4D-2D Correspondence Revisited

now with more fermions

Ryan Thorngren

UC Berkeley

Aspen, Winter 2018

Outline

3+1D Gauge Theory With Bosonic Matter
- Partition Function
- Electric-Magnetic Dualities

Duality and Modularity
- abelian bosonic 4D-2D correspondence
- 4D duality from 2D modularity
- Fermions and Duality

6D CFT and 4D-2D correspondence
- Self-duality and Compactification
- Spin Considerations
- Topological Insulators

Extras
- Bulk-Boundary Correspondences
- Bosonization

3+1D $U(1)$ Gauge Theory

Bosonic Matter

- $Z(X,\tau,\bar{\tau}) = \int DA \exp\left(-\frac{1}{e^2}\int_X F\wedge\star F - \frac{i\theta}{2}\int_X \left(\frac{F}{2\pi}\right)^2\right)$.
- $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ "modular parameter".
- Gauge field A is a connection on a $U(1)$ bundle.
- We can write $A = A_0 + \lambda$ where A_0 satisfies Maxwell's equations and $\lambda \in \Omega^1(X,\mathbb{R})$ is a global 1-form.
- There is a unique A_0 in each topological class $c_1 \in H^2(X,\mathbb{Z})$, so

$$DA = \frac{1}{|\mathcal{G}|}\sum_{c_1\in H^2(X,\mathbb{Z})}\int_{\Omega^1(X,\mathbb{R})} D\lambda$$

- $\mathcal{G}$ = group of small gauge transformations $\sim \Omega^0(X,\mathbb{R})/\mathbb{R}$.

3+1D $U(1)$ Gauge Theory

Bosonic Matter

- $Z(X,\tau,\bar\tau) = \int DA \exp\left(-\frac{1}{e^2}\int_X F\wedge\star F - \frac{i\theta}{2}\int_X\left(\frac{F}{2\pi}\right)^2\right)$
$$= \left(\frac{1}{|\mathcal{G}|}\int D\lambda\ e^{-\frac{1}{e^2}\int_X d\lambda\star d\lambda}\right)\left(\sum_{c_1(A_0)\in H^2(X,\mathbb{Z})} e^{-S(A_0)}\right).$$

- Because the action is quadratic, there are no A_0,λ cross terms.
- λ contribution depends only on $Im\ \tau$, get a factor of $\frac{1}{\sqrt{Im\ \tau}}$ for each physical mode from the determinant.

- $Z(X,\tau,\bar\tau) = (Im\ \tau)^{(b_1-b_0)/2} Z_{vac}(X,\tau,\bar\tau)$.

- Z_{vac} is where the action happens.

E. Witten 95, E. Verlinde 95

3+1D $U(1)$ Gauge Theory

Bosonic Matter

- Z_{vac} is a sum over Chern classes in

$$H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{\text{free}} \oplus H^2(X, \mathbb{Z})_{\text{torsion}}$$

- The torsion part represents flat gauge fields with nontrivial holonomies around noncontractible cycles. The action vanishes on the torsion part. They just contribute a multiplicative constant $b_2^{tors} := |H^2(X, \mathbb{Z})_{\text{torsion}}|$.

- The free part represents gauge fields determined by their curvature, with trivial holonomy around all noncontractible cycles in X.

E. Witten 95, E. Verlinde 95

3+1D $U(1)$ Gauge Theory

Bosonic Matter

- The free part can be given a basis α^I of harmonic integral 2-forms. Then we define

$$G^{IJ} = \int_X \alpha^I \wedge \star\alpha^J \qquad Q^{IJ} = \int_X \alpha^I \wedge \alpha^J.$$

-

$$Z_{vac} = b_2^{tors} \sum_{m_I \in \mathbb{Z}^{b_2}} \exp\left(-\pi(Im\ \tau)m_I G^{IJ} m_J + i\pi(Re\ \tau)m_I Q^{IJ} m_J\right).$$

$$= b_2^{tors} \sum_{m_I \in \mathbb{Z}^{b_2}} q^{\frac{1}{4}m_I(G+Q)^{IJ}m_J} \bar{q}^{\frac{1}{4}m_I(G-Q)^{IJ}m_J}$$

where $q = \exp(2\pi i\tau)$.

- The (anti-)self-dual parts $\alpha \pm \star\alpha$ contribute to the (anti-)holomorphic part of Z_{vac}.

E. Witten 95, E. Verlinde 95

Dualities of 3+1D Gauge Theory

T-Duality

- We first consider the action of $T : \tau \mapsto \tau + 1$. We find that the partition function transforms nontrivially:

$$T : Z_{vac} \mapsto b_2^{tors} \sum_{m_I \in \mathbb{Z}^{b_2}} (-1)^{m_I Q^{IJ} m_J} q^{\frac{1}{4} m_I (G+Q)^{IJ} m_J} \bar{q}^{\frac{1}{4} m_I (G-Q)^{IJ} m_J}.$$

- This is not a problem if the intersection form is even, ie. $m_I Q^{IJ} m_J = 0 \mod 2$ for all m_I. This happens exactly when X is spin!
- However, on a manifold like $\mathbb{CP}^2$ with the Fubini-Study metric, the partition function is

$$Z_{vac} = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2} m^2},$$

and is not T invariant.
- For all manifolds, $T^2 : \tau \mapsto \tau + 2$ is a duality.

E. Witten 95, E. Verlinde 95

- The partition function is also invariant under $S : \tau \mapsto 1/\tau$, by some careful Poisson resummation.
- The reason it works is that the lattice $(H^2(X, \mathbb{Z}), Q)$ is *self-dual*, that is *det* $Q = \pm 1$.
- The combination of S and T generate the group $PSL(2, \mathbb{Z})$. If we consider line operators this is lifted to $SL(2, \mathbb{Z})$.

E. Witten 95, E. Verlinde 95

A 4D-2D correspondence

for bosons

- $$Z_{vac}(X,\tau,\bar{\tau}) = b_2^{tors} \sum_{m_I \in \mathbb{Z}^{b_2}} q^{\frac{1}{4}m_I(G+Q)^{IJ}m_J}\bar{q}^{\frac{1}{4}m_I(G-Q)^{IJ}m_J}$$

  This is very similar in form to the partition function

  $$Z_{2d}(\tau,\bar{\tau}) = Tr\ q^{L_0-b_2/24}\bar{q}^{\bar{L}_0-b_2/24}$$

  of a *two-dimensional* theory of chiral bosons with $c_+ = b_2^+$ right movers and $c_- = b_2^-$ left movers on a spacetime torus with coordinate z defined modulo

  $$z \mapsto z+1$$

  $$z \mapsto z+\tau.$$

E. Witten 95, E. Verlinde 95,

A 4D-2D correspondence

for bosons

- In fact, the correspondence can be made nearly exact if we use the chiral bosons on the boundary of the 2+1D Abelian (Unimodular) Chern-Simons theory

$$\frac{1}{4\pi} Q^{IJ} \int_3 a_I da_J$$

with the 1+1D kinetic term

$$\frac{1}{4\pi} G^{IJ} \int_2 d\phi_I \wedge \star d\phi_J.$$

- For this theory the center-of-mass momentum and winding contribution to the torus partition function is precisely Z_{vac}.
- The only thing missing are some factors of $\eta(\tau)$ from the excited states. We more or less expected this by studying only the vacuum part of Z_{4D}, but they don't match anyway.

E. Witten 95, J. Maldacena, A. Strominger, E. Witten 97

A 4D-2D correspondence

some important examples

- For example, partition function on Fubini-Study $\mathbb{CP}^2$ agrees with the partition function of the chiral boson:

$$Z_{vac}(\mathbb{CP}^2) = \sum_{m\in\mathbb{Z}} q^{\frac{1}{2}m^2},$$

  which sits at the boundary of $U(1)_1$ Chern-Simons theory.
- The partition function on $S^2 \times S^2$ is that of the ordinary boson

$$Z_{vac}(S^2 \times S^2) = \sum_{p,w\in\mathbb{Z}^2} q^{\frac{1}{4}(A_1 p + A_2 w)^2} \bar{q}^{\frac{1}{4}(A_1 p - A_2 w)^2},$$

  (no Chern-Simons req'd).
- There is no smooth 4-manifold that gives rise to the E_8 theory, but the $K3$ manifold gives rise to two E_8's plus three compact bosons.

E. Witten 95, J. Maldacena, A. Strominger, E. Witten 97, A. Scorpan 05

Modularity in 2D and 4D duality

Bosonic Matter

- The coupling parameter τ has become the shape parameter of a torus.
- The *oriented* mapping class group of the torus is generated by $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$, which generates the group $PSL(2, \mathbb{Z})$. With the origin $z = 0$ marked it gives the duality group $SL(2, \mathbb{Z})$.

- Note that the full *unoriented* mapping class group also includes the anti-holomorphic element $R : \tau \mapsto -\bar{\tau}$, and together they generate $SL(2, \mathbb{Z}) \rtimes \mathbb{Z}_2^R$, where the action is

$$RTR = T^{-1}$$

$$RSR = S.$$

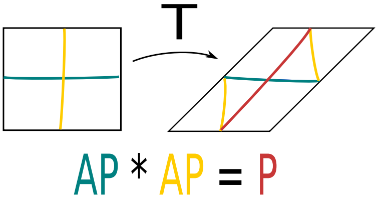
Note that $R : e^2 \mapsto e^2$, $\theta \mapsto -\theta$. We'll return to this point.

B. Farb, D. Margalit 12

Modularity in 2D and Spin Structures

Bosonic Matter?

- The fact that $T : \tau \mapsto \tau + 1$ is not a symmetry of the chiral boson is related to the fact that $U(1)_1$ Chern-Simons theory is a fermionic theory.
- Indeed, the 2D boundary inherits a spin structure from the 3D bulk, and the modular T transformation maps the AP-AP spin structure to the AP-P spin structure:



E. Witten 96, D. Belov, G. Moore 05

Fermionic Monopoles in 3+1D Gauge Theory

At $\theta = 2\pi$

- Correspondingly, the reason why 3+1D $U(1)$ gauge theory is not T invariant on arbitrary manifolds is because $\theta = 2\pi$ is a non-trivial topological term that turns the monopole into a fermion!

- The point is that the $\theta = 2\pi$ term can be written
$$\exp(i\pi w_2 \frac{F}{2\pi})$$
which modifies the quantization condition for the dual field strength to
$$\int_{\Sigma^2 \subset X} \left(\frac{F^\vee}{2\pi} + \frac{1}{2} w_2 \right) \in \mathbb{Z}$$
meaning $A^\vee$ is a Spin^c structure, not an ordinary $U(1)$ gauge field. S-duality for this theory doesn't take us to an ordinary $U(1)$ gauge theory!

O. Aharony, N. Seiberg, Y. Tachikawa 13; A. Kapustin, R.T. 13; R.T. 14

Four 3+1D $U(1)$ Gauge Theories

With Fermionic Matter

- So we recognize four possible gauge theories, defined by their line operator content at $\tau = i$:
- Z_{BB} our vanilla $U(1)$ gauge theory with bosonic charge and monopole.
- Z_{BF} is equivalent to $U(1)$ gauge theory at $\theta = 2\pi$ and has a bosonic charge and a fermionic monopole.
- Z_{FB} has a Spin^c connection as its fundamental degree of freedom, and so it has a fermionic charge and a bosonic monopole.
- Z_{FF} has a Spin^c connection and a sick topological term that causes it to have both a fermionic charge and a fermionic monopole. This theory makes sense at the boundary of a 4+1D invertible TQFT with action $\frac{1}{2} w_2 w_3$, which generates $\Omega_{SO}^5 = \mathbb{Z}_2$. Note this anomaly is curable on spin manifolds.

R.T. 2014

Four 2D CFTs for $\mathbb{CP}^2$

With Fermionic Matter

- We can identify the four vacuum partition functions with four CFT partition functions of chiral *fermions* on a torus with four different spin structures. For $\mathbb{CP}^2$ they are:
- $Z_{BB} \sim \sum_{n\in\mathbb{Z}} q^{n^2}$, AP-AP spin structure.
- $Z_{BF} \sim \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$, AP-P structure.
- $Z_{FB} \sim \sum_{n+1/2\in\mathbb{Z}} q^{n^2}$, P-AP spin structure: zero mode.
- $Z_{FF} \sim \sum_{n+1/2\in\mathbb{Z}} (-1)^{n-1/2} q^{n^2} = 0$ because of the 4D anomaly, and because the P-P spin structure to which it corresponds in nontrivial in Ω_2^{spin}, and cannot be extended to any 3-manifold filling to use for the $U(1)_1$ Chern-Simons bulk theory.
- The $SL(2,\mathbb{Z})$ action on these reproduces the small duality web of 3+1D gauge theories on $X = \mathbb{CP}^2$. All of them are invariant under $R : q \mapsto \bar{q}$, provided we also reverse the orientation of X!

Four 2D CFTs in the General Case

With Fermionic Matter

- The chiral "Luttinger liquid" is formed of boundary modes of the 2+1D Abelian (Unimodular) Chern-Simons theory
$$\frac{1}{4\pi} Q^{IJ} \int_3 a_I da_J$$
and given the 1+1D kinetic term
$$\frac{1}{4\pi} G^{IJ} \int_2 d\phi_I \wedge \star d\phi_J.$$
This depends on spin structure exactly when Q is odd.
- We're interested in the contribution of vertex operators
$$V_I = e^{2\pi i \phi_I}$$
to the partition function. These can be bosons or fermions depending on Q, and their (anti-)periodicity is controlled by the spin structure on the torus.

S. Coleman 75, D. Haldane 81, X-G. Wen 90, D. Belov, G. Moore 05

A 4D-2D correspondence from 6D

for bosons

- The 4D-2D correspondence can be explained from 6D by considering the conformal theory

$$\int dB \wedge \star dB$$

of a 2-form $U(1)$ gauge field B satisfying a (chiral) self-duality constraint on its 3-form field strength $dB = \star dB$. Note that the action above is only schematic.

- When we put the theory on $T^2 \times X^4$, we can take X^4 as small, obtaining a 2D theory with chiral bosons arising from the periods $\phi_\alpha = \int_\alpha B$ for each $\alpha \in H_2(X, \mathbb{Z})$.

E. Verlinde 95, J. Maldacena, A. Strominger, E. Witten 97

A 4D-2D correspondence from 6D

for bosons

- On the other hand we can take T^2 to be small, from which we obtain several fields, among them two 1-form $U(1)$ gauge fields $A_j = \int_j B$, where j indexes two perpendicular 1-cycles of T^2.
- Self-duality of B implies $dA_1 = iIm\ \tau \star dA_2 + Re\ \tau dA_2$, so A_1 and A_2 are not independent. In fact they are the $U(1)$ gauge field and its dual.
- We reproduce the connection between S duality of the 4D theory and S-modularity of the 2D theory by noting that the S transformation of this torus exchanges the two 1-cycles, hence the 4D gauge field and its dual.
- So the $SL(2, \mathbb{Z})$ duality group follows from coordinate invariance of this 6D conformal theory, provided it really exists.

E. Verlinde 95, J. Maldacena, A. Strominger, E. Witten 97

6D-7D Boundary-Bulk Correspondence

speculation ahead

- From considerations of the θ periodicity, we know this can only work out straightforwardly if X^4 is spin. Otherwise we know we will need to choose a spin structure on T^2.
- In fact the 6D theory is chiral and must be realized on the boundary of something like a 7D level 1 Chern-Simons theory for a $U(1)$ 3-form $\sim CdC$, which must be carefully defined.

E. Witten 97, M. Hopkins, I. Singer 02

6D-7D Boundary-Bulk Correspondence

speculation ahead

- $$\int_8 \frac{1}{4\pi} dC \wedge dC + \cdots$$
- Hopkins and Singer showed you can do it this way if the 7-manifold (and hence its 6-manifold boundary) is spin and the theory depends on something called an "integral Wu structure", but not actually on a spin structure.
- I suspect it is possible to do it on a Spin^c 7-manifold as well and to therefore derive the fermionic electric-magnetic dualities from the 6D CFT.

E. Witten 97, M. Hopkins, I. Singer 02

What About The Topological Insulators?

- It was proposed by M. Metlitski and A. Vishwanath that these dualities can also make sense if X is non-orientable, where we define the gauge theory on X using time reversal symmetry.
- Since $T : (e, m) \mapsto (e, -m)$, S duality exchanges the gauge group $U(1)_A \times \mathbb{Z}_2^T$ with $U(1)_{A^\vee} \rtimes \mathbb{Z}_2^T$.
- S-duality can be thought of as a duality between two different gauged topological insulators, if we track T as a global symmetry.

A. Kapustin 14, M. Metlitski, A. Vishwanath 15, M. Metlitski 15

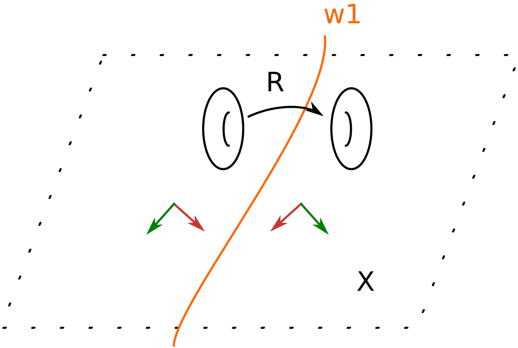
What About The Topological Insulators?

Can we get this duality from 6D?

- The antiholomorphic involution $R : \tau \mapsto -\bar{\tau}$ can be combined with reflection symmetry of X^4 to obtain a symmetry of the 6D theory on $T^2 \times X^4$ for which A is even and $A^\vee$ is odd.
- In fact, this is nothing but a π rotation about a plane in $\mathbb{R}^6$, so we should be able to analyze it using the gravitational part of the 7D Chern-Simons theory, yet to be constructed.

What About The Topological Insulators?

A nonorientable 2D-4D correspondence?



Recap

- The $SL(2,\mathbb{Z})$ duality group of 3+1D $U(1)$ gauge theories on a spin manifold X is a consequence of 1+1D modular invariance of a CFT associated to X.
- A direction relationship between the 4D and 2D systems is by compactification of a 6D theory.
- If X is only Spin^c, then we can still associate to it a CFT, but it will be a Spin-CFT and this complicates the statement of modular invariance, which mirrors the duality web of 3+1D Maxwell theory with fermionic matter.
- 4D time reversal symmetry is implemented by an anti-holomorphic involution on the CFT. Compatibility with $SL(2,\mathbb{Z})$ duality is captured by the full unoriented mapping class group.

Questions and Challenges

- How do we construct the 7D Chern-Simons theory when spacetime is not spin? This will be the first step in proving the dualities from 6D.

- Can we use the anti-holomorphic involution to construct a 2D CFT whose partition function equals that of 4D Maxwell theory on a Pin^c manifold?

- Can we study boundary conditions of the 6D theory to derive 3D dualities?

- Is there a 4D picture of 2D bosonization?

- What about singular 4-manifolds?

(2D-3D)-(4D-5D) Boundary-Bulk Correspondence Correspondence

speculation ahead

- It is known that an orientable 4-manifold admites a filling by a 5-manifold iff its signature vanishes, ie. $b_2^+ = b_2^-$.
- In this case we can regulate the gravitational anomaly of all-fermion electrodynamics Z_{FF} by adding a 5D bulk TQFT $\frac{1}{2}w_2 w_3$.
- Compactifying the 6D-7D system on this 5-manifold with boundary gives the chiral bosons we had before coupled to a TQFT.
- This gives a topological boundary condition for the Q-matrix Chern-Simons theory!

A. Scorpan 05, A. Kapustin, N. Saulina 10

Chiral To Compact Boson

A hidden sum over spin structure

- The radius R compact boson partition function is

$$Z_R = \sum_{p,w} q^{\frac{1}{4}(p/R+wR)^2} \bar{q}^{-\frac{1}{4}(p/R-wR)^2}.$$

- At $R = 1$ we can write

$$(p, w) = \frac{1}{2}(p + w)(1, 1) + \frac{1}{2}(p - w)(1, -1),$$

$$Z_R = \sum_{n \in \frac{1}{2}\mathbb{Z}} q^{\frac{1}{2}n^2} \sum_{m \in \frac{1}{2}\mathbb{Z}} \bar{q}^{\frac{1}{2}m^2}.$$

- The sum is over half-integers, and we can write this as a sum of four terms indexed by $(n, m) \in (\frac{1}{2}\mathbb{Z})^2/\mathbb{Z}^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which labels the four spin structures on the torus. Thus, by reducing to chiral parts, we can pull the spin partition functions out of the bosonic theory.

Classification of Intersection Forms

- By Poincaré duality, the intersection form is unimodular.
- By Donaldson, if the form is definite, then it's diagonalizable, essentially a sum of $\mathbb{CP}^n$'s, get $c_+ = n$ or $c_- = n$ chiral fermions.
- Otherwise, by Serre, if the form is indefinite but odd (so the manifold is not spin), then it is again diagonalizable, so we get some theory of (c_+, c_-) chiral fermions mixed together.
- Finally, if the manifold is spin, then the intersection form is a sum of E_8's and H_2's. For example $K3 = 2E_8 \oplus 3H_2$. These lead to truly bosonic 2D CFTs with no dependence on spin structure and no funny $SL(2, \mathbb{Z})$ action.

A. Scorpan 05