rangle\langle S_x\hat{x}| S_z(\tau)\rangle$$

Here we inserted decomposion of $\mathbb{1}$ in the basis of $\hat{\sigma}^x$:

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|+\hat{z}\rangle + |-\hat{z}\rangle\right) \Rightarrow \langle +\hat{x}|\pm\hat{z}\rangle = \frac{1}{\sqrt{2}}$$

$$|-\hat{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|+\hat{z}\rangle - |-\hat{z}\rangle\right) \Rightarrow \langle -\hat{x}|\pm\hat{z}\rangle = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \langle S_x\hat{x}|S_z\hat{z}\rangle = \frac{1}{\sqrt{2}} e^{i\pi\frac{1-S_x}{2}\cdot\frac{1-S_z}{2}}$$

$$\langle S_z' | e^{\delta\tau\Gamma\hat{\sigma}^x} | S_z \rangle = \sum_{S_x=\pm 1} e^{\delta\tau\Gamma S_x} \cdot \frac{1}{2} e^{i\pi \frac{1-S_x}{2}\left(\frac{1-S_z'}{2} + \frac{1-S_z}{2}\right)} =$$

$$= \frac{1}{2}\left(e^{\delta\tau\Gamma} + e^{-\delta\tau\Gamma} \underbrace{e^{i\pi\left(\frac{1-S_z'}{2}\right)}}_{S_z'} \underbrace{e^{i\pi\left(\frac{1-S_z}{2}\right)}}_{S_z} \right) =$$

$$= \frac{1}{2}\left(e^{\delta\tau\Gamma} + e^{-\delta\tau\Gamma} S_z' S_z\right)$$

(One could have simply used $e^{\alpha\hat{\sigma}^x} = \cosh(\alpha)\,\mathbb{1} + \sinh(\alpha)\,\hat{\sigma}^x = \begin{pmatrix} \mathrm{ch}\alpha & \mathrm{sh}\alpha \\ \mathrm{sh}\alpha & \mathrm{ch}\alpha \end{pmatrix}$ in the S^z basis to verify the above, but the derivation with inserting eigenstates of $\hat{\sigma}^x$ is instructive and useful in other applications.)

Let us try to represent $\boxed{\langle S_z' | e^{\delta\tau\Gamma\hat{\sigma}^x} | S_z \rangle = C e^{J_\tau S_z' S_z}} =$

const.

$$= C\left(\cosh J_\tau + S_z' S_z \sinh J_\tau\right)$$

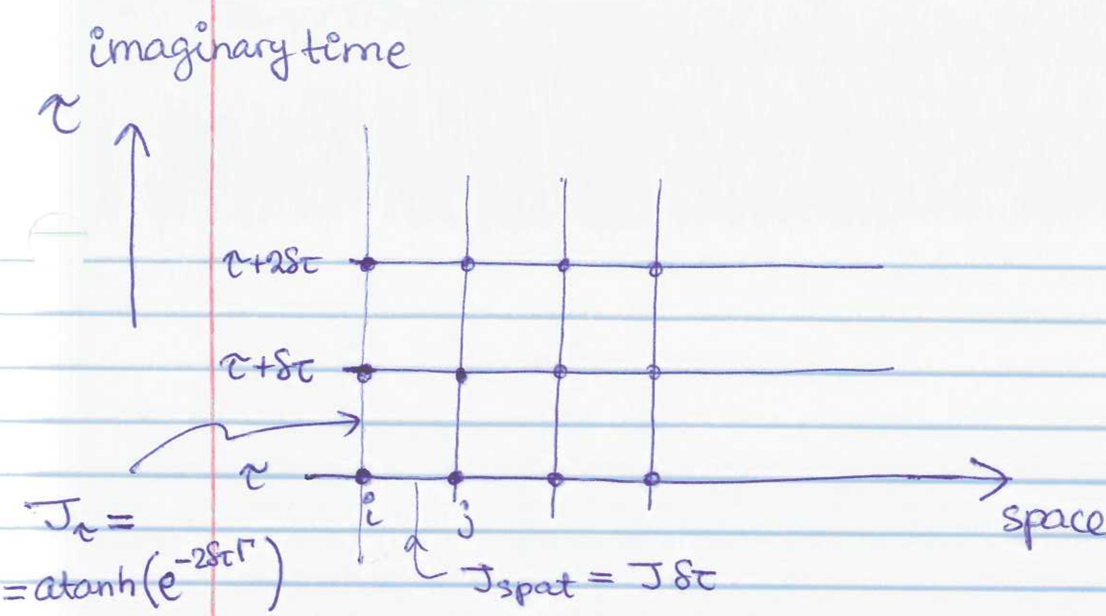
$$\Rightarrow \boxed{\tanh J_\tau = e^{-2\delta\tau\Gamma}}$$

(the value of C is not important, but can easily find $\boxed{C^2 = C^2(\cosh^2 J_\tau - \sinh^2 J_\tau) = \frac{\sinh(2\delta\tau\Gamma)}{2}}$)

Collecting everything, for the transverse field Ising model we have

$$\boxed{Z \approx \sum_{\{S_z(i,\tau_0);\, S_z(i,\tau_1)\ldots,\, S_z(i,\tau_{N-1})\}} C^{\mathrm{Vol}} \times \exp\Big[\underbrace{J\delta\tau}_{\substack{\shortparallel \\ J_{spat}}} \sum_{\tau,\langle ij\rangle} S_z(i,\tau) S_z(j,\tau) + \sum_{\tau,i} J_\tau S_z^{(i)}(\tau+\delta\tau) S_z^{(i)}(\tau) \Big]}$$

$$\boxed{\begin{aligned} J_{spat} &= J\delta\tau \\ J_\tau &= \mathrm{atanh}\left(e^{-2\delta\tau\Gamma}\right) \end{aligned}}$$



Thus, the partition function of d-dim. quantum Ising model at T is represented as a partition fnctn. of d+1-dim. classical Ising model of size $\beta = \frac{1}{T}$ in the imaginary time direction. In the limit $T \to 0$, the classical Ising model has ∞ size in the (d+1)-st dimension.

Remarks

* Such representation of $Z = \mathrm{Tr}\, e^{-\beta\hat{H}}$ is how one would study the quantum Ising model in Quantum Monte Carlo (QMC) — by representing it as classical statmech problem and using usual Monte Carlo to simulate the latter.

* ~~d~~ d-dim quantum Ising at T=0 $\longleftrightarrow$ d+1 classical Ising in the thermodynamic limit (infinite spatial & temporal directions). The couplings in the classical model are anisotropic and depend on the discretization step $\delta\tau$:

$$J_{spat} = J\,\delta\tau$$
$$J_\tau = \mathrm{atanh}\left(e^{-2\delta\tau\Gamma}\right)$$

If we want a numerically accurate representation, we would take small $\delta\tau \to 0 \Rightarrow J_{spat}$ is small, J_τ is large

If we are interested in qualitative physics, we can take reasonable $\delta\tau$ and work with finite J_{spat}, J_τ.

For example, we can try to find $\delta\tau$ s.t.

$$J_{spat} = J\delta\tau = \operatorname{atanh}\left(e^{-2\delta\tau\Gamma}\right) = J_\tau$$

i.e., the classical Ising model is isotropic in space-time. This would fix

$$\underbrace{J_{spat} = J_\tau}_{K} = \text{function of } \left(\Gamma/J\right)$$

$$K = \operatorname{atanh}\left(e^{-2\cdot K\cdot \Gamma/J}\right) \quad \text{- eqn. for } K \text{ in terms of } \Gamma/J.$$

$$\Gamma/J = -\frac{1}{2K}\ln(\tanh K)$$

Note that K is the ~ effective inverse temperature of the classical Ising model, which is determined from the parameter Γ/J of the quantum Ising model. (Alternatively, can take $\delta\tau = 1/J \Rightarrow J_{spat} = 1$, $J_\tau = \operatorname{atanh}\left(e^{-2\Gamma/J}\right)$ - easier to see $\Gamma/J \Leftrightarrow$ effective K correspondence.)

$\circledast$ Correspondence of the phases:

Quantum d-dim model		Classical (d+1)-dim model
large Γ/J quantum disordered phase	$\longleftrightarrow$	small $K \approx e^{-2K\cdot\Gamma/J}$ high-temperature (disordered) phase
small Γ/J quantum ordered phase	$\longleftrightarrow$	large K low-temperature (ordered) phase (for $d+1\geq 2$, i.e. $d\geq 1$)
quantum critical point (for $d\geq 1$)	$\longleftrightarrow$	(d+1)-dim classical Ising critical point

Thus, we can use knowledge of the statmech system to deduce behaviour of the quantum many-body system.

✱ 0-dimensional quantum problem is just single spin
$$\hat{H}_{0dim} = -\Gamma \sigma^x$$
- of course, can solve this easily:

$$\left.\begin{array}{lll} \text{ground state} & |+\hat{x}\rangle, & E=-\Gamma \\ \text{excited state} & |-\hat{x}\rangle, & E=+\Gamma \\ & Z = e^{\beta\Gamma} + e^{-\beta\Gamma} & \end{array}\right\}$$

no phase transition - simple gapped discrete q.m. system

This corresponds to (0+1)d classical Ising model with $J_\tau = \text{atanh}(e^{-2\delta\tau\Gamma})$ - inverse effective temperature.

from last quarter; the transfer matrix method used there is essentially the same as solving 0-dim quantum Hamiltonian

The classical Ising model is disordered for all $J_\tau > 0$, with the correlation length

$$\boxed{\xi = -\frac{1}{\ln \tanh J_\tau} = \frac{1}{2\delta\tau\Gamma}}$$ - "lattice units"

$$\Rightarrow \frac{1}{\xi\delta\tau} = \underbrace{2\Gamma}_{\text{gap in the quantum system}}$$

$$\frac{1}{\xi\delta\tau} = \frac{1}{\tau_{corr.}}$$

This is an explicit example of a general result: Finite gap ΔE in the quantum system corresponds to finite correlation length in the temporal direction

$$\boxed{\tau_{corr} = \frac{1}{\Delta E}}$$

Euclidean path integral for quantum XY rotor model

$$\boxed{\hat{H} = \frac{u}{2}\sum_i \hat{n}_i^2 - J\sum_{\langle ij\rangle} \cos(\varphi_i - \varphi_j)}$$

$\phi_i \in (0, 2\pi)$

$\hat{n}_i = -i\frac{\partial}{\partial\phi_i}$

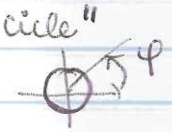
* can model array of Josephson-coupled superconducting islands ($\frac{u}{2}\hat{n}_i^2 \leftarrow$ charging energy

$-J\cos(\varphi_i - \varphi_j) \leftarrow$ Josephson coupling)

[model of bosons at integer filling]

* can model integer-valued quantum spin system

"QM of particle residing on a unit circle"



Single rotor

$\hat{\phi} \in (0, 2\pi)$ — $U(1)$ phase

$\hat{n}$ — conjugate "boson number"

$$\boxed{[\hat{\phi}, \hat{n}] = i}$$

$$\boxed{\hat{n} = -i\frac{\partial}{\partial\phi}}$$

(like momentum $\hat{p} = -i\frac{\partial}{\partial x}$, but ϕ is required to be 2π-periodic).

$|p\rangle \Leftrightarrow e^{ipx}$ in Schrodinger wavefnctn representation

$\hat{p}\,(e^{ipx}) = p\,(e^{ipx})$ $\qquad \psi(x) = e^{ipx} = \langle x|p\rangle$

$|n\rangle \leftrightarrow e^{in\varphi} = \psi(\varphi) = \langle\phi|n\rangle$ — for this to be single-valued, require $\boxed{n = \text{integer-valued}}$

$n = 0, \pm 1, \ldots$

$\hat{n}\left(e^{i\hat{\phi}}\left(e^{in\phi}\right)\right) = \hat{n}\left(e^{i(n+1)\phi}\right) = (n+1)\,e^{i(n+1)\phi}$,

i.e. $e^{i\hat{\phi}}$ raises $\hat{n}$ by 1

$e^{\pm i\hat{\varphi}}$ – raising/lowering operator on $\hat{n}$.

$\leftarrow$ can also be directly verified from the commutation relations

Single rotor $\hat{H} = \frac{u}{2}\hat{n}^2$ - spectrum $E_n = \frac{u}{2} n^2$

$\underbrace{E_0 = \phi}_{\uparrow \text{ ground state}}, \; E_{\pm 1} = \frac{u}{2}, \ldots$

ground state.

Path integral representation in the ϕ-variables:

$$\boxed{\langle \phi(\tau+\delta\tau) | e^{-\delta\tau \frac{u}{2}\hat{n}^2} | \phi(\tau) \rangle} = \sum_{n=-\infty}^{+\infty} \langle \phi(\tau+\delta\tau) | e^{-\delta\tau \frac{u}{2}\hat{n}^2} | n \rangle \langle n | \phi(\tau) \rangle$$

$$= \sum_{n=-\infty}^{+\infty} e^{-\delta\tau \frac{u}{2} n^2} e^{in(\phi(\tau+\delta\tau) - \phi(\tau))}$$

$\leftarrow$ clearly, this is 2π-periodic fnctn of $\Delta\phi$

$$= \text{const} \times \sum_{P\text{-integers}} \exp\left[-\frac{1}{2\delta\tau u} \left(\phi(\tau+\delta\tau) - \phi(\tau) - 2\pi p \right)^2 \right] \approx$$

$\uparrow$ Poisson resummation formula

$\leftarrow$ ensures 2π-periodicity

$$\approx \boxed{\text{const.} \exp\left(\frac{1}{\delta\tau u} \cos(\phi(\tau+\delta\tau) - \phi(\tau)) \right)}$$

$\uparrow$ if $\delta\tau \cdot u \ll 1$

Putting everything together, path integral for the d-dim quantum rotor model

$$\boxed{Z \approx \prod_{i,\tau} \int_0^{2\pi} d\phi(i,\tau) \exp\left[J\delta\tau \sum_{\tau,\langle ij \rangle} \cos(\phi(i,\tau) - \phi(j,\tau)) + \frac{1}{\delta\tau u} \sum_{i,\tau} \cos(\phi(i,\tau+\delta\tau) - \phi(i,\tau)) \right]}$$

- $(d+1)$-dim classical XY model with couplings

$$\boxed{\begin{aligned} J_{spat} &= J\delta\tau \\ J_\tau &= \frac{1}{u\delta\tau} \end{aligned}}$$

Remarks

* general remarks made in the quantum Ising case apply here as well

* To discuss qualitative physics, can take reasonable discretization such that

$J_{spat} = J_\tau = K$ – effective inverse temperature of space-time-isotropic classical XY model

$J\delta\tau = \frac{1}{u\delta\tau}$

$\Rightarrow \delta\tau = \frac{1}{\sqrt{uJ}}$

$$\boxed{K \approx \sqrt{\frac{J}{u}}}$$

Quantum model		(d+1)-dim Classical model
$u \ll J$ ordered quantum phase	$\longleftrightarrow$	K large, i.e., low effective temperature Ordered symmetry-broken phase (assuming $d+1 > 2 \Leftrightarrow d > 1$, or QLRO in $d+1 = 2 \Leftrightarrow d=1$)
$u \gg J$ quantum disordered phase (Mott insulator of bosons)	$\longleftrightarrow$	K small, i.e., high effective temperature; disordered phase

* 0-dim quantum problem – gap $\Delta E = \frac{u}{2}$

$\Leftrightarrow$ (0+1) dim classical XY model with $J_\tau = \frac{1}{u\delta\tau}$ – disordered for all $J_\tau > 0$;

transfer matrix solution $\Rightarrow$ correlation length

$$\xi = 2J_\tau, \quad \underbrace{\xi\delta\tau}_{\tau_{corr}} = \frac{2}{u} = \frac{1}{\Delta E}$$

POISSON RESUMMATION FORMULA

Gaussian integrals:

$$\boxed{\int_{-\infty}^{+\infty} d\vec{x}\, \exp\left[-\frac{1}{2}\vec{x}^T\hat{A}\vec{x} + i\vec{b}^T\vec{x}\right] = \frac{(2\pi)^{N/2}}{\sqrt{\det\hat{A}}}\exp\left[-\frac{1}{2}\vec{b}^T\hat{A}^{-1}\vec{b}\right]}$$

$\vec{x}$ – N-component real vector

$\hat{A}$ – symmetrix matrix with $\mathrm{Re}(\hat{A})$ – positive definite matrix.

Poisson resummation of Gaussian sums:

$$\boxed{\sum_{\vec{m}=-\infty}^{+\infty} \exp\left[-\frac{1}{2}\vec{m}^T\hat{A}\vec{m} + i\vec{b}^T\vec{m}\right] = \frac{(2\pi)^{N/2}}{\sqrt{\det\hat{A}}}\sum_{\vec{p}\in\mathbb{Z}^N}\exp\left[-\frac{1}{2}(\vec{b}+2\pi\vec{p})^T\hat{A}^{-1}(\vec{b}+2\pi\vec{p})\right]}$$

$\vec{m} = (m_1, m_2, \dots, m_N)$ – N-component integer vector

Proof: $$\underbrace{\sum_{m\in\mathbb{Z}}\delta(x-m)}_{\text{periodic fnctn with period 1}} = \underbrace{\sum_{p=-\infty}^{+\infty} e^{i2\pi x p}}_{\text{Fourier representation}}$$

$$\sum_{m\in\mathbb{Z}} f(m) = \int_{-\infty}^{+\infty} dx\, f(x)\sum_{m\in\mathbb{Z}}\delta(x-m) = \int_{-\infty}^{+\infty} dx\, f(x)\sum_{p\in\mathbb{Z}} e^{i2\pi p x}$$

$$\Rightarrow \sum_{\vec{m}\in\mathbb{Z}^N}\exp\left[-\frac{1}{2}\vec{m}^T\hat{A}\vec{m} + i\vec{b}^T\vec{m}\right] = \int_{-\infty}^{+\infty} d\vec{x}\sum_{\vec{p}\in\mathbb{Z}^N}$$

$$\exp\left[-\frac{1}{2}\vec{x}^T\hat{A}\vec{x} + i\vec{b}^T\vec{x} + i2\pi\vec{p}^T\vec{x}\right] = \text{rhs.}\ \blacktriangle$$

One-component:

$$\boxed{\sum_{m=-\infty}^{+\infty} e^{-\frac{1}{2}am^2 + im\theta} = \sqrt{\frac{2\pi}{a}}\sum_{p=-\infty}^{+\infty}\exp\left[-\frac{1}{2a}(\theta+2\pi p)^2\right]}$$

Connection with Feynman path integrals in quantum mechanics

QM particle in potential v: $\hat{H} = \frac{\hat{p}^2}{2m} + v(x)$ $\qquad \hat{x} \in \mathbb{R}$

$\hat{p}$ – momentum, $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$[\hat{p}, \hat{x}] = i\hbar$

Plane wave state $|p\rangle$: $\langle x|p\rangle = e^{i\frac{p}{\hbar}x}$ ← Schrodinger wavefunction $\psi(x)$

(not worrying about normalization, but can do carefully if needed)

Development of the Euclidean path integral:

$$\langle x(\tau+\delta\tau)|\, e^{-\delta\tau\frac{\hat{p}^2}{2m}}\, |x(\tau)\rangle \doteq \int dp\, \langle x(\tau+\delta\tau)|e^{-\delta\tau\frac{\hat{p}^2}{2m}}|p\rangle\langle p|x(\tau)\rangle$$

$$= \int dp\, e^{-\delta\tau\frac{\hat{p}^2}{2m}}\, e^{i\frac{p}{\hbar}(x(\tau+\delta\tau)-x(\tau))} = e^{-\frac{m}{2\delta\tau}\left(\frac{x(\tau+\delta\tau)-x(\tau)}{\hbar}\right)^2}$$

Full path integral:

$$\boxed{Z \approx \int_{-\infty}^{+\infty} \prod_\tau dx(\tau)\, \exp\left\{-\sum_\tau\left[\frac{m}{2}\left(\frac{x(\tau+\delta\tau)-x(\tau)}{\hbar\,\delta\tau}\right)^2 + v(x(\tau))\right]\delta\tau\right\}}$$

$$\boxed{= \int_{-\infty}^{+\infty}\prod_\tau dx(\tau)\, \exp\left\{-\int_0^\beta d\tau\left[\frac{m}{2}\left(\frac{dx}{d(\hbar\tau)}\right)^2 + v(x(\tau))\right]\right\}}$$

$$= -\frac{1}{\hbar}\int_0^{\beta\hbar} du\left[\frac{m}{2}\left(\frac{dx}{du}\right)^2 + v(x(u))\right]$$

can be → used to analyze single-particle QM problems (examples in Feynman, StatMech; particle in a double-well. But the real power is in many-body applications!

Compare with Feynman's path integral in real time:

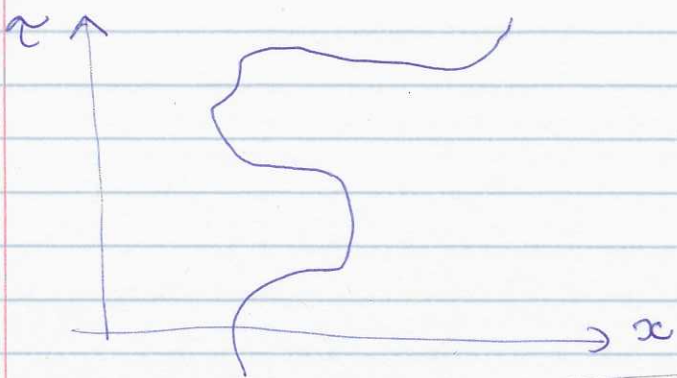
$$\int \mathcal{D}x(t)\, \exp\left\{\frac{i}{\hbar}\int dt\, \underbrace{\mathcal{L}(x(t))}_{=\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - v(x(t))}\right\}.$$

$$\frac{i}{\hbar}\int dt\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - v(x(t))\right] \xrightarrow[t=-iu]{} -\frac{1}{\hbar}\int du\left[\frac{m}{2}\left(\frac{dx}{du}\right)^2 + v(x(u))\right]$$

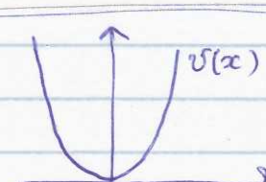
$u = \hbar\tau$ – "imaginary time"!

2 applications:

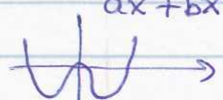
1) Consider $v(x) = 0$ (free particle) : statistical mechanics of line ("polymer") — — "worldline" of a free quantum-mechanical particle



2) Consider confining potential



(or maybe more complex potential like



,)

discrete levels

$\Rightarrow$ we have "0-dimensional" quantum mechanical problem

$x \to \varphi$

$x \to \varsigma$?

$$\frac{p^2}{2m} + a\phi^2 + b\phi^4 .$$

Consider now some d-dim lattice

and imagine having such a "0-dim" q.m. problem at each lattice site:

$$\frac{p_i^2}{2m} + a\phi_i^2 + b\phi_i^4$$
$$\varsigma_i^2 + \varsigma_i^4$$

$$[p_i, \phi_j] = i\hbar\,\delta_{ij}$$

commute on different sites.

Path integral $\Rightarrow$ $S = \sum_i \sum_\tau \Big\{ \#\big(\phi_i(\tau+\delta\tau) - \phi_i(\tau)\big)^2 + a\delta\tau\,\phi_i^2 + b\delta\tau\,\phi_i^4 \Big\}$.

If the original problem also has interaction

$$H_{int} = +J \sum_{\langle ij \rangle} (\phi_i - \phi_j)^2 \quad \text{— "coupled oscillators"}$$

$$\Rightarrow \delta S_{int} = \sum_\tau + J\delta\tau \sum_{\langle ij \rangle} (\phi_i - \phi_j)$$

$\Rightarrow$ S_{tot} is lattice regularization of the ϕ^4 theory in $d+1$.