Ising Models in $d=1,2$

General definitions

- Canonical partition function

$$\boxed{\begin{array}{c} Z(T,H) \equiv \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} e^{-\beta E(S_1,\dots,S_N)} \\ \hline E(S_1,\dots,S_N) \equiv -J\sum_{\langle ij \rangle} S_i S_j - H\sum_i S_i \end{array}}$$

- Helmholtz free energy

$$\boxed{F(T,H) \equiv -k_B T \ln Z = -\frac{1}{\beta}\ln Z}$$

- Internal energy

$$\boxed{\begin{aligned} U(T,H) &\equiv \langle E \rangle = -\frac{1}{Z}\left(\frac{\partial Z}{\partial \beta}\right) = -\frac{\partial \ln Z}{\partial \beta} = \\ &= F + TS = F - T\left(\frac{\partial E}{\partial T}\right)_H \end{aligned}}$$

entropy: $S = -\left(\frac{\partial F}{\partial T}\right)_H$

- Heat capacity

$$\boxed{C_H(T,H) \equiv \left(\frac{\partial U}{\partial T}\right)_H = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_H}$$

- Magnetization

$$\boxed{\begin{aligned} M(T,H) &\equiv \left\langle \sum_i S_i \right\rangle = \frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial H} = \frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial H}\right)_T = \\ &= -\left(\frac{\partial F}{\partial H}\right)_T \end{aligned}}$$

$$dF = -S\,dT - M\,dH$$

- Thermodynamic limit and spontaneous magnetization:

$$\boxed{m(T,H) = \frac{M(T,H)}{N}}$$ – magnetization per site

$$\boxed{m_{\substack{\infty \text{ system};\\ H\geq 0}}(T,H) = \underbrace{\lim_{N\to\infty}}_{\text{thermodynamic limit}} \frac{M(T,H)}{N}}$$

$$\boxed{m_{\infty\text{ system}}(T, H=0+) = \lim_{H\to 0+} m_{\infty \text{ system}}(T,H) = \begin{cases} 0 & \text{– parmagnetic phase} \\ \neq 0 & \text{– ferromagnetic phase} \end{cases}}$$

- Susceptibility per spin

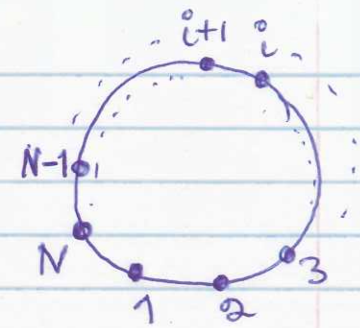
$$\chi(T,H) = \frac{1}{N}\left(\frac{\partial M}{\partial H}\right)_T$$

Methods:
- exact solution in $d=1$ (transfer matrices)
- exact solution in $d=2$ (Onsager)

$d>2$
- Series expansions – next term (high-T expansions; low-T expansions)
- Monte Carlo simulations
- Meanfield qualitatively correct for crit. indices in $d\geq 4$

Ising Chain $(d=1)$

Periodic boundary conditions



$$S_{N+1} \equiv S_1$$

$$Z = \sum_{S_1} \dots \sum_{S_N} \exp\Big[\beta \sum_{i=1}^{N} \underbrace{\Big(J\, S_i S_{i+1} + \tfrac{1}{2} H (S_i + S_{i+1})\Big)}_{\text{"symmetric form" for bond } \langle i, i+1 \rangle}\Big]$$

Let $\hat{T}$ be 2×2 matrix with "indices" labelled by $S = \pm 1$, such that

$$\langle S | \hat{T} | S' \rangle = e^{\beta\left(J S S' + \frac{1}{2} H (S + S')\right)} \quad ; \quad (S, S' = \pm 1)$$

$$\hat{T} = \begin{array}{c c} & \begin{array}{cc} \text{index } S' = +1 & S' = -1 \end{array} \\ \begin{array}{c} \text{index} \\ S = +1 \\ S = -1 \end{array} & \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix} \end{array} \quad \text{– "transfer matrix"}$$

$$Z = \sum_{S_1} \dots \sum_{S_N} \langle S_1 | \hat{T} | S_2 \rangle \langle S_2 | \hat{T} | S_3 \rangle \dots \langle S_N | \hat{T} | S_1 \rangle$$

$$= \sum_{S_1} \langle S_1 | \hat{T}^N | S_1 \rangle = \mathrm{Tr}\left(\hat{T}^N\right) = \lambda_+^N + \lambda_-^N,$$

where $\lambda_\pm$ are eigenvalues of $\hat{T}$; convention: $\lambda_+ > \lambda_-$

$$\hat{T} = \left(\frac{e^{\beta(J+H)} + e^{\beta(J-H)}}{2}\right) \cdot \mathbb{1} + \begin{pmatrix} \frac{e^{\beta(J+H)} - e^{-\beta(J-H)}}{2} & e^{-\beta J} \\ e^{-\beta J} & -\left(\frac{e^{\beta(J+H)} - e^{-\beta(J-H)}}{2}\right) \end{pmatrix}$$

$$\rightarrow e^{\beta J}\left(\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right)$$

$$\boxed{\lambda_\pm = e^{\beta J}\cosh(\beta H) \pm \sqrt{\left(e^{\beta J}\sinh(\beta H)\right)^2 + e^{-2\beta J}}} =$$

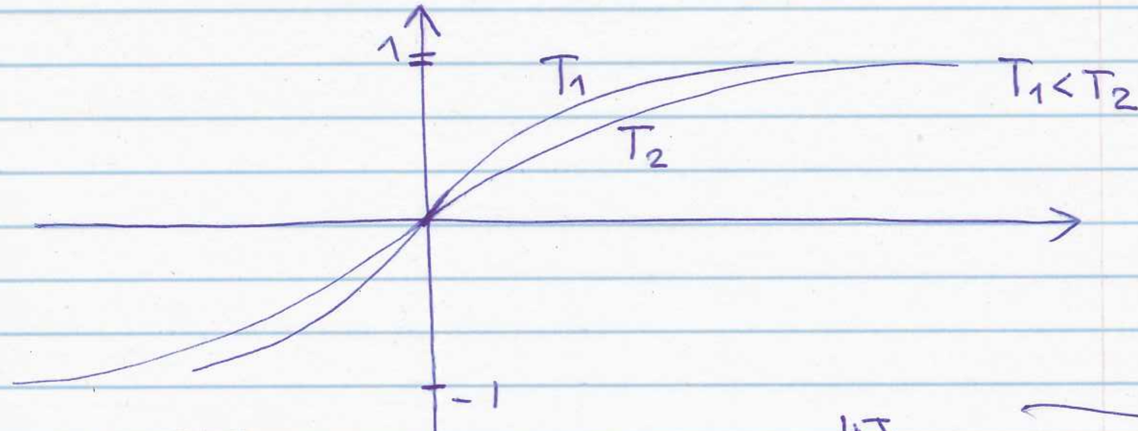
$$Z = \lambda_+^N\left(1 + \frac{\lambda_-^N}{\lambda_+^N}\right) \approx \lambda_+^N \quad \text{for } N \to \infty$$

$$\boxed{\frac{F}{N} = -\frac{1}{\beta}\ln\lambda_+ = -J - \frac{1}{\beta}\ln\left(\cosh(\beta H) + \sqrt{\sinh^2(\beta H) + e^{-4\beta J}}\right)}$$

All thermodynamics follows:

$$\Rightarrow \boxed{m(T,H)} = -\frac{\partial f}{\partial H} = \frac{1}{\beta}\,\frac{\sinh(\beta H)\cdot\beta + \frac{1}{2\sqrt{}}\cdot 2\sinh\cdot\cosh \times\beta}{\cosh(\beta H) + \sqrt{}}$$

$$= \boxed{\frac{\sinh(\beta H)}{\sqrt{\sinh^2(\beta H) + e^{-4\beta J}}}}$$



$$\boxed{m(T,H) \underset{H\to 0}{=} \frac{\beta H}{e^{-4\beta J}} = \frac{e^{\frac{4J}{k_B T}}}{k_B T}\cdot H \xrightarrow{\text{for } H\to 0} 0}$$

for any $T \neq 0$!

- no spontaneous magnetization in $d=1$!

$$\boxed{\chi(T) = \frac{e^{4J/(k_B T)}}{k_B T}}$$

$$\boxed{f(T,H=0) = -k_BT\ln\left(1+e^{-2J/k_BT}\right) - J}$$

$$s = \frac{\text{entropy}}{\text{spin}} = -\frac{\partial f}{\partial T} = k_B\ln\left(1+e^{-2J/k_BT}\right) + k_BT\,\frac{1}{1+e^{-2J/k_BT}}\times$$

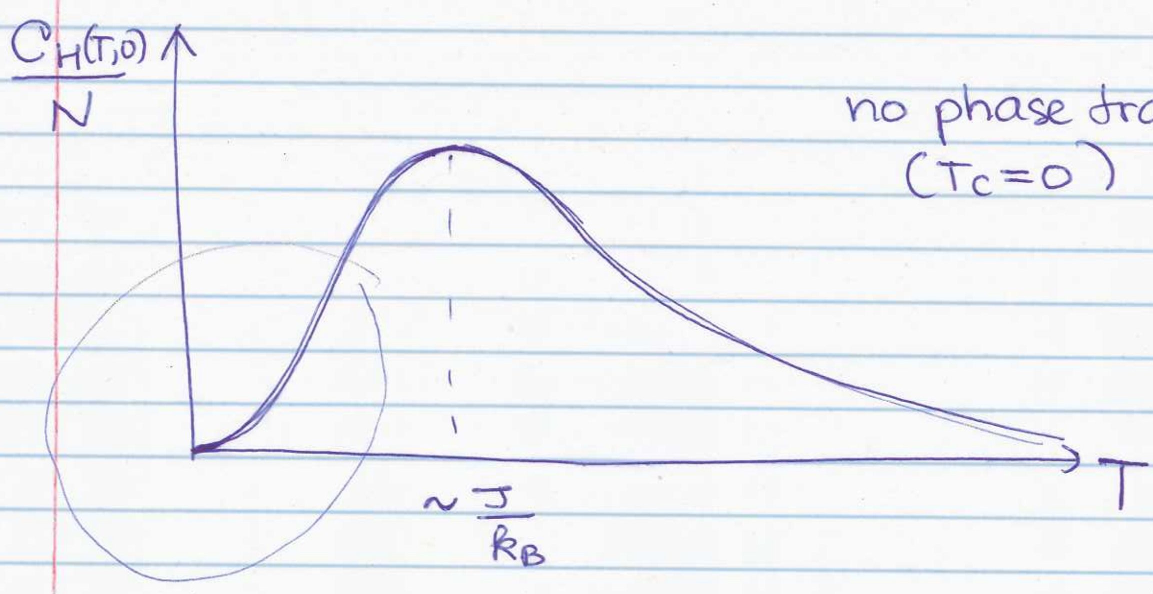
$$\times\, e^{-2J/k_BT}\cdot(-2J)\cdot\frac{-1}{k_BT^2} = k_B\ln\left(1+e^{-2J/k_BT}\right) +$$

$$+\ \frac{2J}{T}\,\frac{1}{e^{2J/k_BT}+1}$$

$$\underset{\text{energy per spin}}{u(T)} = f + Ts = \frac{2J}{e^{2J/k_BT}+1}$$

$$\boxed{\underset{\text{per spin}}{C_H(T)}} = \frac{\partial u}{\partial T} = \frac{2J\cdot(-1)}{\left(e^{2J/k_BT}+1\right)^2}\cdot e^{2J/k_BT}\cdot\frac{-2J}{k_BT^2} =$$

$$= k_B\cdot\left(\frac{2J}{k_BT}\right)^2\frac{1}{\left(2\cosh\frac{J}{k_BT}\right)^2} = \boxed{k_B\,\frac{\left(J/k_BT\right)^2}{\left(\cosh\frac{J}{k_BT}\right)^2}}$$



no phase transition ($T_c=0$)

"2-level form with energy splitting $2J$"

$s_i\cdot s_{i+1} = \sigma_i$ (from solution with open b.c.)

freezing out of the entropy because of correlations among spins

Absence of magnetization in 1d – domain walls!

$T=0$: $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

or

$\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$

domain wall $\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow$ – costs energy $2J$

$T\neq 0$: domain wall can be anywhere along the chain of size N

$\Rightarrow$ entropy $-k_B T \ln N$

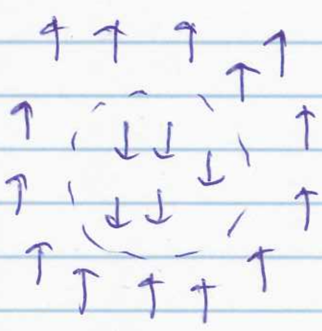
$\Rightarrow$ free energy for adding 1 domain wall:

$$2J - k_B T \underbrace{\ln N}_{\text{large!}} \quad < 0$$

– it is advantegous to introduce domain walls into the system.

At any $T>0$,
Domain walls "proliferate" and scramble the order!

2d



energy of domain wall $\sim 2J \times$ perimeter

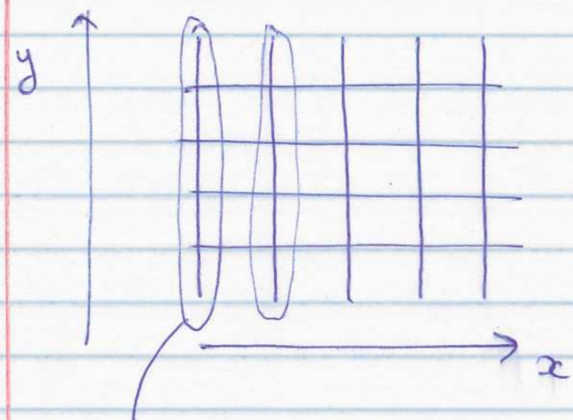
– becomes very large if the flipped domain is large

$\Rightarrow$ can have long range order at sufficiently low T.

Remarks

* n- level system — need $n\times n$ transfer matrix

* Ising model on $L\times L$ square lattice



View as $n = 2^L$ - level system

L such systems arranged in chain along $\hat{x}$-dir, with interactions between them

$$Z = \mathrm{Tr}(\hat{T}^L) \underset{L\to\infty}{\sim} \lambda_+^L$$

where $\hat{T}$ is $2^L \times 2^L$ transfer matrix s.t.

$$\langle S_1, \ldots, S_L | \hat{T} | S_1', \ldots, S_L' \rangle = \prod_i e^{\beta J S_i S_i'} \times e^{\frac{1}{2}\beta H (S_i + S_i')} \times e^{\frac{1}{2}\beta J (S_i S_{i+1} + S_i' S_{i+1}')}$$

λ_+ - largest eigenvalue of $\hat{T}$

$H=0$.

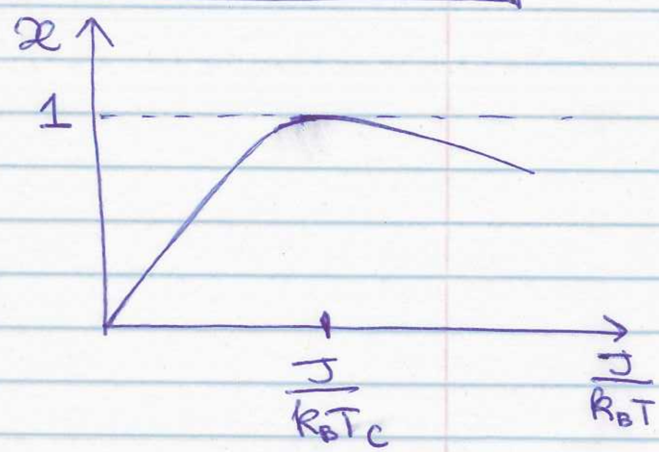
Onsager 1944 — brute-force diagonalization of $\hat{T}$.

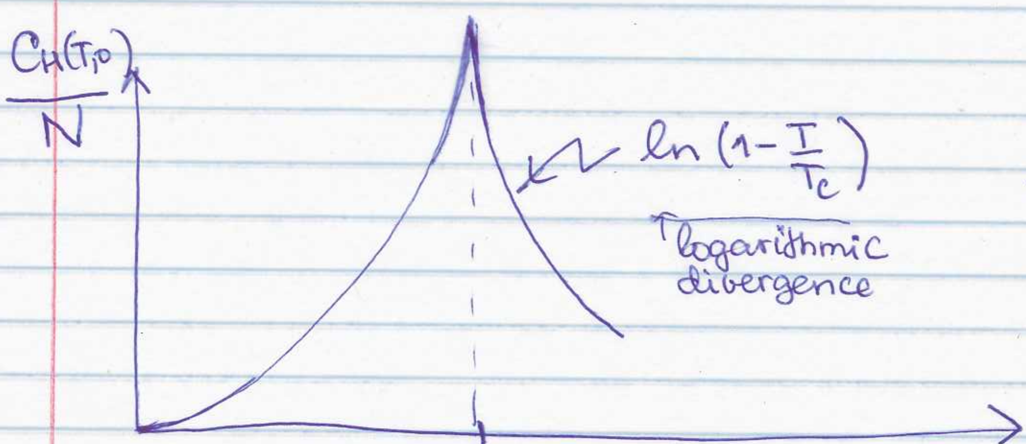
d=2 square lattice : Onsager solution

$H=0$ Onsager 1944

$$\boxed{\frac{F(T,0)}{N} = -\frac{1}{\beta}\ln\left(2\cosh(2\beta J)\right) - \frac{1}{2\pi\beta}\int_0^{\pi} d\phi \, \ln \frac{1}{2}\left(1+\sqrt{1-\varkappa^2\sin^2\phi}\right)}$$

$$\boxed{\varkappa = \frac{2\sinh(2\beta J)}{\cosh^2(2\beta J)}}$$





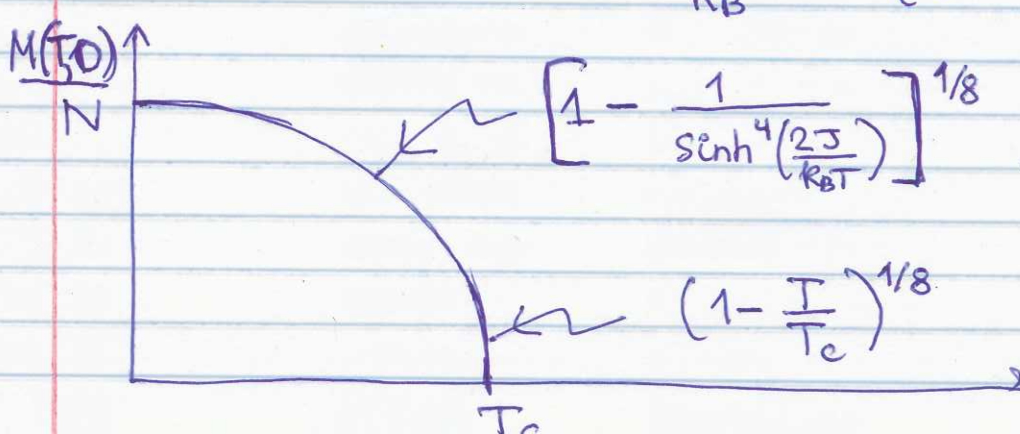
$\ln\left(1-\frac{T}{T_c}\right)$

logarithmic divergence

$\exists$ phase transition

$T_c \approx 2.3 \frac{J}{k_B}$ $\left(2\tanh^2 \frac{2J}{k_B T_c} = 1\right)$

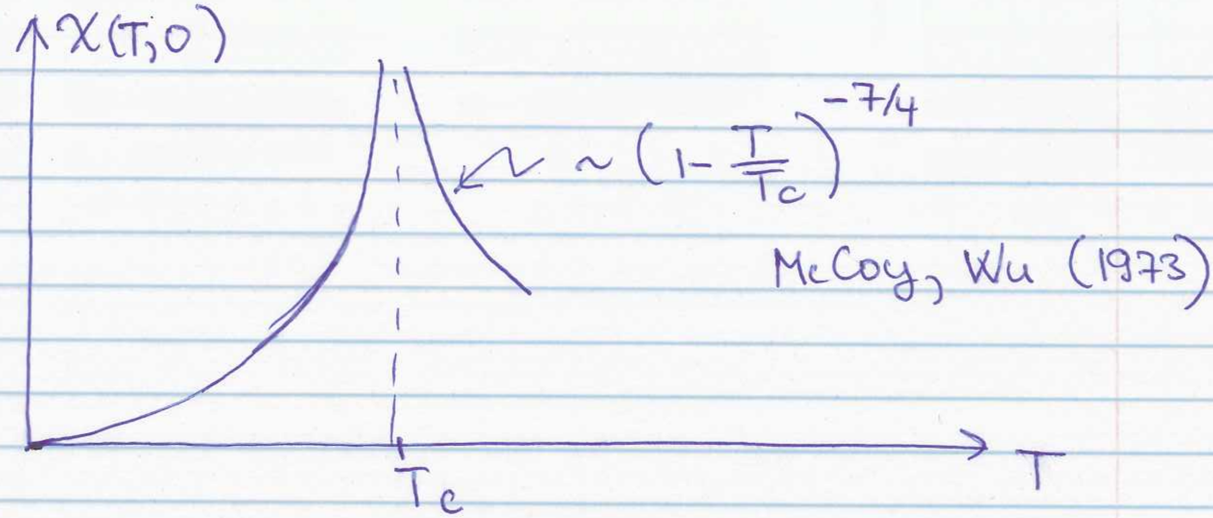
order parameter :



$\left[1-\frac{1}{\sinh^4\left(\frac{2J}{k_B T}\right)}\right]^{1/8}$, $T<T_c$

$\left(1-\frac{T}{T_c}\right)^{1/8}$

Onsager, Yang (1952)



<u>Remarks</u>: ⊛ Onsager's solution was very important for several reasons. First, it showed rigorously that stat. mech. approach can describe phase transitions (singularities in the thermodynamic quantities arise upon taking $N \to \infty$ limit). It also showed that meanfield exponents are wrong:

$$\alpha_{mf} = 0 \qquad \alpha_{exact} = 0$$

$$\beta_{mf} = \frac{1}{2} \qquad \beta_{exact} = \frac{1}{8}$$

$$\gamma_{mf} = 1 \qquad \gamma_{exact} = \frac{7}{4}$$

This forced to think what is wrong w meanfield →
→ fluctuations & Ginzburg criterion
→ development of theory of critical phenomena (Landau-Ginzburg-Wilson theory; scaling and renormalization theory)

⊛ "Modern solutions" – much simpler than original Onsager; Yang; McCoy-Wu
– by mapping to "free fermions"

⊛ 1+1d conformal field theory