1d Ising model

$$\hat{H} = -J \sum_i S_i S_{i+1} - H \sum_i S_i$$

Exact solution

smaller than

$$\approx -J - \frac{1}{\beta}\left((\beta H)^2 + \sqrt{(\beta H)^2 + e^{-4\beta J}} \right)$$

- $f(T, H) = \overbrace{-J - \frac{1}{\beta} \ln\left(\cosh(\beta H) + \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right)}^{\approx -J - \frac{1}{\beta}\left((\beta H)^2 + \sqrt{(\beta H)^2 + e^{-4\beta J}} \right)}$; essential singularity for $T \to 0$

  $f(T, H \to 0) = -J - \frac{1}{\beta} \ln\left(1 + e^{-2\beta J}\right) \approx -J - \frac{e^{-2\beta J}}{\beta}$;
- $\chi(T, H=0) = \frac{\beta}{e^{-4\beta J}} = \beta e^{4\beta J}$ → diverges for $T \to 0$!
- $\xi(T, H=0) = -\frac{1}{\ln \tanh(\beta J)} \approx \frac{e^{2\beta J}}{2}$ → diverges for $T \to 0$.

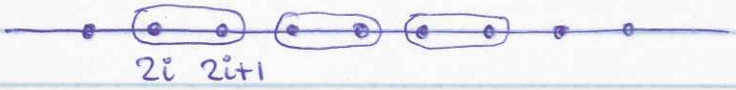
There is no order at any $T \neq 0$, but the properties behave singularly in the $T \to 0$ limit. $T_c = 0$.

The above singularities can be cast into scaling forms (but in somewhat unusual variables $x = e^{-4J/k_B T}$, $h = \beta H$), and the origin of these scaling forms can be understood using idea of Renormalization Group (RG).

Renormalization Group (RG) for 1d Ising model

General idea: "integrate out" high-energy (short-scale) degrees of freedom and focus on the effective interactions of the low-energy (long wavelength) degrees of freedom. Follow the "flow" of these interactions (flow of coupling constants) as we continue going to longer and longer length scales.

1d Ising model – can carry the RG program explicitly



Try "b=2" RG : block (or "coarse grain") two spins into one "block-spin".

Can try e.g.

$$S_{\text{block}}[2i, 2i+1] = \frac{S_{2i}+S_{2i+1}}{2} = \begin{cases} 1, \text{if } S_{2i}=S_{2i+1}=1 \\ -1, \text{if } S_{2i}=S_{2i+1}=-1 \\ \boxed{0} \text{ if } \begin{matrix} S_{2i}=1, & S_{2i+1}=-1 \\ -1 & 1 \end{matrix} \end{cases}$$

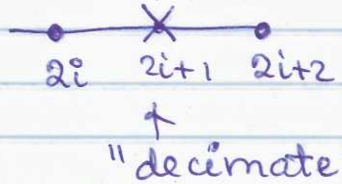
somewhat unfortunate, since S_{block} is not an Ising variable.

use S_{2i} as a "tie-breaker"

$$\Rightarrow S_{\text{block}}[2i, 2i+1] = S_{2i}$$

$$Z = \sum_{\{S_j\}} e^{-\hat{H}/k_BT} = \sum_{\{S_{\text{block}}[2i]\}} \sum_{\{S_j \text{ consistent with } S_{\text{block}} \text{ values}\}} e^{\overset{\bar{H}}{\overset{\equiv}{\boxed{-\beta\hat{H}}}}}$$

$$= \sum_{\{S_{2i}\}} \underbrace{\left(\sum_{\{S_{2i+1}\}} e^{-\beta\hat{H}} \right)}_{\text{effectively "summing out" odd spins ("thinning out" degrees of freedom)}}$$



Write $\overline{H} \equiv -\frac{H}{k_BT} = K\sum_i S_iS_{i+1} + h\sum_i S_i + \underbrace{C\cdot N}_{}$

const shift of energy ; doesnot affect probabilities

$$Z_{orig} = \sum_{\{S_i\}} e^{-\frac{H}{k_BT}} \quad ; \qquad F_{orig} = -k_BT \ln Z_{orig} = -k_BT\ln Z - Nk_BT\ln 2$$

$$Z = \underbrace{\frac{1}{2^N}}_{\prod_{i=1}^{N}\frac{1}{2}} \sum_{\{S_i\}} \underbrace{e^{K\sum_i S_iS_{i+1} + h\sum_i S_i + CN}}_{\prod_{\text{bonds}[i,i+1]} e^{\frac{h}{2}S_i + \frac{h}{2}S_{i+1} + C + KS_iS_{i+1}}}$$

$$\frac{1}{2}\sum_{S_1=\pm1} (\text{bond}[0,1])(\text{bond}[1,2]) =$$

$$= \frac{1}{2}\sum_{S_1=\pm1} e^{\frac{h}{2}S_0 + \frac{h}{2}S_2 + 2C}\; e^{S_1(h+K(S_0+S_2))} =$$

$$= e^{\frac{h}{2}(S_0+S_2)+2C} \cdot \cosh(h+K(S_0+S_2)) \qquad (*)$$

$$\overset{?}{=} e^{\frac{h'}{2}(S_0+S_2)+C'+K'S_0S_2} \qquad (**)$$

~~$e^{\frac{h}{2}(S_0+S_2)+2C}\; e^{\frac{h'-h}{2}(S_0+S_2)+C'-2C+K'S_0S_2} \quad (**)$~~

	$S_0,\ S_2$	$(*)$		$(**)$
a)	$+\ \ +$:	$e^{h+2C}\cosh(h+2K)$	$=$	$e^{h'+C'+K'}$
b)	$-\ \ -$:	$e^{-h+2C}\cosh(2K-h)$	$=$	$e^{-h'+C'+K'}$
c)	$\overset{+}{-}$ or $\overset{-}{+}$:	$e^{2C}\cosh(h)$	$=$	$e^{C'-K'}$

3 equations for 3 unknowns K', h', C'

"RG transformation"

$\frac{a)}{b)}$:
$$e^{2h'} = e^{2h} \cdot \frac{\cosh(2K+h)}{\cosh(2K-h)}$$

$$\Rightarrow \boxed{h' = h + \frac{1}{2}\ln \frac{\cosh(2K+h)}{\cosh(2K-h)}}$$

$\frac{a) \cdot b)}{(c))^2}$:
$$e^{4K'} = \frac{\cosh(2K+h)\cosh(2K-h)}{(\cosh(h))^2}$$

$$\Rightarrow \boxed{K' = \frac{1}{4}\ln \frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2(h)}}$$

$a) \cdot b) \cdot c)^2$:
$$e^{4C'} = e^{8C} \cosh(2K+h)\cosh(2K-h)\cosh^2 h$$

$$\Rightarrow \boxed{C' = 2C + \frac{1}{4}\ln\left(\cosh(2K+h)\cdot\cosh(2K-h)\cdot\cosh^2 h\right)}$$

K', h' – new coupling constants

"C" accumulates contributions to the free energy and does not change probability distributions

New problem

$$\overline{H}_{new} = K'\sum_i \underbrace{S_{2i}}_{S_I}\,\underbrace{S_{2i+2}}_{S_{I+1}} + h'\sum_i \underbrace{S_{2i}}_{S_I} + C'\cdot\frac{N}{2}$$

new coupling constants.

- obtained new Ising model with half the number of spins.

<u>Extracting physics connections between orig & new model</u>

Suppose model $\bar{H} = +K\sum_i S_i S_{i+1} + h\sum_i S_i$

has free energy $f(K,h)$ per lattice site and correlation length $\xi(K,h)$ in lattice units:

$$\langle S_i S_j \rangle = \# \exp\left(-\frac{|i-j|}{\xi(K,h)}\right)$$

Correlation length:

New model has $\xi(K',h')$ in new lattice space units:

$$\langle S_I^{new} S_J^{new} \rangle_{new} = \#' \exp\left(-\frac{|I-J|}{\xi(K',h')}\right)$$

$$\Vert$$

$$\langle S_{2I}^{orig} S_{2J}^{orig} \rangle_{orig} = \# \exp\left(-\frac{|2I-2J|}{\xi(K,h)}\right)$$

$$\Rightarrow \boxed{\begin{array}{l} \xi(K',h') = \frac{1}{2}\xi(K,h) \\ \text{"}X_{new}\text{"} = \frac{\text{"}X_{old}\text{"}}{b} \end{array}} \Rightarrow \boxed{\xi(K,h) = 2\xi(K',h')}$$

Free energy:

~~denote $N\gamma(K,h) \equiv \ln Z_N(K,h) = -N\beta f$~~

$$\underbrace{\ln Z_N(K,h,c)}_{|||} = \ln Z_{N/2}(K',h',c')$$

$$\underbrace{\ln Z_N^{(0)}(K,h)}_{|||} + cN = \ln Z_{N/2}^{(0)}(K',h') + c'\cdot\frac{N}{2}$$

$$N\gamma(K,h) + cN = \frac{N}{2}\gamma(K',h') + c'\frac{N}{2}$$

$$-\beta f \equiv \boxed{\gamma(K,h) = \frac{1}{2}\gamma(K',h') + \frac{c'}{2} - c}$$

$$\frac{1}{\beta}\cdot\left(\frac{c'}{2}-c\right) = k_BT\cdot\frac{1}{8}\ln\left(\cosh(2K+h)\cdot\cosh(2K-h)\cdot\cosh^2 h\right)$$

$$\underset{\substack{\uparrow\\ \text{for } h=0}}{=} k_BT\cdot\frac{1}{4}\ln(\cosh(2K)) \underset{\substack{\uparrow\\ \text{large } K}}{\approx} k_BT\cdot\frac{1}{2}K = \underbrace{\frac{1}{2}J}_{\text{non-singular}}$$

Singular piece comes from

$$\boxed{\gamma_{sing}(K,h) = \frac{1}{2}\gamma_{sing}(K',h')}$$

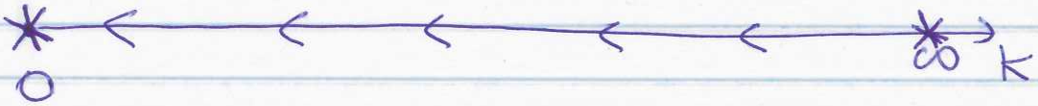
Physics from the RG.

Consider $h=0$

$$\boxed{K' = \frac{1}{2}\ln\cosh(2K)} < K \text{ for any finite } K!$$

- "effective temperature" always grows!
  If K' is sufficiently small, it becomes even smaller very quickly:

small K: $\quad K' \approx \frac{1}{2}\ln\left(1+\frac{1}{2}(2K)^2\right) \approx K^2$



Notion of fixed points: $\quad K^* = R_b[K^*]$

$$K^* = \frac{1}{2}\ln\cosh(2K^*)$$

$\Rightarrow$ only $K^*=0$ or $K^*=\infty$.

$K^*=0$ – high-temperature "disordered fixed point"

- attractor for any $K<\infty$; stable fixed point describing the high-T phase – here all $T>0$

low-temperature

$K^* = \infty$ $(T=0)$ – ordered fixed point – here unstable fixed point. Here this unstable fixed point "controls" the behaviour in the vicinity of $T_c = 0$, as we will see below.

General philosophy for extracting physics using RG: repeat RG ("run RG") until the problem $\left(\underset{\text{here}}{(K',h')}\right)$ becomes simple to solve directly: e.g. $\underset{(\frac{J}{T})_{\text{eff}}}{\underset{\shortparallel}{K'}}$ sufficiently small that spins can be treated as essentially noninteracting (can calculate corrections e.g. in $(\frac{J}{T})_{\text{eff}}$ perturbatively).

Here want behaviour at large K – strong interaction, so pretend cannot solve directly.

For large K, it is convenient to introduce

$$x \equiv e^{-4K} \quad \Big] \leftarrow \left(\begin{array}{l}\text{can be other choices, e.g. } y=e^{-2K}, \\ \text{just need to be consistent throughout}\end{array}\right)$$

$$x' = \frac{1}{\cosh^2(2K)} = \frac{4x}{(1+x)^2}$$

$K^* = \infty$ fixed point corresponds to $x^* = 0$

Write $x = x^* + \underset{\nwarrow \text{ small deviation}}{\delta x}$

$$\boxed{\begin{array}{l} \delta x' \approx \underset{\displaystyle \Lambda_x = b^{y_x}}{\underset{|||}{4}}\,\delta x \\ \qquad b=2 \text{ – rescaling factor} \\ \qquad y_x = 2 \end{array}}$$

(cf. $t' = b^{y_t} t$ in general RG)

$$\xi' = \frac{\xi}{2} = \frac{\xi}{b}$$

Run the RG ℓ times:

$$\delta x_\ell = \Lambda_x^\ell\, \delta x_0 = \left(b^{y_x}\right)^\ell \delta x_0 \underset{\substack{\uparrow \\ \text{here}}}{=} 4^\ell\, \delta x_0$$

$$\underset{\substack{\| \\ \xi(\delta x_\ell)}}{\xi_\ell} = \frac{\xi_0}{b^\ell} = \frac{\xi_0}{2^\ell} \leftarrow \xi(\delta x_0)$$

$$\Rightarrow \boxed{\xi(\delta x_0) = b^\ell\, \xi(\delta x_\ell) = \left(\frac{\delta x_\ell}{\delta x_0}\right)^{1/y_x} \xi(\delta x_\ell)}$$

– relates correlation lengths at strong coupling (small δx_0) to weaker coupling (larger δx)

In particular, if we run the RG until the "final" $\boxed{\delta x_{final} \sim 1,}$ then $\boxed{\xi_{final} = \xi(\delta x_{final}) \sim 1}$

$\uparrow$ weakly-correlated spins; correlation length of order few lattice spacings

$$\Rightarrow \boxed{\underbrace{\xi(\delta x_0) \sim \frac{1}{(\delta x_0)^{1/y_x}}}_{\text{general scaling result}} = \frac{1}{\delta x_0^{1/2}} \sim e^{2J/k_BT}} \text{ — agrees with exact solution}$$

general scaling result, cf. $\boxed{\xi(t) \sim \frac{1}{t^{1/y_t}} = t^{-\nu} \qquad \nu = \frac{1}{y_t}}$

Free energy:

$$\gamma_{sing}(\delta x_0) = \gamma_{sing}(K) \underset{\substack{\uparrow \\ \text{1 step}}}{=} \frac{\gamma_{sing}(K')}{2} = \dots = \frac{1}{b^\ell}\gamma_{sing}(\delta x_\ell) =$$

$$= \left(\frac{\delta x_0}{\delta x_\ell}\right)^{1/y_x} \gamma_{sing}(\delta x_\ell)$$

$\gamma_{sing}(\delta x_\xi) \sim 1$ for $\delta x_\xi \sim 1$

$$\gamma_{sing}(\delta x_0) \sim \underbrace{(\delta x_0)^{1/y_x}} \underset{\substack{\uparrow \\ \text{here } y_x=1}}{=} \delta x_0^{1/2} \sim e^{-2J/k_BT}$$

$$\sim \frac{1}{\xi(\delta x_0)}$$ – "hyperscaling". is satisfied in $d=1$

$$\sim (\delta x_0)^{2-\alpha} \quad , \quad 2-\alpha = \frac{1}{y_x}$$

cf. $2-\alpha = \frac{d}{y_x} = d\nu$
hyperscaling in d dim.

Scaling of the field h:

$h' = h'(K,h)$
$K' = K'(K,h)$

$\underline{K^* = \infty, \ h^* = 0}$
– fixed point.
work around this fixed pt.

$$\boxed{h' = h + \frac{1}{2}\ln\frac{e^{2K}e^{h} + e^{-2K}e^{-h}}{e^{2K}e^{-h} + e^{-2K}e^{h}}} = \text{[illegible]}$$

$$= h + \frac{1}{2}\ln\frac{1+h+x(1-h)}{1-h+x(1+h)} \approx h + \frac{1}{2}\ln\frac{e^{h}}{e^{-h}} = \boxed{h \cdot 2}$$

(at low T, $S_{block} = +1 \leftrightarrow$ ↑↑ – sum the two fields)
$-1 \leftrightarrow$ ↓↓
rarely ↑↓

two spins form a cluster behaving as one larger spin

$$e^{4K'} = \frac{(\cosh(2K)\cosh(h))^2 - (\sinh(2K)\sinh(h))^2}{(\cosh(h))^2} \approx$$

$$\approx (\cosh(2K))^2 + O(h^2)$$, so $\delta x' = 4\delta x$
remains unchanged

$b=2$

$$\delta x' = 4\delta x = \Lambda_x \delta x = b^{y_x} \delta x \qquad y_x = 2$$

$$\delta h' = 2\delta h = \Lambda_h \delta h = b^{y_h} \delta h \qquad y_h = 1$$

$$\delta x_f = \delta x_\ell = \left(b^{y_x}\right)^\ell \delta x_0 \Rightarrow b^\ell = \left(\frac{\delta x_f}{\delta x_0}\right)^{1/y_x}$$

$$\delta h_f = \delta h_\ell = \left(b^{y_h}\right)^\ell \delta h_0 = \left(\frac{\delta x_f}{\delta x_0}\right)^{y_h/y_x} \delta h_0$$

$$\Rightarrow \boxed{\begin{aligned}\xi(\delta x_0, \delta h_0) &= b^\ell \xi(\delta x_f, \delta h_f) = \\ &= \left(\frac{\delta x_f}{\delta x_0}\right)^{1/y_x} \xi\left(\delta x_f, \left(\frac{\delta x_f}{\delta x_0}\right)^{y_h/y_x} \delta h_0\right)\end{aligned}}$$

Choose $\delta x_f = 1$

$$\boxed{\xi(\delta x_0, \delta h_0)} = \underbrace{\frac{1}{(\delta x_0)^{1/y_x}} \xi\left(1, \frac{\delta h_0}{(\delta x_0)^{y_h/y_x}}\right)}_{\displaystyle \frac{1}{(\delta x_0)^\nu} g_\xi\left(\frac{\delta h_0}{\delta x_0^\Delta}\right)} \text{ — scaling form!}$$

$$\nu = \frac{1}{y_x}$$

$$\Delta = \frac{y_h}{y_x}$$

$$= \boxed{e^{2K_0} g_\xi\left(\delta h_0 e^{2K_0}\right)}.$$

Similarly,

$$\boxed{\gamma_{sing}(\delta x_0, \delta h_0)} = \frac{1}{b^\ell} \gamma_{sing}(\delta x_\ell, \delta h_\ell) = \left(\frac{\delta x_0}{\delta x_f}\right)^{1/y_x} \gamma_{sing}\left(\delta x_f, \left(\frac{\delta x_f}{\delta x_0}\right)^{y_h/y_x} \delta h_0\right)$$

$$\sim (\delta x_0)^{1/y_x} g_f\left(\frac{\delta h_0}{\delta x_0^\Delta}\right) \sim \boxed{e^{-2K_0} g_f\left(\delta h_0 e^{2K_0}\right)}$$

compare with

$$\boxed{-\beta f = \text{const} + e^{-2K} \sqrt{\left(h e^{2K}\right)^2 + 1}}$$