Superfluid $^4\text{He}$

Bose-Einstein condensation $\sim$ superfluid $^4\text{He}$

\[ T_{\text{BEC}} = \frac{\hbar^2}{2\pi m k_B} \left( \frac{n}{(2\pi i)^{3/2}} S_{3/2}^{1/2}(1) \right)^{1/3} \sim 3.14 \text{ K} \]

$T_{\text{C}}^{\text{exp}} = 2.18 \text{ K}$

However, the situation in $^4\text{He}$ is much more complicated.

Contrast ideal Bose gas vs real $^4\text{He}$

Ideal Bose gas

- Specific heat

\[ C_v \]

$T_{\text{C}} = 3.14 \text{ K}$

$T_{\text{c}} = 2.18 \text{ K}$

"\(\lambda\) transition": infinite slope at $T_{\text{C}}$, but no divergence

[Older literature - logarithmic divergence]
**Ideal Bose gas**

- **Phase diagram**

  - In the BEC phase, cannot have higher pressure; is decreasing volume, just more molecules will go to $T=0$ state, which exerts no pressure.

  - Pressure dependence of $T_c$:
    
    $T_c \sim g^{\frac{1}{3}}$

    \[
    \frac{dT_c}{dg} < 0
    \]

    \[
    \frac{dT_c}{dg} \sim g^{-\frac{2}{3}} > 0
    \]

    $T_c \uparrow$ for $g \uparrow$

  - $T_c \downarrow$ for $g \uparrow$ on $\lambda$-line

**Real $^4$He**

- Solid
- Normal fluid
- Superfluid

\[ p = \frac{k_B T (2s+1) \langle S_{\frac{5}{2}}^z \rangle}{\chi^3(T)} \sim T^{\frac{5}{2}} \]

\( \lambda \) transition (line of 2nd order transitions)

2.18 K

- Neglects interactions!
  - Of course, one cannot infinitely compress the $T=0$ liquid and feel no pressure.
• Condensate fraction ($\langle n_s \rangle = \# \text{ atoms in the superfluid phase}$)

$$
\frac{\langle n_s \rangle}{\langle N \rangle} = 1 - \left( \frac{T}{T_c} \right)^{3/2}
$$

Linear slope

$$
1 - \left( 1 + \frac{T - T_c}{T_c} \right)^{3/2} = -\frac{3}{2} \frac{T - T_c}{T_c}
$$

[Here assuming $n_s = n_p = 0$]

Andronikashvili experiment:

Liquid $^4$He

disks 0.2 mm apart

$^4$He is dragged along

$\Rightarrow$ moment of inertia $\propto \frac{\langle n_{\text{normal}} \rangle}{\langle n_{\text{s}} \rangle} = 1 - \frac{\langle n_s \rangle}{\langle N \rangle}$

• Zero momentum fraction ($\langle n_0 \rangle = \# \text{ atoms with } \vec{p} = 0$) at $T=0$

$\langle n_0 \rangle = 1$

$\Psi_0(\vec{r}_1, ..., \vec{r}_N) = \text{const}$

$= \text{Perm} \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right)$

$\langle n_0 \rangle \approx 0.1$ (from neutron Compton scattering)

$\Psi_0(\vec{r}_1, ..., \vec{r}_N) \approx \text{const} \cdot \begin{cases} 1 & \text{if } \text{He-He repulsion} \\ \text{otherwise} \end{cases}$

$\Psi_0(\vec{r}_1, ..., \vec{r}_N) \sim e^{-\frac{2}{\lambda} (\vec{r}_1 - \vec{r}_2)}$ "Jost wave"
Energy spectrum of elementary excitations

\[ E_p^2 = \frac{p^2}{2m} \]

from inelastic neutron scattering

\[ c \rho \nu_{\text{sound}} \]

SUPERFLUIDITY

NO SUPERFLUIDITY

\[ \nu_{\text{crit}} = 60 \text{ m/s} \]

Landau's argument for superfluidity

\(*\) Actually currently believed to not be fundamental;
fundamental - generalized rigidity (superfluid stiffness)
when U(1) symmetry is spontaneously broken

Consider whether moving container walls can drag the \(^4\text{He}\) (e.g. moving plates in Andronikashvili experiment)

\[ \text{\#\#\#\#\#} \text{ Vessel: momentum } \overrightarrow{P} \]

\[ \text{\#\#\#\#\#} \text{ mass } M \]

\[ \text{\#\#\#\#\#} \text{ velocity } \vec{v} = \frac{\overrightarrow{P}}{M} \]

\[ \text{\#\#\#\#\#} \]

To drag \(^4\text{He}\), need to create excitations, say momentum \(\overrightarrow{q}\)
energy \(E_q\)

This is possible only if

\[ \frac{\overrightarrow{P}^2}{2M} = \frac{(\overrightarrow{P} - \overrightarrow{q})^2}{2M} + E_q \]

\[ \text{momentum and energy conservation} \]
\[ 0 = \frac{\vec{q}^2}{2m} - \frac{\vec{v} \cdot \vec{J} \cdot \vec{q}}{M} + \varepsilon_q, \]

\[ \varepsilon_q = \vec{v} \cdot \vec{q} - \frac{\vec{q}^2}{2M} \]

negligible \((M \text{ very large})\)

\[ \Rightarrow \varepsilon_q \leq |\vec{v}| \cdot |\vec{q}| \]

need

\[ |\vec{v}| \geq \frac{\varepsilon_q}{|\vec{q}|} \geq \text{critical} \quad - \text{condition for flow with friction} \]

(Vessel at rest and moving He → go to the rest frame of He; i.e. \(1/\nu < \text{critical} \quad - \text{no drag}\)).

Ideal Bose gas:

Real \({}^4\text{He}:

\[ \frac{\varepsilon_q}{|\vec{q}|} = \frac{1}{2m} - \text{can be arbitrary small} \]

- always excitation

- always friction

Origin of phonon-like excitations - Bogoliubov theory of \({}^4\text{He} \text{ excitation spectrum} - \text{subject of many-body physics (condensed matter physics). Here will only say that it is similar to origin of phonon excitations in solids: } c_{\text{phonon}}(\omega) \sim \omega^{3/2} \] - broken continuous symmetry.

When solid forms, translational symmetry is broken → phonon

When a superfluid forms, global U(1) symmetry is broken → phonon

Loosely - \(c \propto (\Phi_1 - \Phi_3)\); global U(1): \(\Phi_1 \rightarrow \Phi_1 + \alpha\)
Statistical mechanics of superfluids

\( T = 0 \) - ground state

**low \( T \) - low-energy excitations** determine low-\( T \) properties

Example: 1) Ideal Bose gas with fixed number of particles

\( T = 0 \) ground state: all bosons in \( k = 0 \)

\[
\begin{align*}
N & \quad \mathbf{k} = 0 \\
N-1 & \quad \mathbf{k} = \mathbf{k}_1 \\
N-2 & \quad \mathbf{k} = \mathbf{k}_1, \mathbf{k}_2
\end{align*}
\]

Excitations: \( \mathbf{R} \) carries momentum \( \mathbf{p} = \frac{\hbar^2 \mathbf{k}^2}{2m} \) and energy \( \epsilon_{\mathbf{p}} = \frac{\hbar^2 \mathbf{k}^2}{2m} \)

General

\[ \{ N_0, n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots \} \quad \text{label simply by} \quad \{ n_k \neq 0 \} \]

\[
\begin{align*}
\mathbf{P} &= \sum_k \mathbf{k} n_k \\
E &= E_g + \sum_k \epsilon_{\mathbf{k}} n_k
\end{align*}
\]

of course, need \( \sum n_k \leq N \)

but for low \( T \) can be safely satisfied.

Canonical partition sum

\[
Z_N = \sum_{\sum n_k \leq N} e^{-\left( E_g + \sum_k \epsilon_{\mathbf{k}} n_k \right)} \approx e^{E_g} \prod_{k=0}^{\infty} \left( \sum e^{-\beta \epsilon_{\mathbf{k}}} \right)
\]

\[ \text{ignoring} \]

\[ Z = e^{-\beta E_g} \prod_{k} \frac{1}{1 - e^{-\beta E_k}} \quad \ln Z = -\beta E_g - \sum_{k} \ln(1 - e^{-\beta E_k}) \]

\[ F = -k_B T \ln Z = E_g + k_B T \sum_{k} \ln(1 - e^{-\beta E_k}) \]

\[ \langle U \rangle = \langle E \rangle - \frac{\partial \ln Z}{\partial \beta} = E_g + \sum_{k} \frac{1}{1 - e^{-\beta E_k}} \cdot (-1) e^{-\beta E_k} \]

\[ = E_g + \sum_{k} \frac{\varepsilon_k}{e^{\beta E_k} - 1} \]

\[ \langle n_k \rangle = \frac{1}{e^{\beta E_k} - 1} \]

- Bose-Einstein distribution with \( \mu = 0 \) - here by explicit calculation. Roughly, because the total number \( \sum_{k} n_k \) is not fixed!

2) Vibrations of solids: by explicit diagonalization of the harmonic crystal problem, obtained collection of decoupled oscillators

\[ E = \sum_{k, s} \hbar \omega_s(k) \left( n_{k, s} + \frac{1}{2} \right) \]

\[ = E_{gs} + \sum_{k, s} n_{k, s} \varepsilon_{k, s} \]

"zero-point energy" \( \hbar \omega_s(k) \) - energy of an elementary excitation - "phonon".

Since \( E_{gs} \) is just an overall constant, for the purposes of thermodynamics can focus entirely on the \( E - E_{gs} = \sum_{k, s} n_{k, s} \varepsilon_{k, s} \) "gas of elementary excitations"
Ground-state can be quite complicated (here, each oscillator is in the ground state wave function, and this still needs to be translated to the original ion coordinates). But for the purposes of thermodynamic calculations, only need to know low-energy excitations, and here can label them as "particle-like".

Ground-state \equiv \text{no excitations} \equiv \text{vacuum of these particles}.

Calculation of thermodynamics — precisely as already described.

\[
U = E_{gs} + \sum \frac{\mathbf{E}_k}{k_B T} \frac{e^{\beta \mathbf{E}_k}}{e^{\beta \mathbf{E}_k} - 1}
\]

this is calculation in canonical ensemble (fixed number of ions or fixed boson number), while the result looks like "grand canonical ensemble with \( \mu = 0 \)" because it is for new "particles" = elementary excitations, whose total number is not restricted when we want to construct the whole spectrum. "Excitations can pop up and disappear" like (thermally excited) photons in vacuum.

Vibrations in solids: \( \omega_s(k) = c_s |\mathbf{k}| \)

\[ C_v \sim T^3 \text{ in 3d} \]

specific heat at low temperature

\[
(U \sim \int d^d k \frac{\mathbf{E}_k}{e^{\beta \mathbf{E}_k}} \sim T^{d+1} \int d^d k \frac{1}{e^{|\mathbf{k}|^2 / k_B T - 1}})
\]

\[ C_v \sim T_d \]
Elementary excitations in superfluid $^4$He

$$E = E_G + \sum_k \varepsilon_k n_k$$

Bogoliubov spectrum

$$\varepsilon_k = \sqrt{(\hbar c k)^2 + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

for weakly interacting Bose gas

$$\approx \hbar c |k|$$ for very small $k$

- similar to phonons in 3d!

Thermodynamics - very similar to low-$T$ properties of vibrations in solids

$$\Rightarrow C \sim T^3 \quad \left(\text{as opposed to } T^{3/2} \text{ for } \varepsilon \sim k^2\right)$$

Landau's insight - focus on elementary excitations; notion of quasiparticles and simplicity of the excitation spectrum.

In fact, it was $C \sim T^3$ observed experimentally that prompted Landau to postulate linearly dispersing elementary excitations $\varepsilon_k = \hbar c |k|$

(and then Landau's argument for superfluidity)

Microscopic origin - Bogoliubov (47-48) quantum mechanical treatment of weakly interacting Bose gas

$$\varepsilon_k = \sqrt{(\hbar c k)^2 + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

from interactions (many-body physics; CMP course)

Modern perspective: Superfluidity - $U(1)$ symmetry breaking - must have Goldstone mode = phonon

Will discuss order parameters and symmetry breaking next term.