Gaussian (free) theory. Gaussian fixed point

Renormalization group for Landau-Ginzburg theory
→ "Landau-Ginzburg-Wilson" perturbative RG

$$S = \int d^d r \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 + u m^4 + \dots \right]$$

– "ϕ^4" field theory.

As a build-up, consider gaussian theory

$$S_0 = \int d^d r \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 - hm \right]$$

– "free theory" (no interaction m^4)

– can analyze fully by doing Gaussian integrals

t=0 – critical point (Gaussian fixed point)

$$Z[t,h]_{K,} = \int [\mathcal{D} m] \exp\left[- \int d^d r \left(\frac{K}{2} (\nabla m)^2 + \underbrace{\frac{t}{2} m^2 - hm}_{\frac{t}{2}\left(m - \frac{h}{t}\right)^2 - \frac{h^2}{2t}} \right) \right]$$

$$= e^{+\mathrm{Vol} \frac{h^2}{2t}} \int [\mathcal{D} \tilde{m}] \exp\left[- \int d^d r \left(\frac{K}{2} (\nabla \tilde{m})^2 + \frac{t}{2} \tilde{m}^2 \right) \right]$$

$$m(r) = \frac{1}{\mathrm{Vol}} \sum_k m(k) e^{i \vec{k} \cdot \vec{r}}, \quad m(k) = \int d^d r \, m(r) e^{-i \vec{k} \cdot \vec{r}}$$

$$\int d^d r \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 \right] = \frac{1}{\mathrm{Vol}} \sum_k \left[\frac{K}{2} k^2 + \frac{t}{2} \right] |m_k|^2 =$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} (K k^2 + t) |m_k|^2$$

$$[\mathcal{D} m] = \prod_r dm_r = \prod_k{}' d(\mathrm{Re}\, m_k) d(\mathrm{Im}\, m_k)$$

$$Z[K,t,h] = \exp\left(+ \mathrm{Vol} \cdot \frac{h^2}{2t} \right) \prod_k{}' \left(\sqrt{\frac{\pi \cdot \mathrm{Vol}}{K k^2 + t}} \right)^2$$

Ornstein-Zernike analysis earlier: $\xi = \sqrt{\frac{K}{t}}$; $\nu = \frac{1}{2}$

$$f \equiv -\frac{1}{Vol}\ln Z = -\frac{h^2}{2t} + \frac{1}{2}\frac{1}{Vol}\sum_k \ln(Kk^2+t)$$

$$= -\frac{h^2}{2t} + \frac{1}{2}\int^{\Lambda} \frac{d^dk}{(2\pi)^d}\ln(Kk^2+t) =$$

$$= -\frac{h^2}{2t} + \underbrace{\frac{1}{2}\int^{\Lambda} \frac{d^dk}{(2\pi)^d}\ln\frac{Kk^2+t}{Kk^2}}_{\left(\sqrt{\frac{t}{K}}\right)^d \int^{\Lambda\sqrt{\frac{K}{t}}} \frac{d^d\tilde{k}}{(2\pi)^d}\ln\left(\frac{\tilde{k}^2+1}{\tilde{k}^2}\right)} + \underbrace{\frac{1}{2}\int^{\Lambda}\frac{d^dk}{(2\pi)^d}\ln(Kk^2)}_{\text{regular piece}}$$

$$= \underbrace{-\frac{h^2}{2t} + t^{d/2}\cdot(\#)}_{t^{d/2}\cdot\left(\# + \frac{h^2}{t^{1+d/2}}\right) = t^{d/2}\cdot g\left(\frac{h}{t^{(2+d)/4}}\right) = t^{2-\alpha} g_s\left(\frac{h}{t^{\Delta}}\right)} + \text{analytic piece}$$

$\Rightarrow \quad 2-\alpha = \frac{d}{2} = d\cdot\nu$ – hyperscaling is satisfied in the Gaussian model

$$\boxed{\begin{aligned}\alpha &= 2-d/2\\ \Delta &= \frac{d+2}{4}\end{aligned}} \Rightarrow$$

β – not meaningful (no $t<0$)

δ – not meaningful (At $t=0$, no control over any h)

$$\gamma = 2\Delta + \alpha - 2 = \frac{d+2}{2} - \frac{d}{2} = 1$$

– of course, could just differentiate $f \sim -\frac{h^2}{2t}$.

Correlation functions @ $t=0$

$$\langle m(r)m(0)\rangle \sim \frac{1}{r^{d-2}}, \quad \eta = 0.$$

Momentum shell RG

Renormalization group solution of the Gaussian model

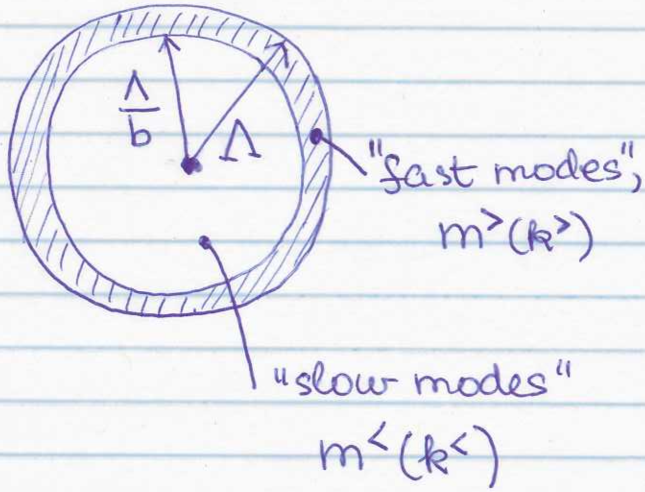
$$S_0 = \int d^d r \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2} m^2 - h m \right] =$$

$$= \frac{1}{Vol} \sum_{k} \frac{1}{2}(Kk^2+t)|m_k|^2 - h\, m_{k=0}$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{2}(Kk^2+t)|m_k|^2 - h\, m_{k=0}.$$

$$Z = \int [\mathcal{D}m]\, e^{-S_0}$$

1) Coarse-grain – "integrate out high momentum modes"



Here $S_0[m^<, m^>] =$

$= S_0[m^<] + S_0[m^>]$

and fast and slow modes do not affect each other

$$Z = \int \mathcal{D}m^< \, e^{-S_0[m^<]} \underbrace{\int \mathcal{D}m^> e^{-S_0[m^>]}}_{e^{-Vol.\, \delta f}}$$

$$\delta f = \frac{1}{2} \int_{\Lambda/b}^{\Lambda} \frac{d^d k}{(2\pi)^d} \ln(Kk^2+t)$$

– accumulates contribs. to the free energy from different length scales as we coarse-grain

2) Rescale the real-space coordinates or equivalently rescale momenta

$$x_{new} = \frac{x_{old}}{b} \qquad\qquad q_{new} = q_{old} \cdot b$$

Formally:

$$S_0[m^<] = \int^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \, \frac{1}{2}(Kq^2+t)|m^<_q|^2 - h\, m_{q=0} =$$

$$\underset{\boxed{q=\frac{\tilde q}{b}}}{=} \frac{1}{b^d} \int^{\Lambda} \frac{d^d \tilde q}{(2\pi)^d} \cdot \frac{1}{2}\left(K\tilde q^2 \cdot \frac{1}{b^2} + t\right) \left|m^<_{(\frac{\tilde q}{b})}\right|^2 - h\, m_{\tilde q=0}$$

3) Renormalize (rescale) the m-field so as to get the gradient term look the same:

$$\boxed{m_{(q)} = b^{\frac{d+2}{2}} \tilde m(\tilde q)}$$

$$\boxed{S_0[m^<] = \int^{\Lambda} \frac{d^d \tilde q}{(2\pi)^d} \, \frac{1}{2}(K\tilde q^2 + b^2 t)\, |\tilde m(\tilde q)|^2 - b^{\frac{d+2}{2}} h\, \tilde m_{q=0}}$$

Such rescaling of m is our choice so as not to keep track of the "flow" of K

$$\boxed{\begin{aligned} K' &= K \\ t' &= b^2 t \\ h' &= b^{\frac{d+2}{2}} h \end{aligned}} \leftarrow \text{RG for the Gaussian model}$$

Gaussian fixed point: $K^* = K, \ t^* = 0, \ h^* = 0$

Linearized equations around the fixed point
(do not really need to linearize here)

$$\delta t' = b^2 \delta t = b^{y_t} \delta t \quad , \qquad \boxed{y_t = 2}$$
$$\delta h' = b^{\frac{d+2}{2}} \delta h = b^{y_h} \delta h \quad , \qquad \boxed{y_h = \frac{d+2}{2}}$$

From the general discussion of RG:

$$\Rightarrow \quad \alpha = 2 - d/y_t = 2 - d/2 \quad \text{- hyperscaling}$$

$$\Delta = y_h / y_t = \frac{d+2}{4}$$

$$\nu = \frac{1}{y_t} = \frac{1}{2}$$

$$\gamma = 2\Delta + \alpha - 2 = \frac{2y_h}{y_t} - \frac{d}{y_t} = \frac{2y_h - d}{y_t} = 1.$$

Power of the fixed-point notion:

δt, δh grow under RG — "relevant" perturbations at the fixed point.

What about other perturbations, e.g. $L \int (\nabla^2 m)^2 d^d r =$

$$= L \int \frac{d^d k}{(2\pi)^d} (k^2)^2 |m_k|^2$$

Under RG : $\quad L' = L \frac{1}{b^d} \frac{1}{b^4} \left(b^{\frac{d+2}{2}} \right)^2 = L b^{-2}$

- decreases as we run the RG.
- irrelevant perturbation!

This is why we often ignored such higher-gradient terms from the start. (Can in principle keep them ($Kq^2 + Lq^4 + \ldots$), but find that long-distance behaviour is controlled entirely by exactly in the Gaussian model)

Quartic terms:

$$u_4 \int d^d r\, m(r)^4 = u_4 \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{d^d q_3}{(2\pi)^d} \frac{d^d q_4}{(2\pi)^d}$$

$$m(q_1) m(q_2) m(q_3) m(q_4) \underbrace{\int d^d r\, e^{i(q_1+q_2+q_3+q_4)\cdot r}}_{(2\pi)^d \delta^{(d)}(q_1+q_2+q_3+q_4)}$$

$$= u_4 \int đ^d q_1\, đ^d q_2\, đ^d q_3\; m(q_1) m(q_2) m(q_3) m(-q_1-q_2-q_3)$$

RG rescaling $\quad \frac{1}{b^{3d}} \quad \tilde{m}^< \quad \tilde{m}^< \quad \tilde{m}^< \quad \tilde{m}^< \quad \left(b^{\frac{d+2}{2}}\right)^4$

$$u_4' = u_4\, b^{4-d}$$

- shrinks if $d>4$ (irrelevant! – meanfield + gaussian fluct. is enough)

- grows if $d<4$ – relevant!

When we allow m^4 interactions, Gaussian fixedpoint obtains new relevant direction

Sixth-power terms m^6

$$v_6 \int d^d r\, m(r)^6 = v_6 \int đ\vec{q}_1\, đ\vec{q}_2 \ldots đ\vec{q}_5\; m(q_1) m(q_2) \ldots m(q_5) \times$$
$$\times\, m(-q_1-q_2-\ldots-q_5)$$

$$v_6' = v_6 \frac{1}{b^{5d}} \cdot \left(b^{\frac{d+2}{2}}\right)^6 = v_6\, b^{6-2d}$$ – shrinks if $d>3$

u_4 is important "earlier" than v_6.

"tree level" analysis