Superfluid ^{4}He

Bose-Einstein condensation $\underset{?}{\sim}$ superfluid ^{4}He

$$\left.\begin{matrix} m_{^4He} \\ \text{experimental} \\ \rho_{liq, He} \end{matrix}\right] \rightarrow T_c^{BEC} = \frac{h^2}{2\pi m k_B}\left(\frac{n}{(2s+1)\,\zeta^+_{3/2}(1)}\right)^{1/3} \approx$$

$$\approx 3.14\,K$$

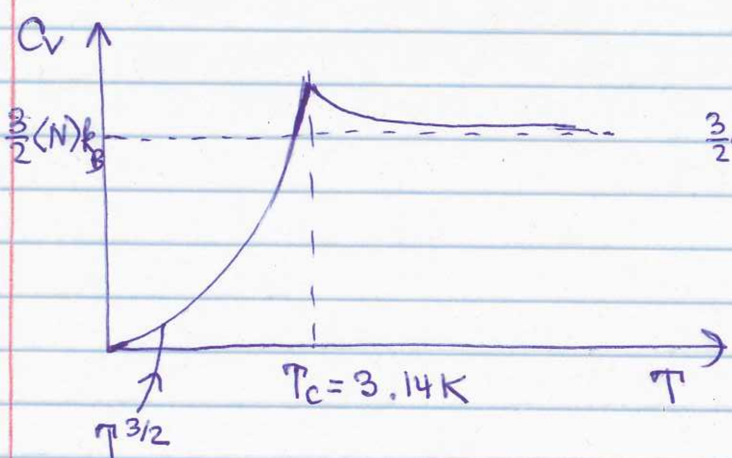
$$T_c^{exp} = 2.18\,K$$

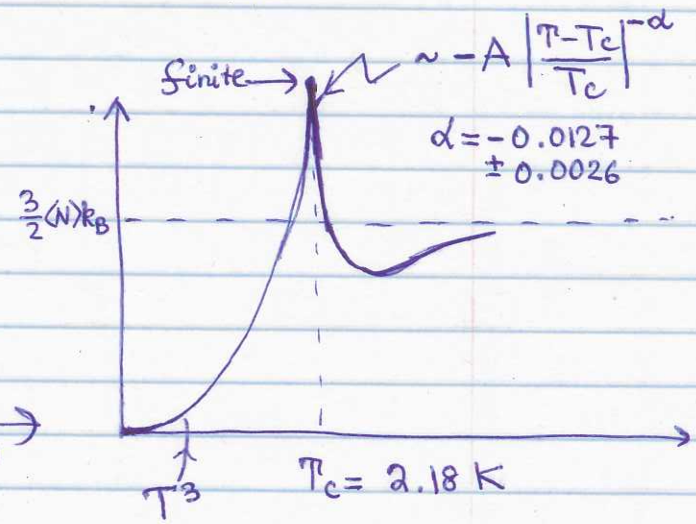
However, the situation in ^{4}He is much more complicated

Contrast ideal Bose gas vs real ^{4}He

Ideal Bose gas | ^{4}He

- Specific heat

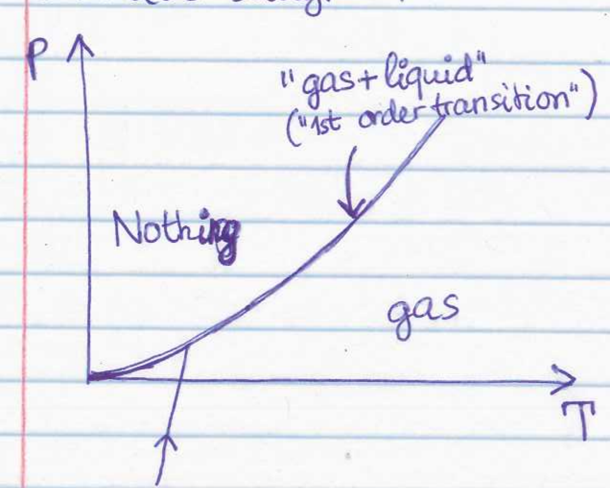


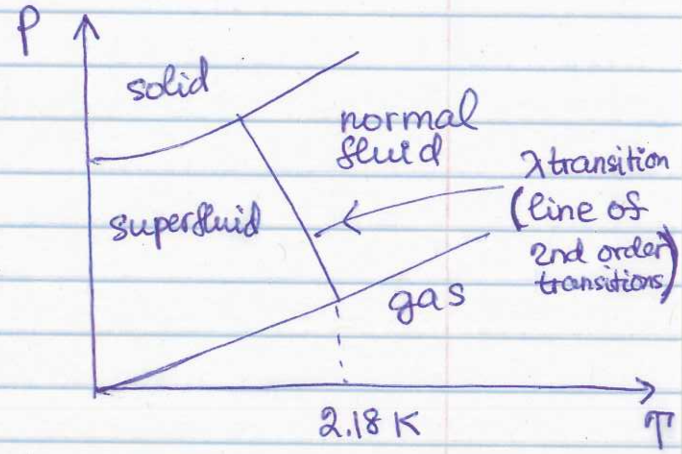


"λ transition"; infinite slope at T_c, but no divergence [older literature – logarithmic divergence]

Ideal Bose gas | Real ^4He

- Phase diagram





$$\underbrace{p = k_B T \frac{(2S+1)}{\lambda^3(T)} f^+_{5/2}(1)}_{\text{in the BEC phase}} \sim T^{5/2}$$

cannot have higher pressure: if decreasing volume, just more molecules will go to $\vec{k}=0$ state which exerts no pressure

neglects interactions! Of course, one cannot infinitely compress the $\vec{k}=0$ liquid and feel no pressure

- Pressure dependence of T_c:

$$T_c \sim \rho^{1/3}$$

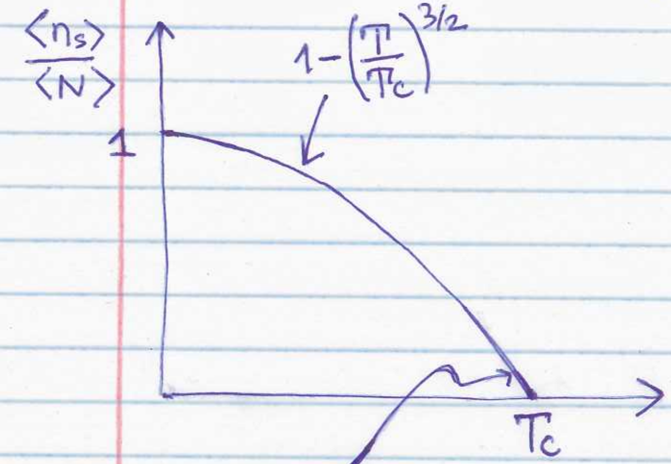
$$\frac{dT_c}{d\rho} \sim \rho^{-2/3} > 0$$

$T_c \uparrow$ for $\rho \uparrow$

$$\frac{dT_c}{d\rho} < 0$$

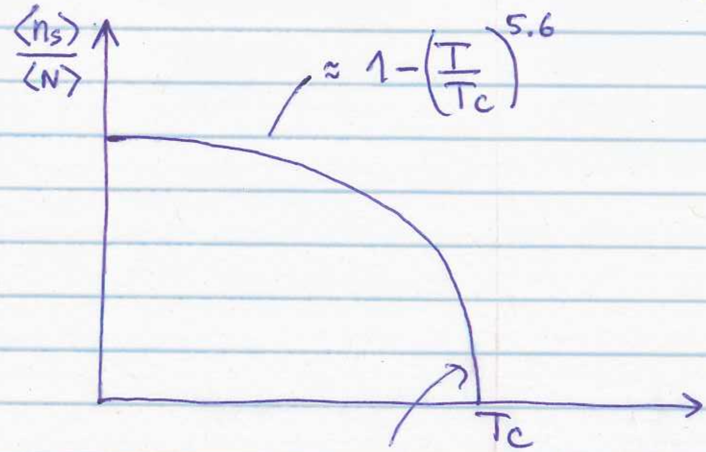
$T_c \downarrow$ for $\rho \uparrow$ on λ-line ($\rho\uparrow$ for $p\uparrow$) – cf. phase diagram

- Condensate fraction ($\langle n_s \rangle$ = # atoms in the superfluid phase)



$1-\left(1+\frac{T-T_c}{T_c}\right)^{3/2} \approx -\frac{3}{2}\frac{T-T_c}{T_c}$] - linear slope

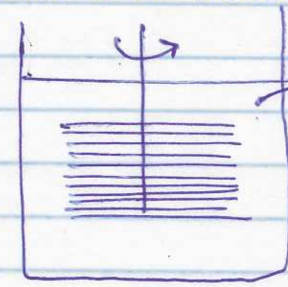
[Here assuming $n_s = n_{\vec{p}=0}$]



$\sim \left(\frac{T_c - T}{T_c}\right)^{\nu}$,

$\nu = 0.674$
± 0.001

Andronikashvili experiment:



liquid ^{4}He

disks 0.2 mm apart
normal ^{4}He is dragged along

$\Rightarrow$ moment of inertia $\propto \frac{\langle n_{normal} \rangle}{\langle N \rangle} =$

$= 1 - \frac{\langle n_s \rangle}{\langle N \rangle}$

- Zero momentum fraction ($\langle n_0 \rangle \equiv$ # atoms with $\vec{p}=0$) at $T=0$ (quantum ground state)

$$\frac{\langle n_0 \rangle}{\langle N \rangle} = 1$$

$$\Psi_0(\vec{r}_1, \dots \vec{r}_N) = \text{const} = \text{Perm}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \end{pmatrix}$$

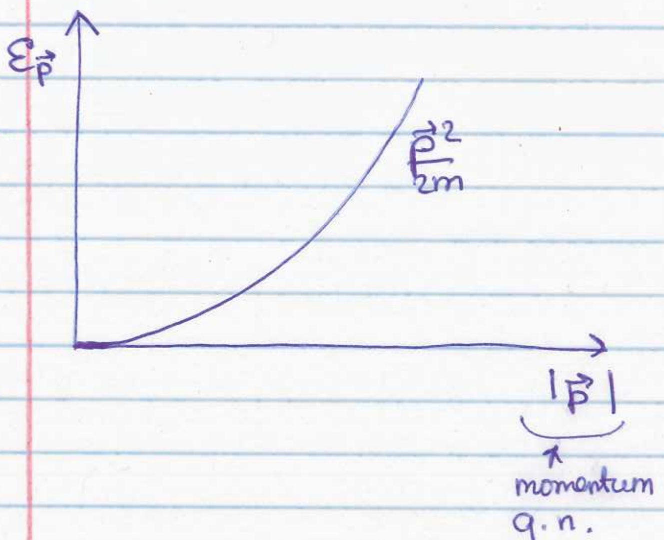
$$\frac{\langle n_0 \rangle}{\langle N \rangle} \approx 0.1$$ (estimate from neutron Compton scattering)

$$\Psi_0(\vec{r}_1, \dots, \vec{r}_N) \approx \text{const} \cdot \begin{cases} 0 & \text{if } |r_i - r_j| < a \\ 1 & \text{otherwise} \end{cases}$$

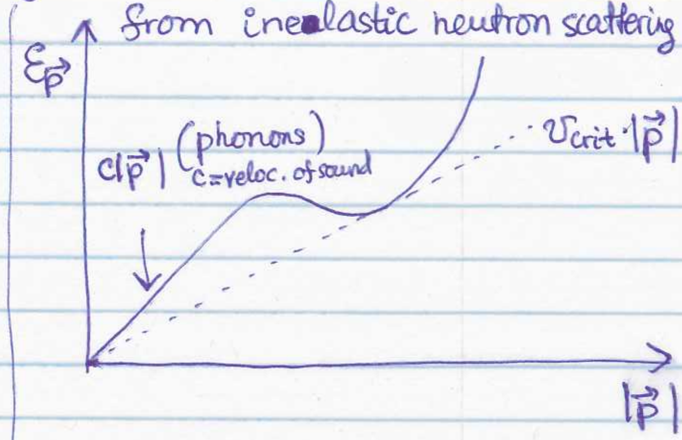
He-He repulsion

$$\Psi_0(\vec{r}_1, \dots, \vec{r}_N) \sim e^{-\sum_{ij} u(r_i - r_j)}$$ – "Jastrow"

- Energy spectrum of elementary excitations



NO SUPERFLUIDITY



SUPERFLUIDITY!

$$v_{crit} = 60 \frac{m}{s}$$

Landau's argument for superfluidity

(* actually currently believed to not be fundamental; fundamental – generalized rigidity (superfluid stiffness) when $U(1)$ symmetry is spontaneously broken)

Consider whether moving container walls can drag the ^4He (e.g. moving plates in Andronikashvili experiment)

vessel : momentum $\vec{P}$
mass M
velocity $\vec{v} = \frac{\vec{P}}{M}$

^4He at rest

To drag ^4He, need to create excitations, say momentum $\vec{q}$
energy $\varepsilon_{\vec{q}}$

This is possible only if

$$\frac{\vec{P}^2}{2M} = \frac{(\vec{P}-\vec{q})^2}{2M} + \varepsilon_{\vec{q}} \quad \Big\} \text{ momentum and energy conservation}$$

$$0 = \frac{\vec{q}^{\,2}}{2m} - \frac{\vec{P}\cdot\vec{q}}{M} + \varepsilon_{\vec{q}}\,,$$

$$\varepsilon_{\vec{q}} = \vec{V}\cdot\vec{q} - \frac{\vec{q}^{\,2}}{2M}$$

negligible (M very large)

$$\Rightarrow \quad \varepsilon_{\vec{q}} = \vec{v}\cdot\vec{q} \le |\vec{v}|\cdot|\vec{q}|$$

need

$$\Rightarrow \quad |\vec{v}| \ge \frac{\varepsilon_{\vec{q}}}{|\vec{q}|} \ge v_{\text{critical}}$$ – condition for flow with friction

(Vessel at rest and moving He – go to the rest frame of He; if $|v| < v_{\text{crit.}}$ – no drag).

Ideal Bose gas:

$$\frac{\varepsilon_{\vec{q}}}{|\vec{q}|} = \frac{|\vec{q}|}{2m}$$ – can be arbitrary small

– always excitation

→ always friction

Real ^{4}He:

if $v < v_{\text{critical}}$ –

– no excitations

Origin of phonon-like excitations – Bogoliubov theory of ^{4}He excitation spectrum – subject of many-body physics (condensed matter physics). Here will only say that it is similar to origin of phonon excitations in solids: $\omega_{\text{phonon}}(k) \sim c|\vec{k}|$

– broken continuous symmetry.

When a solid forms, translational symmetry is broken → phonon

Will discuss much more later in the course

When a superfluid forms, global $U(1)$ symmetry is broken → phonon

Loosely $\quad -\cos(\varphi_i - \varphi_j)$; global $U(1)$: $\varphi_i \to \varphi_i + \alpha$

Statistical mechanics of superfluids

$T=0$ – ground state

low T – low-energy excitations determine low-T properties

Example: 1) Ideal Bose gas with fixed number of particles

$T=0$ ground state: all bosons in $\vec{k}=\vec{0}$

$$\overset{N}{\bullet}_{k=0}$$

Excitations:

$$\overset{N-1}{\bullet}_{k=0} \quad \cdot_{\vec{k}}$$ carries momentum $\vec{k}$ and energy $\varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$

$$\overset{N-2}{\bullet} \quad \cdot_{k_1} \quad \cdot_{k_2}$$ momentum $\vec{k}_1+\vec{k}_2$ and energy $\varepsilon_{k_1}+\varepsilon_{k_2}$

General

$$|N_0, n_{\vec{k}_1}, n_{\vec{k}_2}, \dots\rangle$$ – label simply by $\{n_{k\neq 0}\}$

$$\vec{P} = \sum_k \vec{k}\, n_k$$

$$E = E_G + \sum_{\vec{k}} \epsilon_{\vec{k}} n_{\vec{k}}$$

of course, need $\sum_k n_k \leq N$ but for low T can be safely satisfied.

Canonical partition sum

$$Z_N = \sum_{\substack{\{n_k\} \\ \sum_k n_k \leq N}} e^{-(E_G + \sum_k n_k \epsilon_k)\beta} \approx e^{-\beta E_G} \prod_k \left(\sum_{n_k=0}^{\infty} e^{-\beta n_k \epsilon_k}\right)$$

ignoring

$$Z = e^{-\beta E_G} \prod_k \frac{1}{1-e^{-\beta \varepsilon_k}} \;;\; \ln Z = -\beta E_G - \sum_k \ln\left(1-e^{-\beta\varepsilon_k}\right)$$

$$F = -k_B T \ln Z = E_G + k_B T \sum_k \ln\left(1-e^{-\beta\varepsilon_k}\right)$$

$$U = \langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = E_G + \sum_k \frac{1}{1-e^{-\beta\varepsilon_k}} \cdot (-1) e^{-\beta\varepsilon_k} \cdot (-\varepsilon_k)$$

$$\boxed{= E_G + \sum_k \frac{\varepsilon_k}{e^{\beta\varepsilon_k}-1}} = E_G + \sum_k \varepsilon_k \langle n_k \rangle$$

$$\boxed{\langle n_k \rangle = \frac{1}{e^{\beta\varepsilon_k}-1}}$$

— Bose-Einstein distribution with $\underbrace{\mu = 0}$ — here by explicit calculation.

roughly, because the total number $\sum_{k\neq 0} n_k$ is not fixed!

2) Vibrations of solids: by explicit diagonalization of the harmonic crystal problem, obtained collection of decoupled oscillators

$$E = \sum_{\vec{k},\, s\text{ - "polarization"}} \hbar\omega_s(\vec{k}) \left(n_{s,\vec{k}} + \frac{1}{2}\right)$$

$$= \underbrace{E_{GS}}_{\text{"zero-point energy"}} + \sum_{\vec{k},s} n_{s,\vec{k}} \underbrace{\epsilon_{s\vec{k}}}_{\hbar\omega_s(\vec{k})}$$

$\hbar\omega_s(\vec{k})$ — energy of an elementary excitation — "phonon".

Since E_{GS} is just an overall constant, for the purposes of thermodynamics can focus entirely on the

$$E - E_{GS} = \sum_{k,s} n_{k,s} \epsilon_{k,s} \Big\} \leftarrow \text{"gas of elementary excitations"}$$

Ground-state – can be quite complicated (here, each oscillator is in the ground state wavefunction, and this still needs to be translated to the original ion coordinates). But for the purposes of thermodynamic calculations, only need to know low energy excitations, and here can label them as "particle-like".

Ground-state $\equiv$ no excitations $\equiv$ vacuum of these particles.

Calculation of thermodynamics – precisely as already described.

$$\left(U = E_{GS} + \sum_k \frac{\varepsilon_k}{e^{\beta\varepsilon_k} - 1} \right.$$

this is calculation in canonical ensemble (fixed number of ions or fixed boson number), while the result looks like $\underbrace{\text{"grand canonical ensemble with } \mu = 0\text{"}}$

because it is for new "particles" = elementary excitations, whose total number is not restricted when we want to construct the whole spectrum.

"Excitations can pop up and disappear", like (thermally excited) photons in vacuum.

Vibrations in solids : $\omega_s(k) = c_s |\vec{k}|$

$$\Rightarrow \quad C_V \sim T^3 \text{ in 3d}$$

specific heat at low temperature

$$\left(U \sim \int d^d k \, \frac{\hbar c |k|}{e^{\hbar c |k| / k_B T} - 1} \sim T^{d+1} \int d^d \tilde{k} \, \frac{|\tilde{k}|}{e^{|\tilde{k}|} - 1} \right)$$

$$C_V \sim T^d$$

Elementary excitations in superfluid ^4He

$$E = E_G + \sum_k \varepsilon_k n_k$$

Bogoliubov spectrum for weakly interacting Bose gas

$$\varepsilon_k \approx \sqrt{(\hbar c k)^2 + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

$\approx \hbar c |\vec{k}|$ for very small $\vec{k}$

– similar to phonons in 3d!

Thermodynamics at low T – very similar to low-T properties of vibrations in solids

$\Rightarrow C \sim T^3$] ← observed experimentally! (as opposed to $T^{3/2}$ for $\varepsilon \sim k^2$)

Landau's insight – focus on elementary excitations; notion of quasiparticles and "simplicity" of the excitation spectrum.

In fact, it was $C \sim T^3$ observed experimentally that prompted Landau to postulate linearly dispersing elementary excitations $\varepsilon_k \approx \hbar c |\vec{k}|$ (and then Landau's argument for superfluidity).

Microscopic origin – Bogoliubov (47–48) quantum mechanical treatment of weakly interacting Bose gas.

$$\varepsilon_k \approx \sqrt{\underbrace{(\hbar c k)^2}_{\text{from interactions}} + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

(many-body physics; CMP course)

Modern perspective: superfluidity – U(1) symmetry breaking – must have Goldstone mode = phonon

Will discuss order parameters and symmetry breaking next term.