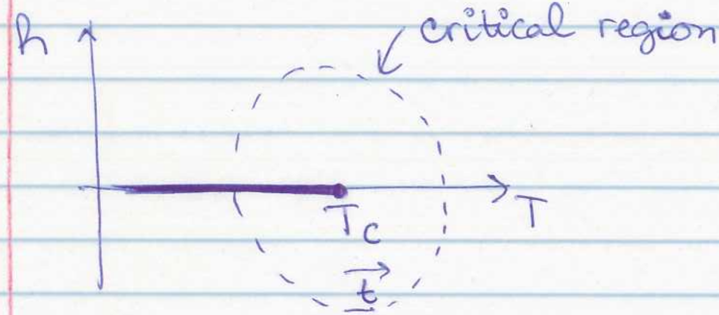
Scaling hypothesis

Scaling hypothesis allows to relate all exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ to two basic exponents.

Introduced empirically by Widom and justified by the phenomenological idea that a single divergent correlation length determines the behavior near T_c.



$$\frac{t}{2} m^2 + u m^4 - h m$$

Mean field:

non-analytic along the cut $h=0, t<0$

$$f(t,h) = \min_m \left[\frac{t}{2} m^2 + u m^4 - h m \right]; \qquad t m + 4 u m^3 - h = 0$$

analytically can calculate

$h=0$: $t>0 \Rightarrow m=0, \; f=0$

$t<0 \Rightarrow \boxed{m = \sqrt{\frac{|t|}{4u}}}; \quad \boxed{f = -\frac{|t|^2}{16u} = f(t<0, h=0)}$

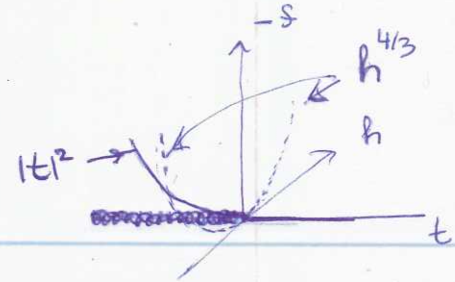
$t=0, \; h \neq 0$: $\Rightarrow \boxed{m = \left(\frac{h}{4u}\right)^{1/3}}; \quad f = \frac{u\, h^{4/3}}{4^{4/3} u^{4/3}} - \frac{h \cdot h^{1/3}}{(4u)^{1/3}} =$

$$\boxed{= -\frac{3}{4 \cdot 4^{1/3}} \frac{h^{4/3}}{u^{1/3}} = f(t=0, h)}$$

$$\boxed{-\frac{\partial f_{mf}}{\partial h} = -\left[\left(t m + 4 u m^3 - h \right) \cdot \frac{dm}{dh} - m \right] = m(h)} \checkmark$$

Omit this and go straight to magnetization

Can unify these forms by



$$f(t,h) = |t|^2 \, g_{s\pm}\left(h/|t|^{3/2}\right),$$

with

$h \ll |t|^{3/2}$

$$g_{s-}\left(x = \frac{h}{|t|^{3/2}} \to 0\right) = -\frac{1}{164}$$

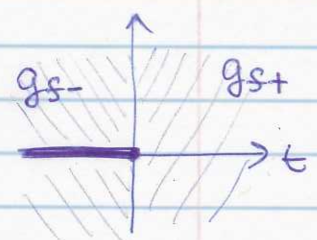
$$g_{s+}(x \to 0) = 0$$

$h \gg |t|^{3/2}$

$$\underbrace{g_{s+}(x \to \infty)}_{\substack{\| \\ g_{s-}(x\to\infty)}} = -\frac{3}{4 \cdot 4^{1/3}} \frac{1}{u^{1/3}} \underbrace{x^{4/3}}_{\frac{h^{4/3}}{|t|^2}}$$

Homogeneity assumption

$$\boxed{f(t,h) = |t|^{2-\alpha} g_{s\pm}\left(\frac{h}{|t|^{\Delta}}\right)}$$



g_{s+} and g_{s-} must match analytically at $t=0, h>0$ and $t=0, h<0$

Because of this, there is only one exponent α (i.e. $\boxed{\alpha_+ = \alpha_- \equiv \alpha}$) and one Δ (i.e. $\boxed{\Delta_+ = \Delta_- \equiv \Delta}$)

Indeed, for $h \neq 0$, $f(t,h)$ must be an analytic fnctn. of t:

$$f(t,h) = A(h) + B(h)t + O(t^2)$$

$$g_{s\pm}(x) \sim A_\pm x^{p_\pm} + B_\pm x^{q_\pm} + \ldots$$

$$|t|^{2-\alpha_\pm} g_{s\pm}\left(\frac{h}{|t|^{\Delta_\pm}}\right) \sim A_\pm h^{p_\pm} \underbrace{|t|^{2-\alpha_\pm - p_\pm \Delta_\pm}}_{t^0} + B_\pm h^{q_\pm} \underbrace{|t|^{2-\alpha_\pm - q_\pm \Delta_\pm}}_{|t|^1}$$

Omit this and go straight to $m(t,h)$

$$2-\alpha_\pm = p_\pm \Delta_\pm \; ; \qquad \left.2-\alpha_\pm = q_\pm \Delta_\pm + 1\right] \Rightarrow \begin{array}{l} \Delta_\pm = \dfrac{1}{p_\pm - q_\pm} \\ \alpha_\pm = 2 - p_\pm \Delta_\pm \end{array}$$

$$= A_\pm h^{p_\pm} + \underbrace{B_\pm h^{q_\pm} |t|}_{\mathrm{sign}(t)\, B_\pm \cdot t}$$

By continuity :

$$\left.\begin{array}{l} A_+ = A_- \,, \quad p_+ = p_- = p \\ B_+ = -B_- \,, \quad q_+ = q_- = q \end{array}\right\} \Rightarrow \begin{array}{l} \Delta_+ = \Delta_- \\ \alpha_+ = \alpha_- \end{array}$$

$$\boxed{f_{\text{sing.}}(t,h) = |t|^{2-\alpha} g_{s\pm}\left(\frac{h}{|t|^\Delta}\right)}$$

* $h=0$:

$$f_{\text{sing}}(t,0) = |t|^{2-\alpha} g_{s\pm}(0)$$

$$\boxed{C_{\text{sing}} \sim -\frac{\partial^2 f}{\partial t^2} \sim \sim (\#)_\pm |t|^{-\alpha}}$$

$$\Rightarrow \boxed{\alpha \equiv \text{specific heat exponents}}$$

(and we have shown that $\alpha_+ = \alpha_-$)

* $h \gg |t|^\Delta$

$$g_{s\pm}(x) \approx A_\pm x^p$$

$$f_{\text{sing}}(t=0,h) = |t|^{2-\alpha-p\cdot\Delta} A_\pm h^p \sim h^{\frac{2-\alpha}{\Delta}}$$

should be indep of t,
i.e. should have

$$p = \frac{2-\alpha}{\Delta}$$

$$m_{\text{sing}}(t=0,h) \sim -\frac{\partial f}{\partial h} \sim h^{\frac{2-\alpha}{\Delta}-1} \sim h^{1/\delta}$$

$$\Rightarrow \boxed{\delta = \frac{\Delta}{2-\alpha-\Delta}}$$

Scaling form of the mean field solution

Mean field equation for m:

$$\boxed{tm + 4um^3 - h = 0} \Rightarrow m(t,h) \quad \left(\text{odd function of } h\right)$$

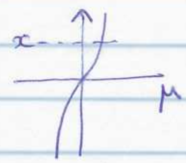
Write
change of vars: $m = \sqrt{\frac{|t|}{4u}}\,\mu$

$$t \cdot \sqrt{\frac{|t|}{4u}}\,\mu + 4u \cdot \frac{|t|}{4u} \cdot \sqrt{\frac{|t|}{4u}}\,\mu^3 - h = 0$$

$$\boxed{\operatorname{sign}(t) \cdot \mu + \mu^3 = \frac{h \cdot \sqrt{4u}}{|t|^{3/2}}}$$

$t>0$

$$\mu + \mu^3 = x \Rightarrow \mu = \mu_+(x), \qquad m(t,h) = \sqrt{\frac{|t|}{4u}}\,\mu_+\!\left(\frac{\sqrt{4u}\,h}{|t|^{3/2}}\right)$$



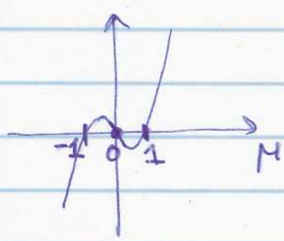
For $x \ll 1$, $\mu_+(x) \approx x$, $m(t,h) \approx \frac{h}{|t|}$

For $x \gg 1$, $\mu_+(x) \approx x^{1/3}$, $m(t,h) \approx \frac{h^{1/3}}{(4u)^{1/3}}$

$t<0$

$$-\mu + \mu^3 = x \Rightarrow \mu = \mu_-(x), \qquad m(t,h) = \sqrt{\frac{|t|}{4u}}\,\mu_-\!\left(\frac{\sqrt{4u}\,h}{|t|^{3/2}}\right)$$

solution with the same sign as x



$\mu_-(x) \approx \operatorname{sign}(x) + \frac{1}{2}x + \ldots$

For $x \ll 1$, $\underset{x>0}{\mu_-(x)} \approx 1 + \frac{x}{2}$, $m(t,h) \approx \bar{m} \times \left(1 + \frac{1}{2}\frac{h \cdot \sqrt{4u}}{|t|^{3/2}}\right) =$

$$-(1+\delta\mu) + (1+\delta\mu)^3 \approx 2\delta\mu = x$$

$$= \bar{m} + \frac{h}{2|t|}$$

For $x \gg 1$, $\mu_-(x) \approx x^{1/3}$, $m(t,h) \approx \frac{h^{1/3}}{(4u)^{1/3}}$

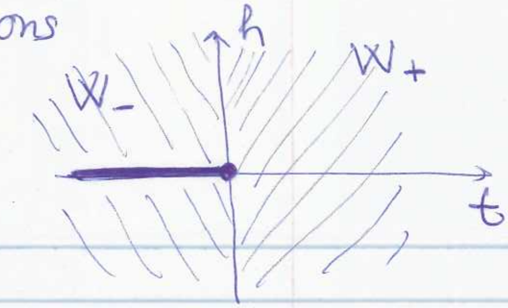
Need to be careful: $t \to 0+$ & $t \to 0-$ at fixed h must give the same m with the chosen form where $x \sim \frac{h}{|t|^{3/2}}$ — will see how this matching occurs

Beyond meanfield including fluctuations

Assume

Generalize → scaling form

$$\boxed{m_{\text{sing}}(t,h) = |t|^{\beta}\, \mathcal{W}_{\pm}\left(\frac{h}{|t|^{\Delta}}\right)}$$



β – magnetiz. exponent

can try $\quad |t|^{\beta_\pm}\, \mathcal{W}_{\pm}\left(\frac{h}{|t|^{\Delta_\pm}}\right)$

$\mathcal{W}_+$ & $\mathcal{W}_-$ match analytically at $t=0$, $h\neq 0$

At nonzero h, $\underline{m(t,h)}$ must be smooth function of t:

$$m(t, h\neq 0) \simeq A(h) + B(h)\,t + O(t^2)$$

$$\mathcal{W}_{\pm}(x \gg 1) \simeq A_{\pm}\, x^{p_\pm} + B_{\pm}\, x^{q_\pm} + \dots$$

leading terms in the asymptotic expansion

$$m(t\to 0, h) = |t|^{\beta_\pm}\mathcal{W}_{\pm}\left(\frac{h}{|t|^{\Delta_\pm}}\right) \approx A_{\pm} \underbrace{|t|^{\beta_\pm - p_\pm \Delta_\pm}}_{\text{lowest power of } t;\ \text{must be } t^0} h^{p_\pm} + B_{\pm} \underbrace{|t|^{\beta_\pm - q_\pm \Delta_\pm}}_{\text{must be } t^1} h^{q_\pm}$$

$$\boxed{\begin{aligned}\beta_\pm - p_\pm \Delta_\pm &= 0\\ \beta_\pm - q_\pm \Delta_\pm &= 1\end{aligned}}$$

$$p_\pm = \frac{\beta_\pm}{\Delta_\pm} \qquad q_\pm = \frac{\beta_\pm - 1}{\Delta_\pm}$$

$$= A_{\pm}\, h^{p_\pm} + \left(\operatorname{sign}(t)\, B_{\pm}\right) h^{q_\pm} \cdot t$$

$A_\pm h^{p_\pm} = A(h) = \text{const} \cdot h^{1/\delta}$

$\operatorname{sign}(t)\, B_\pm = \pm B_\pm$; $\quad (\operatorname{sign}(t)\, B_{\pm})\, h^{q_\pm} = B(h)$

$$\boxed{\delta = \frac{1}{p} = \frac{\Delta}{\beta}}$$

$$\Rightarrow \boxed{\begin{aligned}p_+ &= p_-\\ A_+ &= A_-\end{aligned}} \quad \boxed{\begin{aligned}q_+ &= q_-\\ B_+ &= -B_-\end{aligned}} \Rightarrow \boxed{\begin{aligned}\beta_+ &= \beta_-\\ \Delta_+ &= \Delta_-\end{aligned}}$$

Explicitly for mean field:

$p_+ = p_- = \frac{1}{3} = \frac{1/2}{3/2}$ ✓

$q_+ = q_- = -\frac{1}{3} = \frac{\frac{1}{2}-1}{3/2}$

$B_+ = -B_-$

- $\mu_+ = (x - \mu_+)^{1/3} = x^{1/3}\left(1 - \frac{\mu_+}{x}\right)^{1/3} \approx x^{1/3}\left(1 - \frac{\mu_+}{x}\cdot\frac{1}{3}\right) \simeq$ (large x)

  $\approx x^{1/3} - \frac{1}{3}x^{-1/3}$

  $\mu_- = (x + \mu_-)^{1/3} = x^{1/3}\left(1 + \frac{\mu_-}{x}\right)^{1/3} \approx x^{1/3} + \frac{1}{3}x^{-1/3}$

Susceptibility exponent

$$m(t,h) = |t|^{\beta}\, W_{\pm}\left(\frac{h}{|t|^{\Delta}}\right)$$

At $t>0$ fixed, $m(t,h)$ is an analytic fnctn. of h

$$m(t,h) = \chi(t)\,h + (\#)\,h^3 + \dots$$

$$\Rightarrow W_+(x) \approx b_+ \cdot x + \dots \qquad (\text{mf } \mu_+(x) \approx x \checkmark)$$

$$\boxed{m(t,h) \approx \frac{b_+ \cdot h}{|t|^{\Delta-\beta}}} \Bigg] \Rightarrow \chi(t) \sim \frac{1}{|t|^{\Delta-\beta}} \sim \frac{1}{|t|^{\gamma}}$$

$$\boxed{\gamma_+ \equiv \Delta - \beta}$$

At $t<0$ fixed

$$m(t,h) = \bar{m}\,\mathrm{sign}(h) + \chi_-(t)\,h + \dots$$

$$W_-(x) \approx a_-\,\mathrm{sign}(x) + b_- \cdot x + \dots$$

$$m(t,h) \approx a_- |t|^{\beta} + \underbrace{\left(\frac{b_-}{|t|^{\Delta-\beta}}\right)}_{=\,\chi_-(t)} \cdot h$$

$$\Rightarrow \boxed{\gamma_- = \gamma_+ = \Delta - \beta}$$

Summary: By assuming scaling form

$$m(t,h) = |t|^{\beta_{\pm}}\, W_{\pm}\left(\frac{h}{|t|^{\Delta_{\pm}}}\right)$$

and requiring analyticity across $t=0$ at any $h \neq 0$

$\Rightarrow$ $\beta_+ = \beta_- \equiv \beta$ – identified as magnetization exponent

$\Delta_+ = \Delta_- \equiv \Delta$

meanfield satisfies the scaling form, so satisfies

$\Rightarrow$ $\boxed{\delta = \frac{\Delta}{\beta}}$ – exponent for nonlinear response to h @ T_c.

$\Rightarrow$ $\boxed{\gamma_+ = \gamma_- = \Delta - \beta}$ – susceptibility exponent.

Scaling form for the free energy

Mean field:

$$\boxed{f_{mf}(t,h)} = \frac{t}{2}\frac{|t|}{4u}\mu^2 + u\frac{|t|^2}{16u^2}\mu^4 - h\sqrt{\frac{|t|}{4u}}\,\mu$$

$$= \text{sign}(t)\cdot\frac{|t|^2}{8u}\,\mu^2 + \frac{|t|^2}{16u}\mu^4 - \frac{|t|^2}{4u}\,\frac{h\cdot\sqrt{4u}}{|t|^{3/2}}\,\mu$$

$$= \frac{|t|^2}{16u}\left(\mu^4 + 2\,\text{sign}(t)\,\mu^2 - \frac{h\cdot\sqrt{4u}}{|t|^{3/2}}\,\mu\right)$$

$$= \boxed{\frac{|t|^2}{16u}\left(\mu_\pm^4(x) \pm 2\mu_\pm^2(x) - x\mu_\pm(x)\right),} \qquad x = \frac{h\sqrt{4u}}{|t|^{3/2}}$$

$$= |t|^2\, g_{f\pm}\left(\frac{h}{|t|^{3/2}}\right)$$

General scaling form

$$\boxed{f(t,h) = |t|^{2-\alpha}\, g_{f\pm}\left(\frac{h}{|t|^{\Delta}}\right)}$$

(as before, can argue $\alpha_+ = \alpha_- = \alpha$, $\Delta_+ = \Delta_- = \Delta$)

α is the specific heat exponent $\left(C_{sing} \sim -\frac{\partial^2 f(t,h=0)}{\partial t^2} \sim |t|^{-\alpha}\right)$.

Δ is the same exponent as in the magnetization scaling form:

$$m(t,h) = -\frac{\partial f}{\partial h} = -|t|^{2-\alpha}\, g'_{f\pm}\left(\frac{h}{|t|^\Delta}\right)\cdot\frac{1}{|t|^\Delta} =$$

$$= |t|^{\overbrace{2-\alpha-\Delta}^{\beta}}\cdot\underbrace{(-1)\cdot g'_{f\pm}\left(\frac{h}{|t|^\Delta}\right)}_{\to\, \mathcal{W}_\pm\left(\frac{h}{|t|^\Delta}\right)}$$

$$\Rightarrow \boxed{\beta = 2 - d - \Delta} \quad \Rightarrow \quad \boxed{\begin{aligned} \delta &= \frac{\Delta}{2-d-\Delta} = \frac{\Delta}{\beta} \\ \gamma &= \Delta - \beta = 2\Delta + d - 2 \end{aligned}}$$

These relations are always satisfied.

Summary: Consequences of the scaling form assumption (homogeneous form)

* All bulk exponents can be obtained from two indep., e.g. d & Δ

* Critical indices above and below T_c are the same

* Exponent identities trivially follow from above expressions

$$\alpha + 2\beta + \gamma = d + 2\cdot(2-d-\Delta) + 2\Delta + d - 2 = 2$$

(Rushbrooke's identity)

$$\delta - 1 = \frac{\Delta}{2-d-\Delta} - 1 = \frac{2\Delta + d - 2}{2-d-\Delta} = \frac{\gamma}{\beta}$$

(Widom's identity).

* 3d MC estimates of exponents → satisfy these experimental estimates ⟶ identities

* 2d Ising model

$$\underbrace{\alpha = 0 \quad \beta = \frac{1}{8}} \qquad \gamma = \frac{7}{4} \qquad \delta = 15 \qquad \nu = 1 \qquad \eta = \frac{1}{4}$$

$\Downarrow$

$\Delta = \frac{15}{8}$, $\delta = \frac{\Delta}{\beta} = 15$ ✓

$\gamma = \Delta - \beta = \frac{7}{4}$ ✓

Correlation length &

Scaling form for the correlation function

$$G(r,t,h) = \frac{1}{r^{d-2+\eta}}\, g\left(\frac{r}{\xi}, \frac{h}{|t|^{\Delta}}\right) = \frac{1}{r^{d-2+\eta}}\, \tilde{g}\left(r|t|^{\nu}, h/|t|^{\Delta}\right).$$

$$\xi(t, h=0) \sim |t|^{-\nu}$$

$$\xi(t,h) \sim |t|^{-\nu} X\left(\frac{h}{|t|^{\Delta}}\right)$$

$h = 0$.

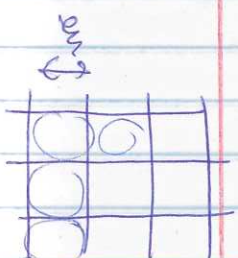
$$\chi(t) = \int G(r,t,h=0)\, d^d r = \int \frac{1}{r^{d-2+\eta}}\, \tilde{g}(r|t|^{\nu}, 0)\, d^d r =$$

$$= \underbrace{|t|^{(\eta-2)\nu}}_{\sim |t|^{-\gamma}} \int \frac{d^d y}{y^{d-2+\eta}}\, \tilde{g}(y, 0)$$

$\boxed{\gamma = (2-\eta)\nu}$ – holds generally.

Hyperscaling: NOT SATISFIED FOR $d>4$
(aka Josephson's relation) (not satisfied by meanfield)

Satisfied for $d<4$: If fluctuations dominate



$$\left. f_{sing} \sim \frac{k_B T_c}{(\xi)^d} \sim |t|^{+\nu d} \quad \stackrel{!}{=} |t|^{2-\alpha} \right\} \Rightarrow \boxed{\nu d = 2-\alpha}$$

if we assume that $\xi(t)$ is the only important length in the problem and is solely responsible for the singular behaviour of the free energy.

$O(1)$ free energy per block of size ξ – will justify in the RG thinking

$d>4$: saddle point
m.f. contrib is $|t|^2 \Theta(t)$
– more singular