Landau – Ginzburg – Wilson theory

Perturbative RG treatment and Wilson – Fisher fixed point

$$\boxed{\begin{aligned} S &= S_0 + \delta S \\ S_0 &= \int d^d r \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2} m^2\right] = \int đ\vec{q}\ \frac{1}{2}(Kq^2 + t)|m_q|^2 \\ \delta S &= \int d^d r\ u\, m(r)^4 = u \int đ\vec{q}_1\, đ\vec{q}_2\, đ\vec{q}_3\, đ\vec{q}_4\ m_{q_1} m_{q_2} m_{q_3} m_{q_4} \times \\ &\quad \times \delta^{(d)}(q_1+q_2+q_3+q_4) \end{aligned}}$$

Momentum shell RG :

$$m(r) = \int^{\Lambda} đ\vec{q}\, e^{i\vec{q}\cdot\vec{r}}\, m(\vec{q}) = \int^{\Lambda/b} đ\vec{q}\, e^{i\vec{q}\cdot\vec{r}} \underbrace{m(\vec{q})}_{\text{denote } m^< \text{ – "slow modes"}} + \int_{\Lambda/b}^{\Lambda} đq\, e^{i\vec{q}\cdot\vec{r}} \underbrace{m^>}_{\text{"fast modes"}}$$

$$= m^<(r) + m^>(r)$$

- $S_0 = S_0[m^<] + S_0[m^>]$ – fast and slow modes are not coupled in the Gaussian theory

- $\delta S = \int d^d r\ u\left(m^<(r)^4 + 4(m^<)^3 m^> + 6(m^<)^2(m^>)^2 + 4m^<(m^>)^3 + (m^>)^4\right) = u\Big[\int đ\vec{q}_1^{\,<}\, đ\vec{q}_2^{\,<}\, đ\vec{q}_3^{\,<}\, đ\vec{q}_4^{\,<}\ m^<_{q_1} m^<_{q_2} m^<_{q_3} m^<_{q_4} \cdot \delta(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)$

slow and fast modes are coupled!

~~+ ... – fast and slow modes do~~

$$+ \int đ\vec{q}_1^{\,<}\, đ\vec{q}_2^{\,<}\, đ\vec{q}_3^{\,<}\, đ\vec{q}_4^{\,>}\ 4 m^<_{q_1} m^<_{q_2} m^<_{q_3} m^>_{q_4}\, \delta(q_1+q_2+q_3+q_4)$$

$$+ \int đq_1^<\, đq_2^<\, đq_3^>\, đq_4^>\ 6\, m^<_{q_1} m^<_{q_2} m^>_{q_3} m^>_{q_4}\, \delta(q_1+q_2+q_3+q_4)$$

$$+ \ldots \Big]$$

$$Z = \int \mathcal{D}m^< \mathcal{D}m^> \, e^{-S[m^<, m^>]} = \int \mathcal{D}m^< \, e^{-S_0[m^<]} \times$$

$$\times \int \mathcal{D}m^> \, e^{-S_0[m^>] - \delta S[m^<, m^>]} \times \frac{\int \mathcal{D}m^> e^{-S_0[m^>]}}{\int \mathcal{D}m^> e^{-S_0[m^>]}}$$

$$\cancel{e^{-\delta S_{eff}[m^<]}} \quad \cancel{e^{-\delta S}} \qquad e^{-\delta f \cdot Vol}$$

$$e^{-\delta S_{eff}[m^<]} \equiv \frac{\int \mathcal{D}m^> \, e^{-S_0[m^>] - \delta S[m^<, m^>]}}{\int \mathcal{D}m^> \, e^{-S_0[m^>]}} = \left\langle e^{-\delta S} \right\rangle_{0,>} \approx$$

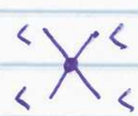
$$\approx \left\langle \left(1 - \delta S + \frac{\delta S^2}{2} + \ldots \right) \right\rangle_{0,>} = 1 - \langle \delta S \rangle_{0,>} + \frac{1}{2} \langle \delta S^2 \rangle_{0,>} + \ldots$$

$$\approx \exp\left(- \langle \delta S \rangle_{0,>} + \frac{1}{2} \left[\langle \delta S^2 \rangle_{0,>} - \left(\langle \delta S \rangle_{0,>} \right)^2 \right] \right)$$

$$\boxed{\delta S_{eff} = \langle \delta S \rangle_{0,>} - \frac{1}{2} \left[\langle \delta S^2 \rangle_{0,>} - \left(\langle \delta S \rangle_{0,>} \right)^2 \right]}$$

✱ Leading order in u:

$$\delta S^{(1)}_{eff} = \langle \delta S \rangle_{0,>} = \underbrace{\int d\vec{r} \, u (m^<)^4}_{\text{no fast modes; "tree level"}} + \underbrace{6u \int d\vec{r} \, (m^<(r))^2 \langle m^>(r)^2 \rangle_{0,>}}_{\text{renormalizes quadratic term}}$$



$$+ \underbrace{\int dr \, u \langle (m^>)^4 \rangle_{0,>}}_{\text{contribution to } \delta f; \text{ does not affect coupling constants.}}$$

effectively pushes "t" up, and requires to go to lower T to actually undergo the transition (fluctuations suppress the order)

$$\frac{\delta(q+q')}{Kq^2+t}$$

$$\|$$

$$\boxed{\langle m^>(r)^2\rangle_{0,>}} = \int đq^> e^{i\vec{q}\cdot\vec{r}} \int đq'^> e^{i\vec{q}'\cdot\vec{r}} \langle m^>(q)\, m^>(q')\rangle_{0,>} =$$

area unit sphere

$$= \int đq^> \frac{1}{Kq^2+t} = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{Kq^2+t} = \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{q^{d-1}}{Kq^2+t}\, dq$$

$$\approx \frac{S_d}{(2\pi)^d} \frac{\Lambda^{d-1}}{K\Lambda^2+t} \cdot \left(\Lambda - \frac{\Lambda}{b}\right) = \boxed{\frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{K\Lambda^2+t}\, \delta l}$$

infinitesimal $b \approx e^{\delta l} = 1+\delta l$

Thus, to leading order in u:

(since we habe $\frac{t}{2}m^2$)

$$t'(\text{before rescaling}) = t + 6u \cdot \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{q^{d-1}dq}{Kq^2+t} \times 2 =$$

$$= t + 6u \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{K\Lambda^2+t} \delta l \times 2$$

$$u'^{\text{ before rescaling}} = u$$

$$t'^{\text{ after rescaling}} = b^2\left[t + 12u \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{q^{d-1}dq}{Kq^2+t}\right] \approx$$

$$\approx (1+2\delta l)\left(t + 12u \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{K\Lambda^2+t} \delta l\right) =$$

$$= t + \delta l \cdot \left(2t + 12u \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{K\Lambda^2+t}\right)$$

$$u'^{\text{ after rescaling}} = b^{4-d} u \approx (1+(4-d)\cdot\delta l)\cdot u = u + (4-d)u\,\delta l$$

Infinitesimal flow equations:

$$\boxed{\begin{aligned} \frac{dt}{dl} &= 2t + 12u \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{K\Lambda^2+t} \\ \frac{du}{dl} &= (4-d)u \end{aligned}}$$

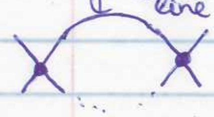
— flow away from the Gaussian fixed point if $d<4$

⊛ Quadratic order in u:

Need $\boxed{\langle \delta S^2 \rangle_{0,>} - \left(\langle \delta S \rangle_{0,>}\right)^2} = \Big\langle \Big(\underset{\substack{\alpha \\ \uparrow \\ \text{"pieces" of } \delta S}}{\sum} \delta S_\alpha\Big)^2 \Big\rangle_{0,>} - \Big\langle \sum_\alpha \delta S_\alpha \Big\rangle^2_{0,>}$

$$= \sum_{\alpha,\beta} \underbrace{\Big[\langle \delta S_\alpha \, \delta S_\beta \rangle_{0,>} - \langle \delta S_\alpha \rangle_{0,>} \langle \delta S_\beta \rangle_{0,>} \Big]}_{\substack{\text{III} \\ \langle \delta S_\alpha \, \delta S_\beta \rangle_{0,>,\text{connected}}}}$$

at least one connecting line



$$\boxed{= \sum_\alpha \langle \delta S_\alpha^2 \rangle_{>,\text{conn}} + 2 \sum_{\alpha<\beta} \langle \delta S_\alpha \, \delta S_\beta \rangle_{>,\text{conn}} .}$$

$\to \delta S = \delta S_{4<,0>} + \delta S_{3<,1>} + \delta S_{2<,2>} + \delta S_{1<,3>} + \delta S_{0<,4>}$

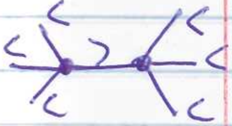
Can analyze all terms one by one

- $\underbrace{\langle \delta S_{4<,0>} \, \underset{\substack{\uparrow \\ \text{any}}}{\delta S_\beta} \rangle_{0,>}}_{\langle \delta S_{4<,0>} \rangle \langle \delta S_\beta \rangle_>} - \langle \delta S_{4<,0>} \rangle_{0,>} \langle \delta S_\beta \rangle_{0,>} = 0$

- $\langle \delta S_{3<,1>} \cdot \delta S_{3<,1>} \rangle_{>,\text{conn}}$ – 6 slow modes; generates v_6 term

Explicitly: $u^2 \int d\!\!{}^-q_1^< \, d\!\!{}^-q_2^< \, d\!\!{}^-q_3^< \, d\!\!{}^-q_4^> \, \delta(q_1+q_2+q_3+q_4)$

$\int d\!\!{}^-q_1'^< \, d\!\!{}^-q_2'^< \, d\!\!{}^-q_3'^< \, d\!\!{}^-q_4'^> \, \delta(q_1'+q_2'+q_3'+q_4')$



$m^<_{q_1} m^<_{q_2} m^<_{q_3} m^<_{q_1'} m^<_{q_2'} m^<_{q_3'} \underbrace{\langle m^>_{q_4} m^>_{q_4'} \rangle}_{\frac{\delta(q_4+q_4')}{K\Lambda^2+t}}$

upon doing q_4 & q_4' integrals
$\downarrow$

$= u^2 \int d\!\!{}^-q_1^< \, d\!\!{}^-q_2^< \, d\!\!{}^-q_3^< \; \delta(q_1+q_2+q_3+q_1'+q_2'+q_3')$

$d\!\!{}^-q_1'^< \, d\!\!{}^-q_2'^< \, d\!\!{}^-q_3'^< \; m^<_{q_1} m^<_{q_2} m^<_{q_3} m^<_{q_1'} m^<_{q_2'} m^<_{q_3'}$

- $\left\langle \delta S_{3<,1>} \; \delta S_{1<,3>} \right\rangle_{>,\text{conn}} =$

$$= u^2 \int \bar{d}q_1^< \, \bar{d}q_2^< \, \bar{d}q_3^< \, \bar{d}q_4^> \, \delta(q_1+q_2+q_3+q_4)$$
$$\int \bar{d}q_1'^< \, \bar{d}q_2'^> \, \bar{d}q_3'^> \, \bar{d}q_4'^> \, \delta(q_1'+q_2'+q_3'+q_4')$$
$$m_{q_1}^< m_{q_2}^< m_{q_3}^< \cdot m_{q_1'}^< \underbrace{\left\langle m_{q_4}^> \, m_{q_2'}^> \, m_{q_3'}^> \, m_{q_4'}^> \right\rangle_>}_{\text{will force, say } q_3'+q_4'=0}$$

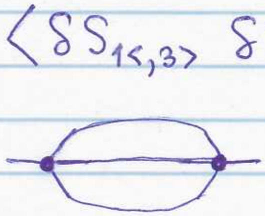
$$\Rightarrow \underset{\substack{\uparrow \\ \text{fast}}}{q_2'} = -\underset{\substack{\uparrow \\ \text{slow}}}{q_1'}$$

cannot equal!

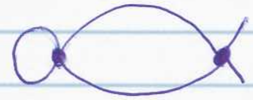
$$= \emptyset$$

The above covers all possible terms involving $\delta S_{3<,1>}$

- $\left\langle \delta S_{1<,3>} \; \delta S_{1<,3>} \right\rangle_{>,\text{conn}}$ – 2 slow modes | provides $O(u^2)$ contribution to t, while we already had $O(u)$ contrib, so ignore.



- $\left\langle \left(\delta S_{0<,4>}\right)^2 \right\rangle_{>,\text{conn}}$ – just a number
- $\left\langle \delta S_{0<,4>} \; \delta S_{2<,2>} \right\rangle_{>,\text{conn}}$ – 2 slow modes | – $O(u^2)$ contrib to t



The only remaining term is $\left\langle \delta S_{2<,2>} \; \delta S_{2<,2>} \right\rangle_{>,\text{conn}}$

$$\boxed{\left\langle \delta S_{2<,2>}\, \delta S_{2<,2>} \right\rangle_{>,\text{conn.}}} = u^2 \cdot 36 \cdot \int đq_1^< đq_2^< \, đq_3^> đq_4^> \int đq_1'^< đq_2'^< \, đq_3'^> đq_4'^>$$

$$\delta(q_1+q_2+q_3+q_4) \quad \delta(q_1'+q_2'+q_3'+q_4')$$

$$m_{q_1}^< m_{q_2}^< \, m_{q_1'}^< m_{q_2'}^< \left\langle m_{q_3}^> m_{q_4}^> \, m_{q_3'}^> m_{q_4'}^> \right\rangle_{>,\text{conn}} . \implies$$

Wick's theorem $\rightarrow \; \parallel$

$$\langle m_{q_3} m_{q_4} \rangle_> \langle m_{q_3'} m_{q_4'} \rangle_> + \langle m_{q_3} m_{q_3'} \rangle_> \langle m_{q_4} m_{q_4'} \rangle_> + \langle m_{q_3} m_{q_4'} \rangle_> \langle m_{q_4} m_{q_3'} \rangle_>$$

$$- \langle m_{q_3} m_{q_4} \rangle_> \langle m_{q_3'} m_{q_4'} \rangle_> \longrightarrow$$

$$\longrightarrow \; 2 \langle m_{q_3} m_{q_3'} \rangle_{0,>} \langle m_{q_4} m_{q_4'} \rangle_{0,>} = 2 \, \frac{\delta(q_3+q_3')}{Kq_3^2+t} \, \frac{\delta(q_4+q_4')}{Kq_4^2+t} .$$

$$\implies \boxed{72 u^2 \int đq_1^< đq_2^< đq_1'^< đq_2'^< \, m_{q_1}^< m_{q_2}^< m_{q_1'}^< m_{q_2'}^< \, \delta(q_1+q_2+q_1'+q_2') \int đq_3^> \, \Theta(q_4 = -q_1-q_2-q_3 \text{ fast}) \times \frac{1}{Kq_3^2+t} \times \frac{1}{Kq_4^2+t}}$$

$u(q_1, q_2, q_1', q_2')$

q_1 & q_2 are remaining "long-wavelength" momenta and eventually we are interested in the small $q_1, q_2, \ldots$ limit. Can imagine expanding around $q_1 = q_2 = 0$ and focusing on the dominant term (the subdominant ones $\sim \nabla m \nabla m \cdot m \cdot m$ irrelevant by power counting).

$$\longrightarrow \int đq_3^> \frac{1}{(Kq_3^2+t)^2} = \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{(K\Lambda^2+t)^2} \delta\ell$$

set $q_1 = q_2 = 0$

Contribution to u' before rescaling:

$$\delta u'_{\substack{\text{before}\\ \text{rescaling}}} = -\frac{1}{2}\cdot 72u^2 \cdot \int_{q>} \frac{1}{(Kq^2+t)^2} = -36u^2 \cdot \frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{(K\Lambda^2+t)^2}\delta l$$

$$u'_{\text{after rescaling}} = \underbrace{b^{4-d}}_{1+(4-d)\delta l}\left(u + (\;)\right) \approx u + \left[(4-d)u - 36u^2\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{(K\Lambda^2+t)^2}\right] \times \delta l .$$

Infinitesimal flow equations

$$\boxed{\begin{aligned}\frac{dt}{dl} &= 2t + 12u\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{K\Lambda^2+t} - O(u^2)\\ \frac{du}{dl} &= (4-d)u - 36u^2\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{(K\Lambda^2+t)^2}\end{aligned}}$$

Introduce $\bar{t} = \dfrac{t}{K\Lambda^2}$, $\quad \bar{u} = \dfrac{u\Lambda^{d-4}}{K^2}$

$$\cancel{K\Lambda^2}\frac{d\bar{t}}{dl} = 2\cancel{K\Lambda^2}\bar{t} + 12\frac{\cancel{K^2}}{\cancel{\Lambda^{d-4}}}\bar{u}\cdot\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\cancel{K\Lambda^2}(1+\bar{t})}$$

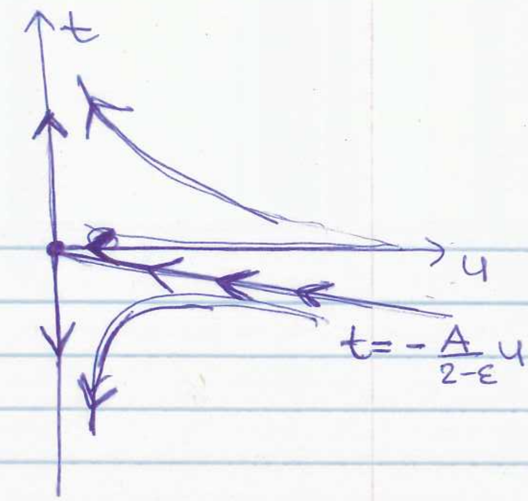
$$\frac{d\bar{u}}{dl} = (4-d)\bar{u} - 36\bar{u}^2\frac{K^2}{\Lambda^{d-4}}\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{K^2\Lambda^4(1+\bar{t})^2}$$

(*) $\Rightarrow$
$$\boxed{\begin{aligned}\frac{d\bar{t}}{dl} &= 2\bar{t} + 12\frac{S_d}{(2\pi)^d}\frac{\bar{u}}{1+\bar{t}}\\ \frac{d\bar{u}}{dl} &= \varepsilon\bar{u} - 36\frac{S_d}{(2\pi)^d}\frac{\bar{u}^2}{(1+\bar{t})^2}\end{aligned}}$$

$\varepsilon = 4-d$

Flows:

$d > 4 \Rightarrow \varepsilon < 0$



$\bar{t}^* = \bar{u}^* = 0$
fixed pt.

$$\begin{pmatrix} \frac{d\bar{t}}{dl} \\ \frac{d\bar{u}}{dl} \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & A \\ 0 & \varepsilon \end{pmatrix}}\begin{pmatrix} \bar{t} \\ \bar{u} \end{pmatrix}$$

Eigenvalues 2 & $\varepsilon < 0$: 1 relevant direction and 1 irrelevant

Eigenstates : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{A}{2-\varepsilon} \\ 1 \end{pmatrix}$

$$\begin{pmatrix} \bar{t} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{A}{2-\varepsilon} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tau \\ \vartheta \end{pmatrix} \quad \Bigg\| \quad \begin{aligned} \tau &= \bar{t} + \frac{A}{2-\varepsilon}\bar{u} \\ \vartheta &= \bar{u} \end{aligned}$$

$$\begin{pmatrix} \frac{d\bar{t}}{dl} \\ \frac{d\bar{u}}{dl} \end{pmatrix} = \left[2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varepsilon\begin{pmatrix} -\frac{A}{2-\varepsilon} \\ 1 \end{pmatrix} \right]\begin{pmatrix} \tau \\ \vartheta \end{pmatrix}$$

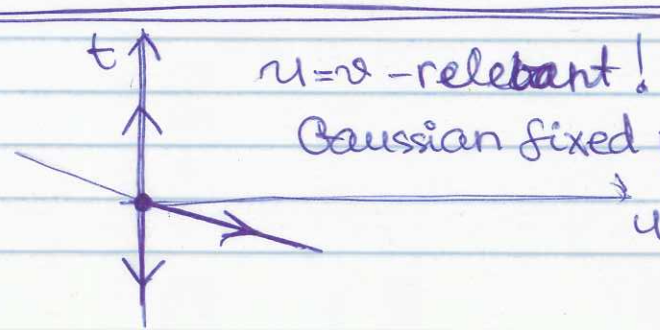
direct check

$$\frac{d\tau}{dl} = \frac{d\bar{t}}{dl} + \frac{A}{2-\varepsilon}\frac{d\bar{u}}{dl}$$
$$= 2\bar{t} + A\bar{u} + \frac{A}{2-\varepsilon}\cdot\varepsilon\bar{u}$$
$$= 2\left(\bar{t} + \frac{A}{2-\varepsilon}\bar{u}\right)$$
$$= 2\tau$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{d\tau}{dl} + \begin{pmatrix} -\frac{A}{2-\varepsilon} \\ 1 \end{pmatrix}\frac{d\vartheta}{dl} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot\tau + \varepsilon\begin{pmatrix} -\frac{A}{2-\varepsilon} \\ 1 \end{pmatrix}\vartheta$$

$$\left.\begin{aligned} \frac{d\tau}{dl} &= 2\tau \\ \frac{d\vartheta}{dl} &= \varepsilon\vartheta \text{ - irrelevant} \end{aligned}\right] \Rightarrow \text{scaling forms with meanfield exponents.}$$

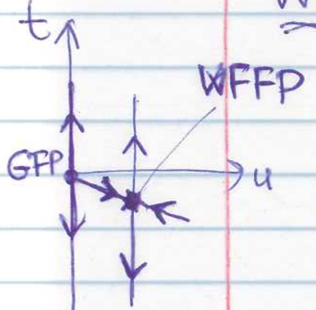
$d < 4 \Rightarrow \varepsilon > 0$



$u = \vartheta$ - relevant!
Gaussian fixed point is unstable!

$\underline{d<4 \Rightarrow \varepsilon>0}$ $\quad t^*=u^*=0$ fixed point now has two relevant directions — Gaussian fixed point is unstable and is not describing generic 2nd-order transition

Fortunately, there is a stable fixed point nearby still accessible to perturbative treatment if ε is small!



Wilson–Fisher fixed point:

$$\boxed{\bar{u}^* = \frac{\varepsilon\cdot(1+\bar{t}^*)^2}{36\, S_d/(2\pi)^d}} = \frac{\varepsilon\left(1-\frac{\varepsilon}{6+\varepsilon}\right)^2}{36\, S_d/(2\pi)^d}$$

$$2\bar{t}^* + 12\frac{S_d}{(2\pi)^d}\frac{1}{1+\bar{t}^*}\cdot\frac{\varepsilon(1+\bar{t}^*)^2}{36\, S_d/(2\pi)^d} = 0$$

$$2\bar{t}^* + \frac{\varepsilon}{3}(1+\bar{t}^*) = 0 \Rightarrow \boxed{\bar{t}^* = -\frac{\varepsilon}{6+\varepsilon}}$$

Linearize the RG equation near t^*, u^*:

$$\frac{d\delta t}{dl} = 2\delta t + 12\frac{S_d}{(2\pi)^d}\left(\frac{\delta u}{1+t^*} - \frac{u^*}{(1+t^*)^2}\delta t\right) =$$

$$= \Big(2 - \underbrace{12\frac{S_d}{(2\pi)^d}\frac{u^*}{(1+t^*)^2}}_{=\frac{\varepsilon}{3}}\Big)\delta t + 12\frac{S_d}{(2\pi)^d}\frac{1}{1+t^*}\delta u$$

$$\frac{d\delta u}{dl} = \varepsilon\,\delta u - 36\frac{S_d}{(2\pi)^d}\left(\frac{2u^*\delta u}{(1+t^*)^2} - \frac{u^{*2}\cdot 2\delta t}{(1+t^*)^3}\right) =$$

$$= -\varepsilon\,\delta u + \underbrace{\left(36\frac{S_d}{(2\pi)^d}\frac{u^{*2}\cdot 2}{(1+t^*)^3}\right)}_{=\ \varepsilon\cdot\frac{2u^*}{1+t^*} = \frac{\varepsilon^2(1+t^*)}{18\, S_d/(2\pi)^d}}\delta t$$

$$\boxed{\begin{aligned} \frac{d\delta t}{dl} &= \left(2-\frac{\varepsilon}{3}\right)\delta t + 12\frac{S_d}{(2\pi)^d}\frac{1}{1+t^*}\delta u \\ \frac{d\delta u}{dl} &= \frac{\varepsilon^2(1+t^*)}{18\frac{S_d}{(2\pi)^d}}\delta t - \varepsilon\delta u \end{aligned}}$$

Diagonalizing the 2×2 matrix $\Rightarrow$ relevant exponent $y_t^{>0}$
and first irrelevent $y_{u} < 0$

"ε expansion"

So far we have analyzed the flow equations (⊗) exactly and for such flows will have exact characterization of the new fixed point (t^*, u^*) from .
However, (⊗) were derived working perturbatively in u and the treatment is "controllable" if u^* is (self-consistent) small. If we want to see just qualitative picture of the appearance of new stable fixed point and get crude estimate of new exponents, the above is in principle sufficient. On the other hand, if we want to be systematic (e.g. have controllable way of estimating ~~[illegible]~~ further corrections), we can proceed in formal "ε expansion" treating $\varepsilon = 4-d$ as small.

To leading order in ε

$$\begin{pmatrix} \frac{d\delta t}{dl} \\ \\ \frac{d\delta u}{dl} \end{pmatrix} = \begin{pmatrix} 2-\frac{\varepsilon}{3} & 12\frac{S_d}{(2\pi)^d}\frac{1}{1+t^*} \\ \\ O(\varepsilon^2) & -\varepsilon \end{pmatrix} \begin{pmatrix} \delta t \\ \\ \delta u \end{pmatrix}$$

$$\begin{pmatrix} 2-\frac{\varepsilon}{3} & * \\ O(\varepsilon^2) & -\varepsilon \end{pmatrix}$$ ← not important (only determines "direction")

$$\Rightarrow \quad y_t = 2 - \frac{\varepsilon}{3} \quad \Rightarrow \quad \boxed{\nu = \frac{1}{y_t} = \frac{1}{2(1-\frac{\varepsilon}{6})} \approx \frac{1}{2} + \frac{\varepsilon}{12}}$$

$$y_u = -\varepsilon$$

$$\boxed{\begin{aligned} \alpha = 2 - d\nu &= 2-(4-\varepsilon)\nu \approx \\ &\approx 2 - 4\left(\tfrac{1}{2}+\tfrac{\varepsilon}{12}\right) + \varepsilon\cdot\tfrac{1}{2} \approx \frac{\varepsilon}{6} \end{aligned}}$$

Example

$d=3$, $\varepsilon=1$ $\Rightarrow$ $\nu \approx 0.5833\ldots$

$\alpha \approx 0.166\ldots$

Monte Carlo estimate

$\nu_{3D\ \text{Ising}} = 0.63$

$\alpha_{3D\ \text{Ising}} = 0.11$

Other critical exponents: can fudge a bit by noting that we did not need to rescale the fields at this level of analysis (no $O(u)$ contribution to K from Kq^2)

$\Rightarrow$ $\eta = 0$ to leading order in ε

Then use $\boxed{\gamma = (2-\eta)\nu \approx 1 + \frac{\varepsilon}{6}}$

Rushbrooke's identity

$$\alpha + 2\beta + \gamma = 2 \quad \Rightarrow \quad \boxed{\beta = \frac{1}{2}\left(1 - \frac{\varepsilon}{3}\right) = \frac{1}{2} - \frac{\varepsilon}{6}}$$

Widom's identity

$$\delta = 1 + \frac{\gamma}{\beta} \quad \Rightarrow \quad \boxed{\delta = 1 + \frac{1+\frac{\varepsilon}{6}}{\frac{1}{2}-\frac{\varepsilon}{6}} \approx 1 + 2\left(1+\frac{\varepsilon}{6}\right)\left(1+\frac{\varepsilon}{3}\right) \approx 3 + \varepsilon}$$

$$y_h = \frac{d+2}{2} + O(\varepsilon^2)$$

$$\gamma = \frac{(2y_h - d)}{y_t} = (2y_h-d)\nu \approx 2\nu \approx 1 + \frac{\varepsilon}{6}$$