Ideal Bose gas. Bose-Einstein condensation

General properties:

μ is determined from $\longrightarrow$ and can range $-\infty < \mu < \varepsilon_1$

$$\boxed{\begin{aligned} -pV = \Omega &= k_B T \sum_\alpha \ln\left(1 - e^{-\beta(\varepsilon_\alpha - \mu)}\right) \\ N &= \sum_\alpha \langle n_\alpha \rangle = \sum_\alpha \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} - 1} \\ U &= \sum_\alpha \frac{\varepsilon_\alpha}{e^{\beta(\varepsilon_\alpha - \mu)} - 1} \end{aligned}}$$

Non-relativistic gas in 3d:

$$\alpha = \{\vec{k}, \sigma\}$$

$$\varepsilon_\alpha = \frac{\hbar^2 k^2}{2m}$$

Formulas upon $\sum_k \to V\int \frac{d^3k}{(2\pi)^3}$

$$(\circledast)\quad \boxed{\begin{aligned} N &= \frac{V(2S+1)}{\lambda^3} f^+_{3/2}(z) \\ U &= \frac{3}{2} pV \\ \frac{p}{k_B T} &= \frac{(2S+1)}{\lambda^3} f^+_{5/2}(z) \end{aligned}}$$

$$f^+_\nu(z) = \frac{1}{\underbrace{(\nu-1)!}_{\Gamma(\nu)}} \int_0^\infty dy \frac{y^{\nu-1}}{z^{-1} e^y - 1}$$

High-temperature behaviour – considered generally (boson/fermion) earlier (bosons & ferms differ only "slightly" – the sign of the 1st correction)

Low-temperature behaviour – behaviour of fermions and bosons is drastically different.

$T = 0$ – put all bosons into the lowest energy state, i.e. macroscopic occupation of a single state!

$T > 0$ but small – in 3d, macroscopic occupation of the state $\varepsilon_{p=0}$ still persists.

Consider  $n = \frac{N}{V} = \frac{(2S+1)}{\lambda^3} f^+_{3/2}(z) = (2S+1)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} f^+_{3/2}(z)$

Notation:

$f^+_m(1) = \zeta_m$

Riemann

- "zeta function"

$\zeta(m) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{t^{m-1}}{e^t - 1} dt$

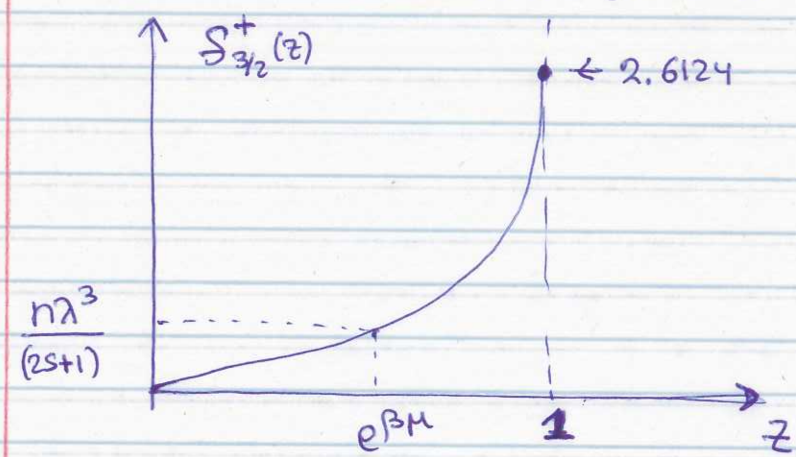
Must have

$\mu < 0$

$z = e^{\beta\mu} \to 1$ when $\mu \to 0$

$$f^+_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{z^{-1} e^y - 1} \longrightarrow$$

$$\longrightarrow \frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty dy \frac{y^{1/2}}{e^y - 1}}_{\text{converges at small } y} = 2.6124 \equiv \zeta_{3/2}$$



But $\frac{n\lambda^3}{(2S+1)} = \frac{n}{(2S+1)} \cdot \left(\frac{h}{\sqrt{2\pi m k_B T}} \right)^3$ - continues to increase without limit as we lower T.

- equation $\frac{n\lambda^3}{(2S+1)} = f^+_{3/2}(z)$

has no solution below T_c s.t. $\boxed{\frac{n\lambda^3(T_c)}{(2S+1)} = f^+_{3/2}(1)}$

$$T_c = \frac{h^2}{2\pi m k_B} \left(\frac{n}{(2S+1)\zeta_{3/2}} \right)^{2/3}$$

For $T > T_c$ - normal Bose gas described by ⊛.

$T < T_c$ - Bose-Einstein condensation forms

The difficulty with (*) for $T<T_c$ is that for $\mu \to 0$,

$\sum_{\vec{k}} \to V\int \frac{d^3k}{(2\pi)^3}$ is no longer accurate –

– need to treat separately $\vec{k}=0$ and $\vec{k}\neq 0$

$$n_{\vec{k}=0} = \frac{1}{e^{\beta(0-\mu)}-1} = \frac{1}{e^{-\beta\mu}-1} \approx \frac{1}{-\beta\mu} \to \infty \quad \text{as } \mu \to 0-$$

$$\boxed{N = n_{k=0} + \sum_{k\neq 0} n_k = n_{k=0} + \underbrace{V\int \frac{d^3k}{(2\pi)^3} n_k}_{\text{converges}}}$$

$$n_{k=0} = N - V\int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\hbar^2k^2/2m}-1}$$

↑ condensate fraction

No other $\vec{k}\neq 0$ gets macroscopic occupation:

$$\mu \sim \frac{1}{-\beta N} \sim \frac{1}{-\beta L^3}, \quad \mu<0$$

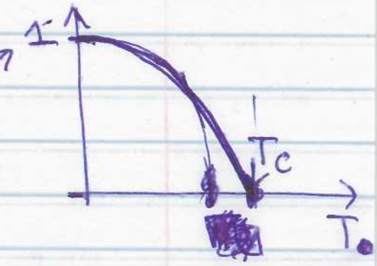
$$\varepsilon_{k\neq 0} - \mu \sim \frac{\hbar^2 k_{min}^2}{2m} - \mu \sim \frac{\hbar^2 (2\pi/L)^2}{2m} - \mu$$

$$\frac{1}{e^{\beta(\varepsilon_{k\neq 0}-\mu)}-1} \approx \frac{1}{\beta(\varepsilon_{k\neq 0}-\mu)} \sim \underbrace{\frac{1}{\beta}\frac{L^2}{\frac{\hbar^2}{2m}}}_{\text{sub-macroscopic!}} \ll L^3$$

$$N_{k>0}^{(T<T_c)} = V \frac{(2S+1)}{\lambda^3(T)} f_{3/2}^{+}(1)$$

$$\frac{N_{k>0}(T<T_c)}{N} = \frac{N_{k>0}(T<T_c)}{N_{k>0}(T_c)} = \frac{\lambda^3(T_c)}{\lambda^3(T)} = \left(\frac{T}{T_c}\right)^{3/2}$$

$$\boxed{\frac{n_{k=0}}{N} = 1 - \frac{N_{k>0}}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$



Thermodynamics of the BEC state $T<T_c$

Check if $k=0$ is special: ↑ NOT in U & Ω

$$U = \sum_\alpha \frac{\varepsilon_\alpha}{e^{\beta(\varepsilon_\alpha - \mu)} - 1} = \frac{\overset{0}{\overset{\shortparallel}{\varepsilon_{k=0}}}}{e^{\beta(\varepsilon_\alpha-\mu)}-1} + \sum_{k\neq 0} \dots = \sum_{k\neq 0} \dots$$

$$= V\int \frac{d^3k}{(2\pi)^3} \dots$$

$$-pV = \Omega = k_B T \sum_\alpha \ln\left(1 - e^{-\beta(\varepsilon_\alpha - \mu)}\right) = k_B T \left\{ \ln\left(1 - e^{-\beta(\varepsilon_{k=0}-\mu)}\right) + \sum_{k\neq 0} \dots \right\}$$

$$\ln \underbrace{\left(1 - e^{-\beta(\varepsilon_{k=0}-\mu)}\right)}_{e^{-\beta\mu}(e^{\beta\mu}-1)} \sim \ln(N) \text{ - negligible compared to } O(N)$$

$\Rightarrow$ expressions
$$\boxed{\begin{aligned} \frac{p}{k_B T} &= \frac{2S+1}{\lambda^3} f_{5/2}^{+}(1) \\ U &= \frac{3}{2} pV \end{aligned}}$$
still hold.

$$p = k_B T \frac{(2S+1)}{\lambda^3} \underbrace{f_{5/2}^{+}(1)}_{1.3415} \sim T^{5/2}$$ - is independent of the N and V!

$$U = \frac{3}{2} pV \sim T^{5/2}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{5}{2}\frac{U}{T} = \frac{15}{4}\frac{pV}{T} = \frac{15}{4} k_B (2s+1) \frac{V}{\lambda^3} \times$$

$$\times f^+_{5/2}(1)$$

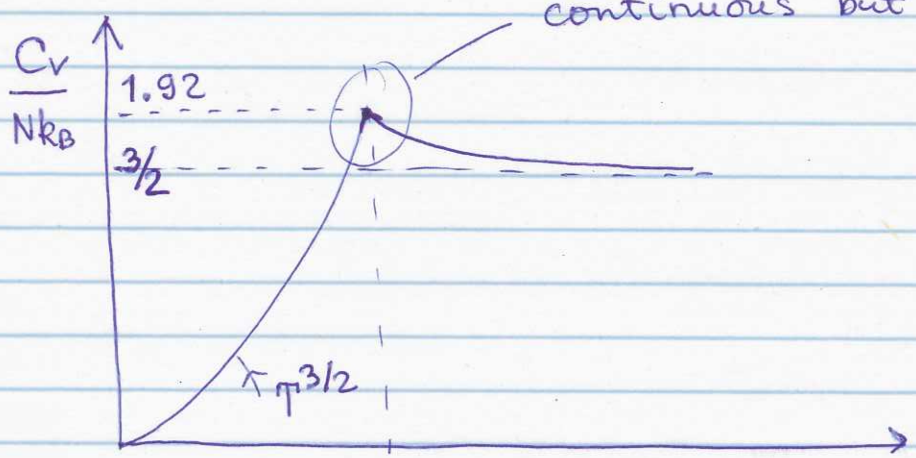
$$\boxed{\frac{C_V}{N} = k_B \frac{15}{4} (2s+1) f^+_{5/2}(1) \frac{1}{n\lambda^3}} \sim T^{3/2}$$

Remarks : * pressure depends on the temperature only!

Pressure - created only by $k \neq 0$ states, and particle density per volume in such excited states is determined solely by T.

* Internal energy $\sim pV$ depends on T & V but not on N.

* continuous but change in slope



Low - T behaviour : $$U \simeq V \int \frac{d^d k}{(2\pi)^d} \frac{\hbar^2 k^2/2m}{e^{(\hbar^2 k^2/2m)\frac{1}{k_B T}} - 1}$$

$$k_T \sim \left(\frac{m k_B T}{\hbar^2}\right)^{1/2}$$

$$\sim T^{(2+d)/2} = T^{1+d/2}$$

$$k_T^d \cdot k_T^2 \sim T^{(d+2)/2}$$

$$C_V \sim T^{d/2}$$

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{(2S+1)}{n\lambda^3(T)} f^+_{5/2}(1) = \frac{15}{4}\frac{\lambda^3(T_c)}{\lambda^3(T)}\frac{(2S+1)}{n\lambda^3(T_c)} f^+_{5/2}(1)$$

T_c condition: $\boxed{\frac{(2S+1)}{n\lambda^3(T_c)} \cdot f^+_{3/2}(1) = 1}$

$$\Rightarrow \boxed{\frac{C_V}{Nk_B} = \frac{15}{4}\frac{\lambda^3(T_c)}{\lambda^3(T)}\frac{f^+_{5/2}(1)}{f^+_{3/2}(1)} = \frac{15}{4}\frac{f^+_{5/2}(1)}{f^+_{3/2}(1)}\left(\frac{T}{T_c}\right)^{3/2}}$$

Experimental observations of BEC

$$N = V\frac{(2S+1)}{\lambda^3(T_c)} f^+_{3/2}(1)$$

Consider $S=0$ bosons. $\lambda(T) = \frac{h}{\sqrt{2\pi m k_B T}} = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$

$$k_B T_c = \left(\frac{N}{V}\right)^{2/3}\frac{1}{\left(f^+_{3/2}(1)\right)^{2/3}}\frac{h^2}{2\pi m} = \frac{1}{2\pi}\frac{1}{(2.6124)^{2/3}} \times \frac{h^2}{m}\left(\frac{N}{V}\right)^{2/3}$$

$$= 0.061\,\frac{h^2}{m}\left(\frac{N}{V}\right)^{2/3}$$

2 strategies for observing BEC:

* (high) Density typical of everyday liquids; weak interactions so that no solidification; small mass so that the quantum effects are large

liquid He^4 $\quad n = 1.5\cdot 10^{22}\,cm^{-3}$; $T_c \sim 3K$

* Dilute systems (gases) so that interactions are not important and very low temperatures

attainable by laser cooling

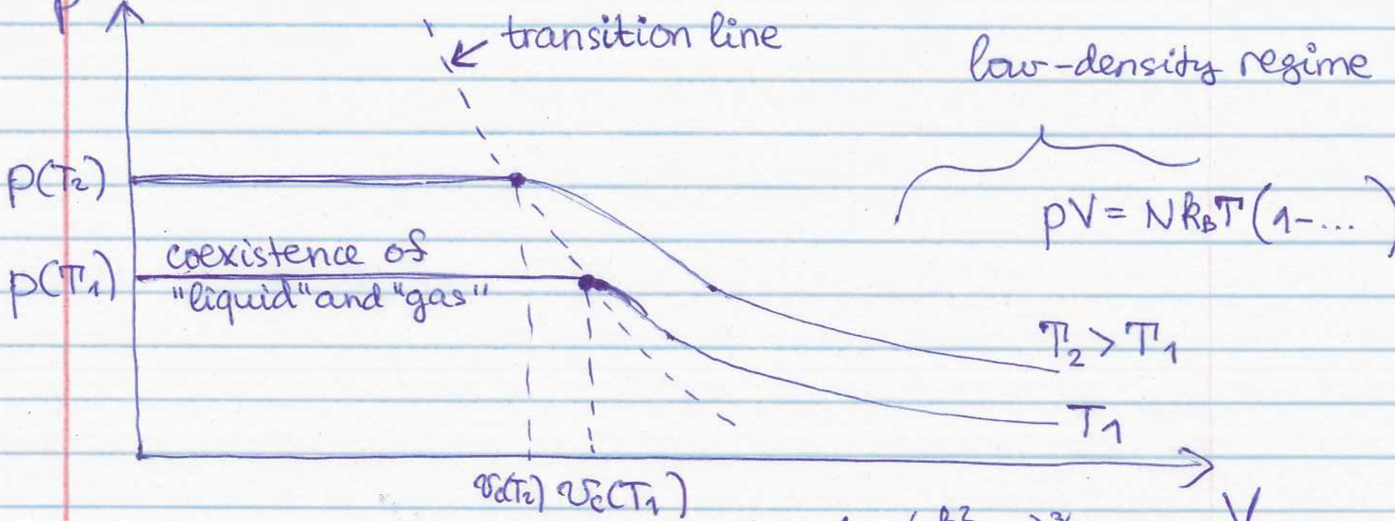
e.g. $5 \cdot 10^5$ sodium atoms, trapped at density $10^{14}\,cm^{-3}$ at $10^{-6}\,K$

$\rightarrow$ BEC of cold atomic gases

2001 Nobel prize in physics (lectures on the web)

Isotherms for ideal Bose gas

Fixed number of particles N



$$v_c(T) = \frac{1}{f^+_{3/2}(1)} \frac{1}{(2S+1)} \left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}$$

Equation for T_c:

$$\boxed{\frac{N}{V} = \frac{(2S+1)}{\lambda^3(T_c)} f^+_{3/2}(1) = (2S+1)\left(\frac{2\pi m k_B T_c}{h^2}\right)^{3/2} f^+_{3/2}(1)}$$

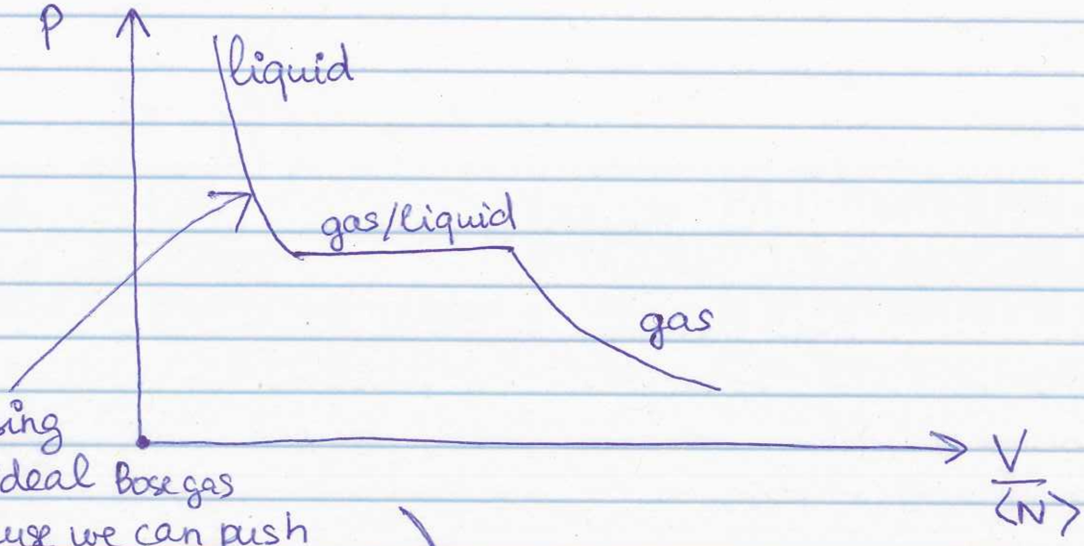
* For fixed $\frac{N}{V}$, varying T, have BEC for $T < T_c(\rho)$; the higher $\rho = \frac{N}{V}$, the higher T_c is.
* For fixed T, N, varying V, have BEC for $\boxed{\left(\frac{N}{V}\right) > (2S+1)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \times f^+_{3/2}(1)}$

For $V < V_c$, we have coexistence of "condensate" ($k=0$) and "normal fluid" ($k \neq 0$), with more fluid converted from normal to condensate as we decrease V.

At the same time, the pressure remains fixed

$$\boxed{p = k_B T (2S+1)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} f^+_{5/2}(1)}$$

In comparison, normal liquid/gas isotherm



missing for ideal Bose gas because we can push arbitrary many particles in $\vec{k}=0$ state even for $\frac{V}{\langle N \rangle} \to 0$, and this liquid creates no pressure

not possible for interacting particles – will have enormous pressures when molecules start to overlap

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BEC

Liquid – gas coexistence has aspects of first-order transition.

In the BEC

$$U = \frac{3}{2} pV = \frac{3}{2} k_B T (2S+1) \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} f^{+}_{5/2}(1) \cdot V$$

$$N_{gas} = N_{k>0} = (2S+1) \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} f^{+}_{3/2}(1) \cdot V$$

Increasing V from V_1 to V_2 (but still on the coexistence line = BEC phase): – converting $\Delta N = \frac{(2S+1)}{\lambda^3(T)} f^{+}_{3/2}(1) \Delta V$ particles from liquid to gas

$$\left.\begin{aligned} \Delta U_{\text{internal energy}} &= \frac{3}{2} p \Delta V \\ \Delta W_{\text{work}} &= p \Delta V \end{aligned}\right\} \Rightarrow \Delta Q = \frac{5}{2} p \Delta V$$

latent heat per particle

$$\boxed{\frac{\Delta Q}{\Delta N} = \frac{5}{2} k_B T \frac{f^{+}_{5/2}(1)}{f^{+}_{3/2}(1)}} \equiv \ell$$

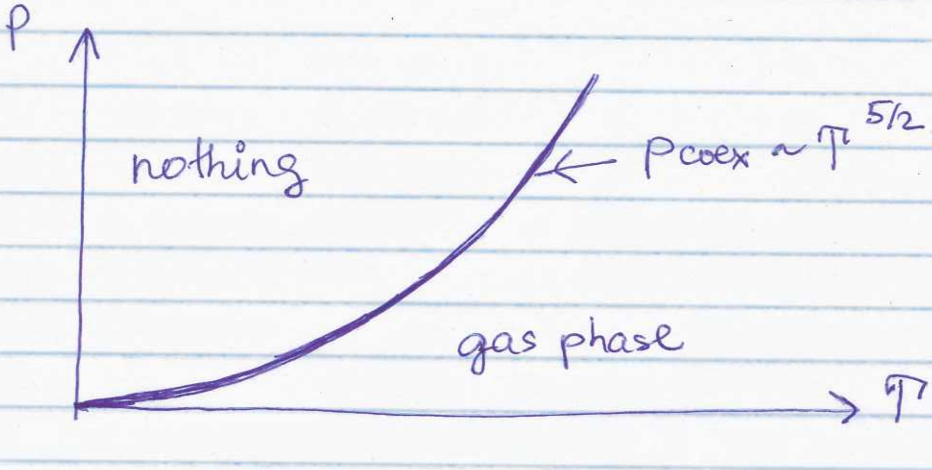
Clausius-Clapeyron equation for the BEC as coexistence of liquid ($k=0$) and gas ($k>0$):

$$p_{coex}(T) = k_B T\,(2S+1)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} f^+_{5/2}(1)$$

$$\frac{dp_{coex}}{dT} = \frac{5}{2}\frac{p_{coex}}{T} = \frac{5}{2}k_B(2S+1)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} f^+_{5/2}(1)$$

$$\stackrel{!}{=} \frac{\ell}{T(v_{gas}-v_{liq})} \underset{\uparrow}{=} \frac{5}{2}k_B\frac{(2S+1)}{\lambda^3(T)} f^+_{5/2}(1) \checkmark$$

Here
$$\boxed{v_{liq}=0, \quad v_{gas} = \frac{V}{N_{gas}} = \frac{\lambda^3(T)}{(2S+1)f^+_{3/2}(1)}, \quad \ell = \frac{5}{2}k_B T\frac{f^+_{5/2}(1)}{f^+_{3/2}(1)}}$$



(BEC) liquid phase lies on the's transition line.
The transition line does not end at any T since for any T ∃ crit. density $\rho_c(T)$ above which condensation occurs.