Fluctuations in the Ginzburg-Landau theory

Ornstein-Zernike theory: correlation functions in the Ginzburg-Landau theory

Consider first case with one-component order param. (Ising-like order param):

$$\beta F_{eff}(\{m(r)\}) = \int d^d r \left[\frac{K}{2}(\vec{\nabla} m)^2 + \frac{t}{2} m^2 + u m^4 \right]$$

$T > T_c$: simply ignore um^4 term and calculate correlation function $\langle m(r)\, m(r')\rangle$

$T < T_c$: $m = \bar{m} + \delta m$, $4u\bar{m}^2 = -t$

$$\frac{t}{2} m^2 + u m^4 \approx \text{const} + \frac{(t + 12u\bar{m}^2)\,\delta m^2}{2} =$$

$$= \frac{1}{2} \cdot 2|t|\,\delta m^2$$

$$\beta F_{eff}(\{\delta m\}) = \int d^d r \Big[\frac{K}{2}(\vec{\nabla}\delta m)^2 + \frac{2|t|}{2}\delta m^2 + \underbrace{\ldots}_{\text{ignore higher-order terms}} \Big]$$

Want to calculate $\langle \delta m(r)\, \delta m(r')\rangle$ in this theory. The structure is very similar to the $T > T_c$ case, only need to replace $t \to 2|t|$.

Once we have the correlation fnctn $\langle m(r)\, m(r')\rangle$, we will discuss when it is justifiable to neglect the m^4 (or $(\delta m)^{p>2}$ terms) (The expectation is that this is OK to do for $T \gg T_c$ or $T \ll T_c$, since then m or δm are indeed very small.)

$$\boxed{\underset{\substack{\text{Gaussian}\\ \text{action}}}{S} = \int d^d r \left[\frac{K}{2} (\vec{\nabla} m)^2 + \frac{t}{2} m^2 \right]}$$

box of volume $Vol = L_x L_y \ldots$

Imagine periodic boundary conditions

$$\boxed{m(r) = \frac{1}{Vol} \sum_k m(k)\, e^{i\vec{k}\cdot\vec{r}}} \qquad \vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \ldots \right)$$

- $$\int d^d r\, m^2 = \frac{1}{Vol^2} \sum_{k k'} m_k m_{k'} \underbrace{\int d^d r\, e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}}_{Vol \cdot \delta_{k+k'=0}} =$$

$$= \frac{1}{Vol} \sum_k |m_k|^2$$

Here used

$m(r)$ real $\Rightarrow \quad m_k = \int d^d r\, m(r) e^{-i\vec{k}\cdot\vec{r}} = m^*_{-k}$

- $$\int d^d r\, (\vec{\nabla} m)^2 = \frac{1}{Vol^2} \sum_{k,k'} m_k m_{k'}\, i\vec{k} \cdot i\vec{k}' \cdot Vol \cdot \delta_{k+k'} =$$

$$= \frac{1}{Vol} \sum_k \vec{k}^2 |m_k|^2$$

$$\boxed{S = \frac{1}{Vol} \sum_k \frac{1}{2} (K\vec{k}^2 + t) |m_k|^2 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} (K\vec{k}^2 + t) \times |m_k|^2}$$

Will argue now that

$$\langle m_k m_{k'} \rangle = \delta_{k+k'=0} \langle |m_k|^2 \rangle = \frac{\delta_{k+k'=0} \cdot Vol}{K\vec{k}^2 + t} = \frac{(2\pi)^d \delta^{(d)}(\vec{k}+\vec{k}')}{K\vec{k}^2 + t}$$

Schematicaly, can appeal to equipartition theorem:

$$\rho = \frac{1}{k_B T} \quad \frac{p^2}{2m} \quad \Rightarrow \quad \left\langle \frac{p^2}{2m} \right\rangle = \frac{k_B T}{2} \quad \text{or} \quad \left\langle \frac{p^2}{k_B T \cdot 2m} \right\rangle = \frac{1}{2}$$

$$\Rightarrow \quad \left\langle \frac{K\vec{k}^2 + t}{2 \cdot \mathrm{Vol}} |m_k|^2 \right\rangle = \frac{1}{2}$$

schematically
m_k are indep
variables.

But let's be more careful, since m_k & $m_{-k} = m_k^*$ are not independent variables.

Write

$$\rho = \sum_k{}' \frac{1}{\mathrm{Vol}} \frac{1}{2} (K\vec{k}^2 + t) \left((\mathrm{Re}\, m_k)^2 + (\mathrm{Im}\, m_k)^2 \right) \times 2$$

sum over one half of k-values,
e.g. $k_x > 0$, k_y any
k_z any
...

Now $\mathrm{Re}\, m_k$ & $\mathrm{Im}\, m_k$ are indep. variables. Can appeal to equipartition theorem (or just do Gaussian integral)

$$\langle (\mathrm{Re}\, m_k)^2 \rangle = \frac{1}{2} \frac{\mathrm{Vol}}{K k^2 + t}$$

$$\langle (\mathrm{Im}\, m_k)^2 \rangle = \frac{1}{2} \frac{\mathrm{Vol}}{K k^2 + t}$$

(The necessary Gaussian integral is e.g.

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{+\infty} dx \, \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \cdot x^2}{\int_{-\infty}^{+\infty} dx \, \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}} = \sigma^2$$

"normal distribution" or "error distribution fnctn")

$$\langle \mathrm{Re}\, m_k \, \mathrm{Im}\, m_{k'} \rangle \overset{= 0 \text{ always;}}{;} \quad \langle \mathrm{Re}\, m_k \, \mathrm{Re}\, m_{k'} \rangle = \langle \mathrm{Im}\, m_k \, \mathrm{Im}\, m_{k'} \rangle$$

$$= 0 \quad \text{for } k \neq k' \text{ and both from the "one half of } k\text{-values"}$$

It is convenient not to have to divide m_k into Re & Im, and summation over k into such halves; summarize the above results for any k & k':

$$\underbrace{\langle |m_k|^2 \rangle}_{\langle |m_{-k}|^2 \rangle \,=\, \langle m_k m_{-k} \rangle} = \langle (\mathrm{Re}\, m_k)^2 + (\mathrm{Im}\, m_k)^2 \rangle = \frac{\mathrm{Vol}}{Kk^2 + t} ;$$

$$\langle m_k m_k \rangle = \langle (\mathrm{Re}\, m_k)^2 - (\mathrm{Im}\, m_k)^2 \rangle = 0$$

$$\langle m_{-k} m_{-k} \rangle = 0$$

$$\Rightarrow \boxed{\langle m_k m_{k'} \rangle = \delta_{k+k'} \cdot \frac{\mathrm{Vol}}{Kk^2 + t} = \frac{(2\pi)^d \delta^{(d)}(\vec{k} + \vec{k}')}{Kk^2 + t}}$$

$$\boxed{\delta_{q=0} \cdot \mathrm{Vol} = (2\pi)^d \delta^{(d)}(\vec{q})}$$

in the sense that

$$\frac{1}{\mathrm{Vol}} \sum_q f(\vec{q}) \underset{\text{LHS}}{(\quad)} = f(\vec{q} = 0)$$

$$\frac{1}{\mathrm{Vol}} \sum_q f(\vec{q}) \, (\text{RHS}) = \int \frac{d^d q}{(2\pi)^d} f(\vec{q}) \cdot (\text{RHS}) =$$

$$= f(\vec{q} = 0)$$

$$\boxed{\langle m(r)\, m(r')\rangle} = \int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{r}} \int \frac{d^dk'}{(2\pi)^d}\, e^{i\vec{k}'\cdot\vec{r}'} \langle m(k)\, m(k')\rangle$$

$$= \boxed{\int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{K\vec{k}^2+t}}$$

Set $\vec{r}'=0$; calculate the integral $\int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{K\vec{k}^2+t} =$

$$= \frac{1}{K} \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+k_0^2}, \qquad k_0^2 = \frac{t}{K} = \frac{1}{\xi^2}$$

for several dimensionalities:

$\underline{d=1}$: $\boxed{\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{k^2+k_0^2}} \underset{\uparrow \text{ table integral}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\cos kx}{k^2+k_0^2}\, dk = \boxed{\frac{1}{2} \frac{e^{-k_0|x|}}{k_0}}$

$\underline{d=3}$: $\boxed{\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+k_0^2}} = \frac{1}{(2\pi)^3}\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^{\infty} k^2 dk\, \frac{e^{ikr\cos\theta}}{k^2+k_0^2}$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{k^2 dk}{k^2+k_0^2} \underbrace{\int_{1}^{-1} d(-\cos\theta)\, e^{ikr\cos\theta}}_{\int_{-1}^{1} dx\, e^{ikr\cdot x} = \frac{2\sin(kr)}{kr}} =$$

$$= \frac{1}{2\pi^2}\frac{1}{r}\int_0^{\infty} \frac{k\sin(kr)}{k^2+k_0^2}\, dk \underset{\uparrow \text{ table integral}}{=} \boxed{\frac{e^{-k_0 r}}{4\pi r}}$$

— "Yukawa" (screened Coulomb)

Table integrals used above:

$$\boxed{\begin{aligned} \int_{-\infty}^{+\infty} \frac{\cos(kx)}{k^2+k_0^2}\,dk &= \frac{\pi}{k_0}\, e^{-k_0|x|} \\ \int_{-\infty}^{+\infty} \frac{k\sin(kx)}{k^2+k_0^2}\,dk &= \pi\, e^{-k_0|x|}\cdot \mathrm{sign}(x)\,. \end{aligned}}$$

General dimensionality: cannot do $\int d^dk \ldots$ analytically, but can argue that

e.g. Kardar:

$$\left\{ \begin{array}{l} \underline{* \text{ for } r \gg \xi} \qquad \displaystyle\int \frac{d^dk}{(2\pi)^d}\, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+k_0^2} \sim \frac{e^{-r/\xi}}{r^{(d-1)/2}} \\ \quad (d=1,3 \text{ - checks}) \\ \underline{* \text{ for } r \ll \xi} \qquad \displaystyle\int \frac{d^dk}{(2\pi)^d}\, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+k_0^2} \sim \frac{1}{r^{d-2}} \end{array} \right.$$

Schematic argument for $r \ll \xi$:

$$\int^{\Lambda} \frac{d^dk}{(2\pi)^d}\, \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+k_0^2} \underset{k=\frac{\tilde{k}}{r}}{=} \frac{1}{r^{d-2}} \int^{\Lambda r} \frac{d^d\tilde{k}}{(2\pi)^d}\, \frac{e^{i\tilde{\vec{k}}}}{\tilde{k}^2+(k_0 r)^2}$$

$$= \frac{1}{r^{d-2}}\cdot \mathrm{func}(\underbrace{k_0 r}_{=\frac{r}{\xi}})$$

For $\xi \to \infty$ or $\frac{r}{\xi} \ll 1$ $\Big|$ $\sim \frac{1}{r^{d-2}}$ (#). $\Big\}$ "critical correlations" (correlations on scales $\ll \xi$)

Kardar's argument for the asymptotic behaviour of

$$\boxed{f(\vec{r}) = + \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + k_0^2}}$$

$$\vec{\nabla}^2 f(\vec{r}) = \int \frac{d^dk}{(2\pi)^d} \frac{-\vec{k}^2}{\vec{k}^2+k_0^2} e^{i\vec{k}\cdot\vec{r}} = -\int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}}$$

$$+ k_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+k_0^2} = -\delta^{(d)}(\vec{r}) + k_0^2 f(\vec{r})$$

Look for a spherically symmetric solution

$$\boxed{\vec{\nabla}^2 f(|\vec{r}|)} = \sum_\mu \frac{\partial^2}{\partial x_\mu^2} f\left(\sqrt{\Sigma x_\mu^2}\right) = \sum_\mu \frac{\partial}{\partial x_\mu}\left(f'(|\vec{r}|)\frac{x_\mu}{|\vec{r}|}\right) =$$

$$= \sum_\mu \left[\underbrace{\frac{d}{d|\vec{r}|}\left(\frac{f'(|\vec{r}|)}{|\vec{r}|}\right)}_{\frac{f''}{|\vec{r}|} - \frac{f'}{|\vec{r}|^2}} \cdot \frac{x_\mu}{|\vec{r}|} \cdot x_\mu + \frac{f'(|\vec{r}|)}{|\vec{r}|}\right] = f''(|\vec{r}|) + \boxed{+ \frac{(d-1)f'(|\vec{r}|)}{|\vec{r}|}}$$

$$\boxed{f''(|\vec{r}|) + (d-1)\frac{f'(|\vec{r}|)}{|\vec{r}|} - k_0^2 f(|\vec{r}|) = -\delta^{(d)}(\vec{r})}$$

For $|\vec{r}| \neq 0$: $\quad f''(r) + (d-1)\frac{f'(r)}{r} - k_0^2 f(r) = 0$

Try a solution decaying as $\frac{e^{-k_0 r}}{r^p}$:

$$LHS = \frac{d}{dr}\left(\frac{-k_0 e^{-k_0 r}}{r^p} - \frac{p e^{-k_0 r}}{r^{p+1}}\right) + \frac{(d-1)}{r}\left(\frac{-k_0 e^{-k_0 r}}{r^p} - \frac{p e^{-k_0 r}}{r^{p+1}}\right)$$

$$- k_0^2 \frac{e^{-k_0 r}}{r^p} = \cancel{\frac{k_0^2 e^{-k_0 r}}{r^p}} - k_0 e^{-k_0 r} \cdot \frac{(-p)}{r^{p+1}} \cdot 2 + \quad +$$

$$+ \frac{p(p+1)e^{-k_0 r}}{r^{p+2}} - \frac{(d-1)k_0 e^{-k_0 r}}{r^{p+1}} - \frac{p(d-1)e^{-k_0 r}}{r^{p+2}} - \cancel{k_0^2 \frac{e^{-k_0 r}}{r^p}} =$$

$$= e^{-k_0 r}\left(k_0 \cdot \frac{(2p-(d-1))}{r^{p+1}} + p \cdot \frac{\overbrace{(p+1-(d-1))}^{p-d+2}}{r^{p+2}} \right)$$

Large $r \gg \xi$

* Setting $p = \frac{d-1}{2}$ makes the leading $\frac{1}{r^{p+1}}$ term vanish (For $d=3 \Rightarrow p=1$ and $\frac{1}{r^{p+2}}$ also vanishes – this is exact solution).

$\Rightarrow$ For large $r \gg \frac{1}{k_0} = \xi$, $\boxed{f(r) \sim \frac{e^{-r/\xi}}{r^{\frac{d-1}{2}}}}$

* Small $r \ll \xi$: set $p = d-2$, $\boxed{f(r) \sim \frac{1}{r^{d-2}}}$

If formally set $k_0 = 0$ ($\xi \to \infty$):

$$f''(r) + (d-1)\frac{f'(r)}{r} = 0$$ is solved by

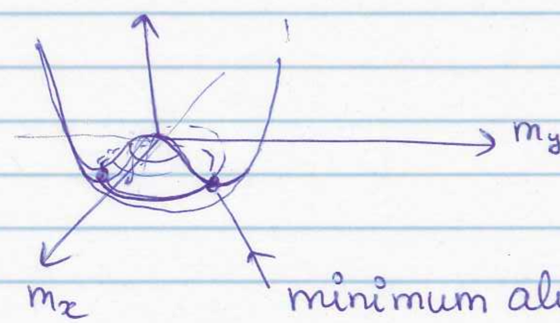
$f(r) = \frac{A}{r^{d-2}}$ – basically, power counting

Continuous symmetry breaking and Goldstone modes

$$F = \int d^d r \left\{ \frac{K}{2}(\vec{\nabla}\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u\vec{m}^4 \right\}$$

Vector order parameter $\vec{m} = (m_x, m_y, \dots)$

$t<0$: $\quad f_{uni} = \underbrace{u\left(\vec{m}^2 - \frac{|t|}{4u}\right)^2}_{\text{"Mexican hat potential"}} - \frac{|t|^2}{16u}$



minimum along surface $|\vec{m}| = \sqrt{\frac{|t|}{4u}}$

The system chooses one state out of continuously many
$\Rightarrow$ continuous symmetry breaking

Suppose $\overline{\vec{m}} = \left(\sqrt{\frac{|t|}{4u}}, 0, 0, \dots\right)$; consider

$$\delta\vec{m} = (\delta m_x, \delta m_y, \delta m_z, \dots)$$

δm_x: longitudional $\quad$ $\delta m_y, \delta m_z$: transverse

- $|\vec{m}|^2 = \left(\frac{|t|}{4u} + \sqrt{\frac{|t|}{u}}\,\delta m_x + \delta m_x^2\right) + \delta m_y^2 + \dots$

$$\Rightarrow u\left(|\vec{m}|^2 - \frac{|t|}{4u}\right)^2 - \frac{|t|^2}{16u} = \text{const} + u\left(\sqrt{\frac{|t|}{u}}\,\delta m_x + \delta\vec{m}^2\right)^2$$

$$= \text{const} + |t|\,\delta m_x^2 + 2\sqrt{u|t|}\,\delta m_x\,\delta\vec{m}^2 + u(\delta\vec{m}^2)^2$$

• $(\vec{\nabla} \vec{m})^2 = (\vec{\nabla} \delta m_x)^2 + (\vec{\nabla} \delta m_y)^2 + \dots$

$$\Rightarrow \boxed{\begin{aligned} F = \int d^d r \Big\{ & \underbrace{\frac{K}{2}(\vec{\nabla}\delta m_x)^2 + |t|(\delta m_x)^2}_{\text{"massive" longitudional mode similar to the fluctuations in the Ising model}} + O(\delta m_x^3, \delta m_x^4) \\ & + \sum_{\mu=2}^{d} \underbrace{\frac{K}{2}(\vec{\nabla}\delta m_\mu)^2}_{\text{"massless" — no } \#\cdot(\delta m_\mu)^2 \text{ term — Goldstone mode transverse}} + O(\delta m_x \delta m_\mu^2, \delta m_\mu^4) \Big\} \end{aligned}}$$

Consequences for thermodynamics at low temperature

Consider thermodynamics with one gapless mode

e.g. $\hat{H} = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)$

$\approx \frac{J}{2} \sum_{\langle ij \rangle} (\varphi_i - \varphi_j)^2$ + const.

e.g., Kardar Problem 3.1

$$\frac{J}{2} \int d^d r \, (\vec{\nabla}\varphi)^2 = \frac{1}{Vol} \sum_k \frac{J}{2} \vec{k}^2 |\varphi_k|^2$$

$$Z = \int [\mathcal{D}\varphi(r)] \, e^{-\frac{1}{k_B T} \cdot \frac{1}{Vol} \sum_k \frac{J}{2} \vec{k}^2 |\varphi_k|} =$$

$$= \int \prod_k{}' \, d\mathrm{Re}\varphi_k \, d\mathrm{Im}\varphi_k \; e^{-\frac{1}{k_B T} \cdot \frac{1}{Vol} \sum_k{}' J k^2 \left((\mathrm{Re}\varphi_k)^2 + (\mathrm{Im}\varphi_k)^2\right)}$$

$$= \prod_k{}' \sqrt{\frac{\pi}{Jk^2/(k_B T \cdot Vol)}} \cdot \sqrt{\frac{\pi}{Jk^2/(k_B T \cdot Vol)}}$$

$$U \underset{\uparrow}{=} \sum_k{}' \left(\frac{k_B T}{2} + \frac{k_B T}{2} \right) = \sum_k \frac{k_B T}{2} = N_{sites} \cdot \frac{k_B}{2}$$

by equipartition theorem

$\Rightarrow$ Specific heat $\approx \frac{k_B}{2}$ per site

Consequences of Goldstone modes for correlations at long distances

Correlations of longitudinal fluctuations — same
(i.e., amplitude fluctuations)
as computed for the Ising model:

$$\langle \delta m_\ell(r)\, \delta m_\ell(r') \rangle \sim \begin{cases} \dfrac{e^{-|r-r'|/\xi}}{|r-r'|^{\frac{d-1}{2}}} & \text{for } |r-r'| \gg \xi \\ \dfrac{1}{|r-r'|^{d-2}} & \text{for } |r-r'| \ll \xi. \end{cases}$$

For the transverse fluctuations, there is no mass term and "ξ_t" $= \infty$ for all $T<T_c$:

$$\langle \delta m_t(r)\, \delta m_t(r') \rangle \sim \frac{1}{|r-r'|^{d-2}} \quad !$$

$$\Big(\Rightarrow \langle \delta \vec{m}(\vec{r}) \cdot \delta \vec{m}(r') \rangle = \langle \delta m_x(r)\, \delta m_x(r') \rangle + \langle \delta m_y(r)\, \delta m_y(r') \rangle + \dots$$

$$\sim \frac{1}{|r-r'|^{d-2}}$$ — dominated by the transverse fluctuations & Goldstone modes $\Big)$

Correspondingly, longitudinal susceptibility

$$\chi_\ell(q) \sim \int d^d r \, \langle \delta m_\ell(r)\, \delta m_\ell(0) \rangle \, e^{-i\vec{q}\cdot\vec{r}}$$

$$\sim \frac{1}{q^2 + 1/\xi^2} \to \text{const for } q \to 0$$

while the transverse susceptibility

$$\chi_t(q) \sim \int d^d r \, \langle \delta m_t(r)\, \delta m_t(0) \rangle \, e^{-i\vec{q}\cdot\vec{r}} \sim \frac{1}{q^2} \to \infty$$

for $q \to 0$.