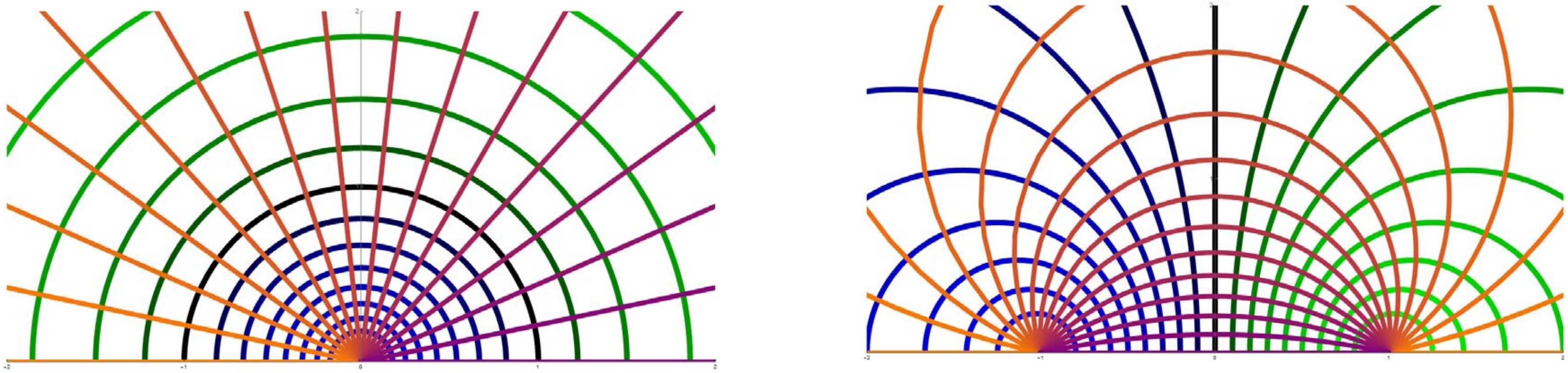
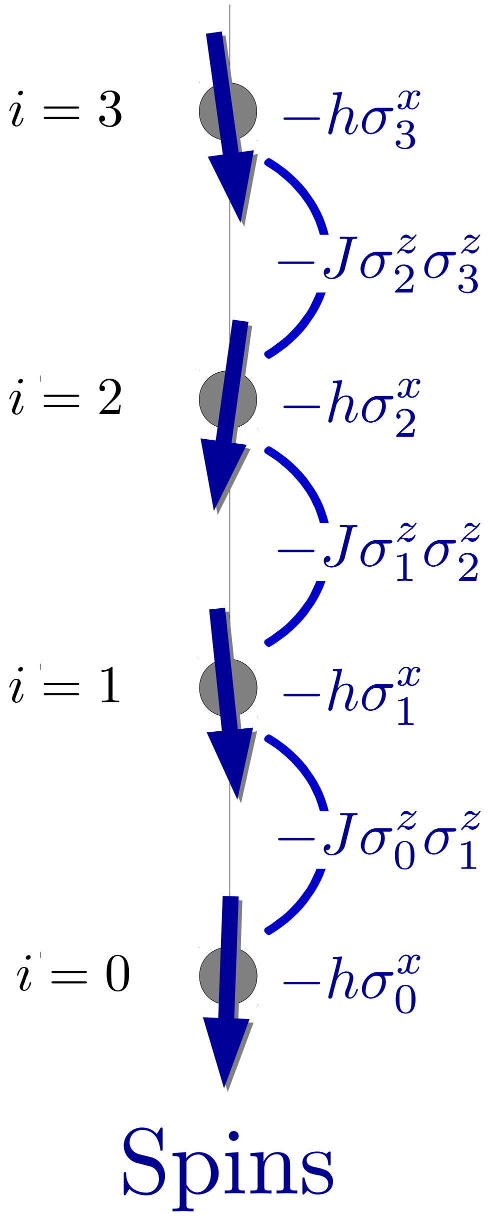
Symmetries and Dualities of Dirac Fermions in 2+1 Dimensions

David F. Mross



DFM, Alicea, Motrunich, PRL 117, 016802 (2016)
DFM, Alicea, Motrunich, PRX 7, 041016 (2017)

Warm up in 1+1 dimensions: Quantum Ising model



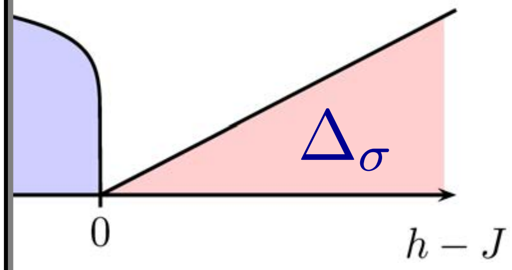
Duality Transformation

$$\sigma_i^z \sigma_{i+1}^z = \tau_{i+\frac{1}{2}}^x$$

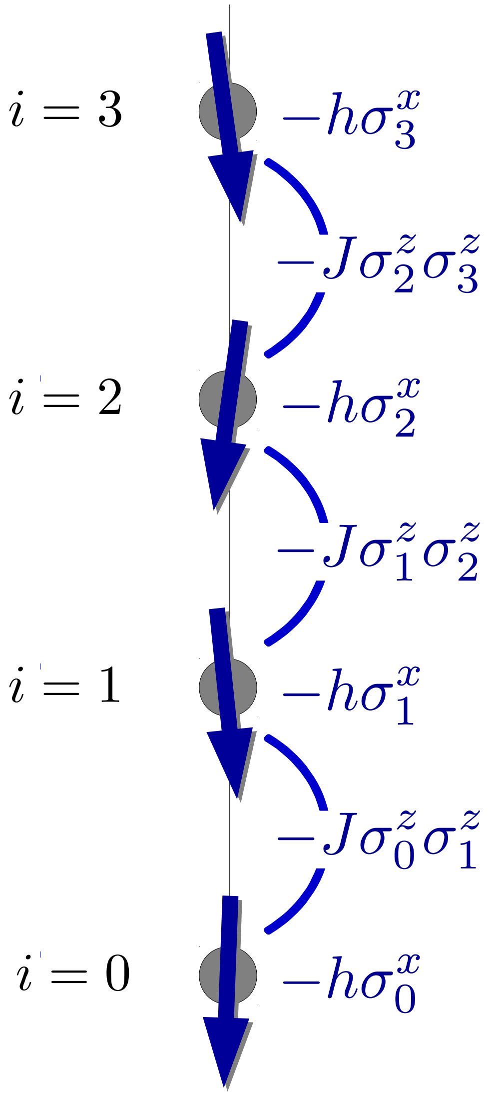
$$\sigma_i^x = \tau_{i-\frac{1}{2}}^z \tau_{i+\frac{1}{2}}^z$$

non-local:

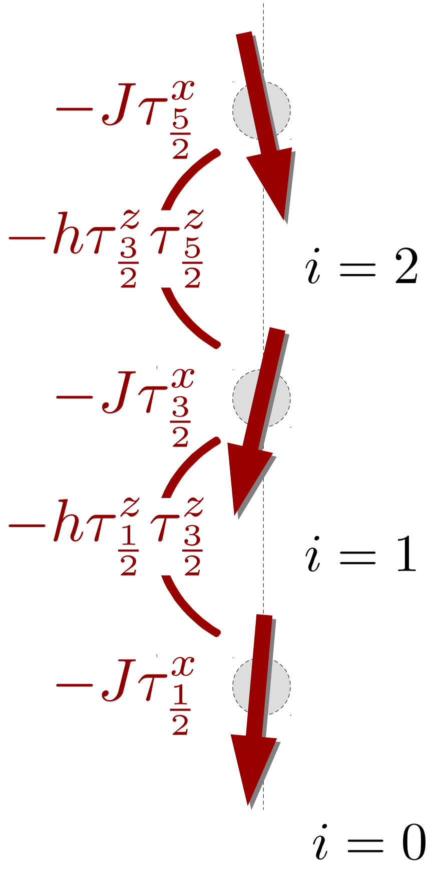
$$\sigma_i^z = \prod_{j<i} \tau_{j+\frac{1}{2}}^x$$



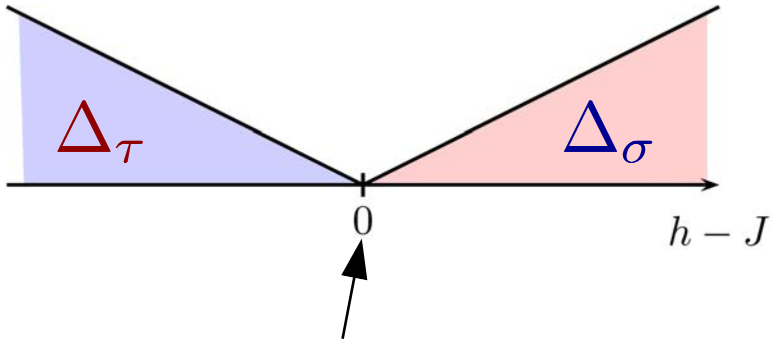
Warm up in 1+1 dimensions: Quantum Ising model



Spins

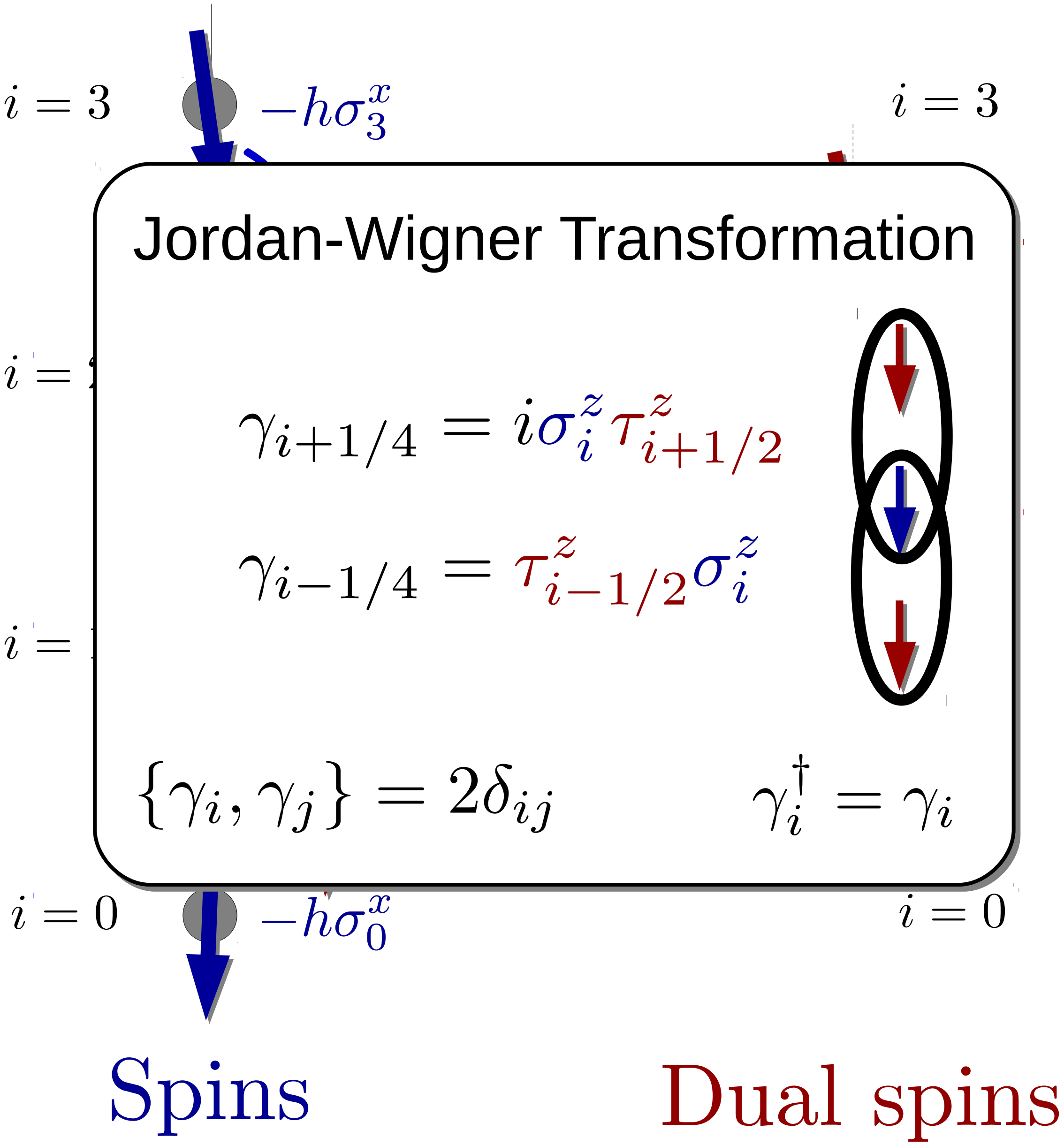


Dual spins



Self-duality ensures criticality at $J = h$

Warm up in 1+1 dimensions: Quantum Ising model



Jordan-Wigner Transformation

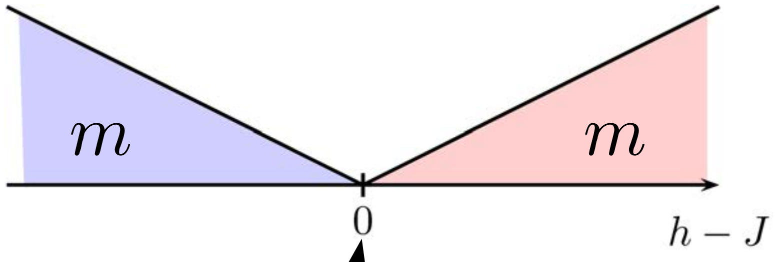
$$\gamma_{i+1/4} = i\sigma_i^z \tau_{i+1/2}^z$$

$$\gamma_{i-1/4} = \tau_{i-1/2}^z \sigma_i^z$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \qquad \gamma_i^\dagger = \gamma_i$$

Gapped fermions

$$m = \Delta_\sigma + \Delta_\tau$$



Self-duality ensures criticality at $J = h$

Warm up in 1+1 dimensions: Quantum Ising model

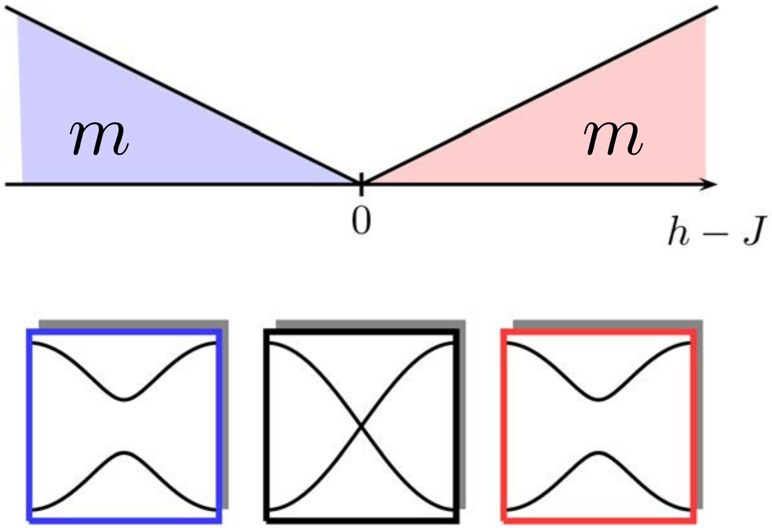
$i = 3$

$i = 2$

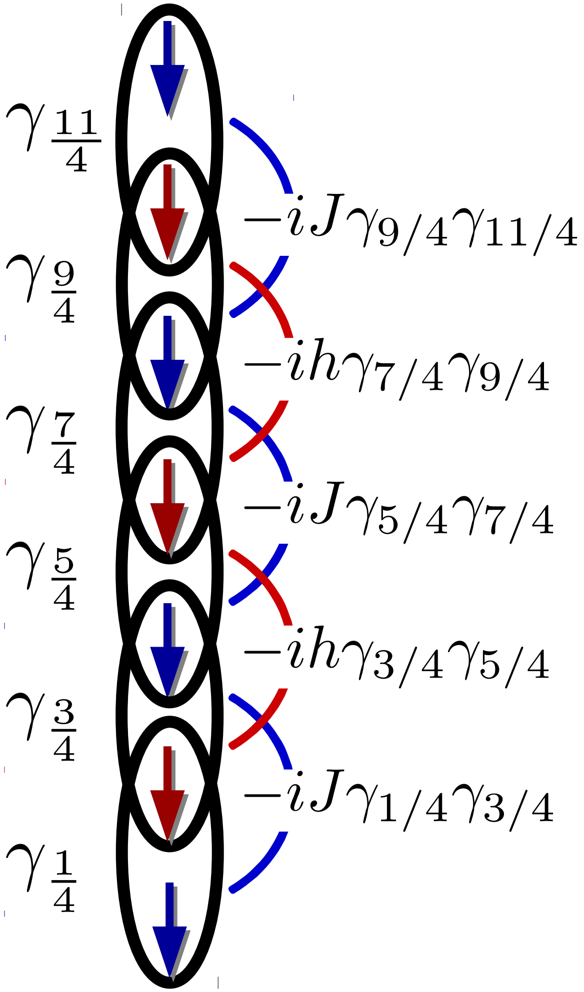
$i = 1$

$i = 0$

Self-duality ensures criticality at $J = h$



Translation symmetry yields gapless spectrum for $J = h$



Sp

Fermions

Warm

model

$i = 3$

$i = 2$

$i = 1$

$i = 0$

$4\gamma_{11/4}$

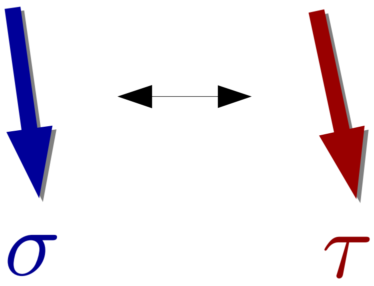
$4\gamma_{9/4}$

$4\gamma_{7/4}$

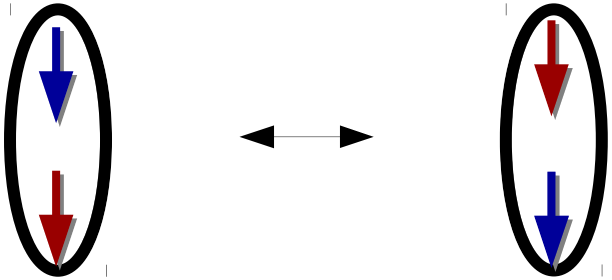
$4\gamma_{5/4}$

$4\gamma_{3/4}$

- Duality Transformation on spin variables



- Translation by ½ of Majorana fermions



$$\gamma_{i-1/4} = \tau^z_{i-1/2}\sigma^z_i \qquad \gamma_{i+1/4} = i\sigma^z_i\tau^z_{i+1/2}$$

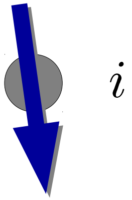
- Can interpret **duality** as a **symmetry** of new degrees of freedom
- Symmetry is **anomalous** (impossible in microscopic 1D system).

Sp

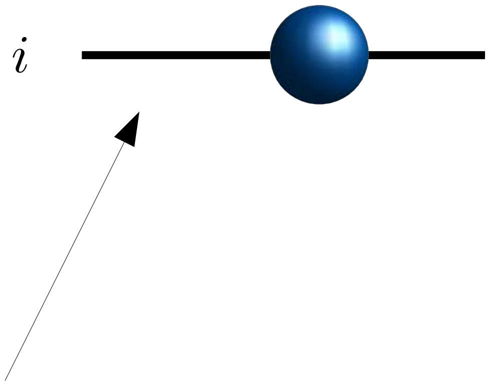
2+1 dimensions: Particle-vortex duality

Spins on sites labelled by i

Bosons on wires labelled by i



$$\sigma_i^z \longleftrightarrow b_i(x) \sim e^{i\varphi_i}$$



Long-wavelength description (1D bosonization)

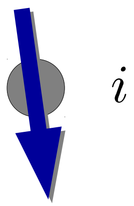
$$b(x) \sim e^{i\varphi(x)} \qquad \rho(x) = \bar{\rho} + \frac{1}{\pi}\partial_x\theta$$

$$\mathcal{L}_{\text{kin}} \sim \frac{i}{\pi}\partial_x\theta\partial_\tau\varphi + g(\partial_x\varphi)^2 + g^{-1}(\partial_x\theta)^2$$

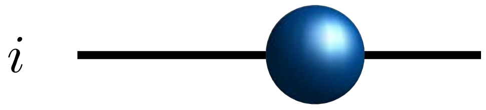
2+1 dimensions: Particle-vortex duality

Spins on sites labelled by i

Bosons on wires labelled by i



$$\sigma_i^z \longleftrightarrow b_i(x) \sim e^{i\varphi_i}$$



$$\sigma_i^x \longleftrightarrow p_i(x) \sim e^{i2\theta_i}$$

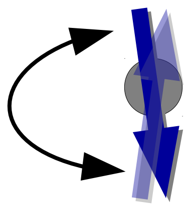
Creates defects (spin flips) in ferromagnet

Creates defects (phase-slips) in superfluid

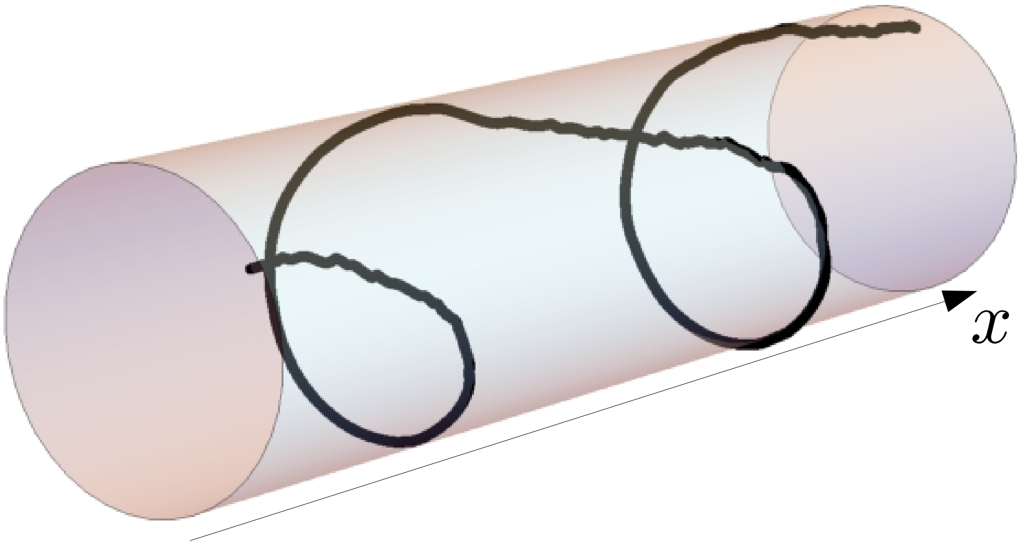
$\langle \sigma_i^z \rangle \neq 0$: Ferromagnet

$\langle b_i \rangle \neq 0$: Superfluid

σ_i^x:



$p_i(x)$:



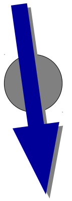
$\langle \sigma_i^z \rangle = 0$: Paramagnet

$\langle b_i \rangle = 0$: Mott Insulator

2+1 dimensions: Particle-vortex duality

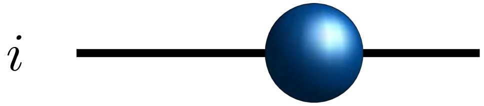
Spins on sites labelled by i

Bosons on wires labelled by i



$$\sigma_i^z \longleftrightarrow b_i(x) \sim e^{i\varphi_i}$$

$$\sigma_i^x \longleftrightarrow p_i(x) \sim e^{i2\theta_i}$$



Ising duality

$$\sigma_i^z \sigma_{i+1}^z = \tau^x_{i+\frac{1}{2}}$$

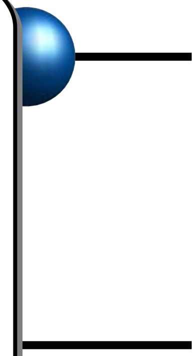
$$\sigma_i^x = \tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}}$$

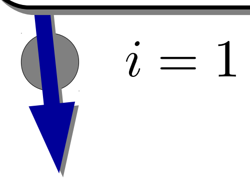
Boson-vortex duality

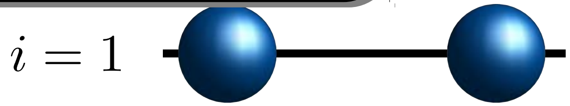
Phase-slip is vortex hopping

$$b_i^\dagger b_{i+1} = \tilde{p}_{i+1/2}$$

$$p_i = \tilde{b}^\dagger_{i+\frac{1}{2}} \tilde{b}_{i-\frac{1}{2}}$$







2+1 dimensions: Particle-vortex duality

Spins on dual sites labelled by $i+\frac{1}{2}$

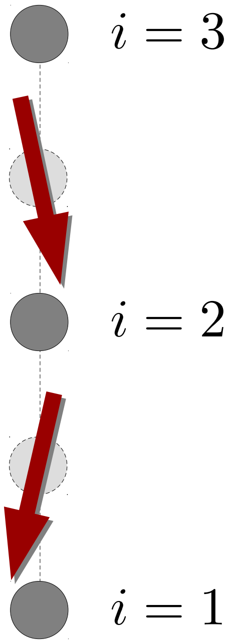
Bosons on dual wires labelled by $i+\frac{1}{2}$

$$\tau^z_{i+\frac{1}{2}} \longleftrightarrow \tilde{b}_{i+\frac{1}{2}}(x)$$

Creates defects (spin flips) in dual ferromagnet →

$$\tau^x_{i+\frac{1}{2}} \longleftrightarrow \tilde{p}_{i+\frac{1}{2}}(x)$$

Creates defects (phase-slips) in dual superfluid

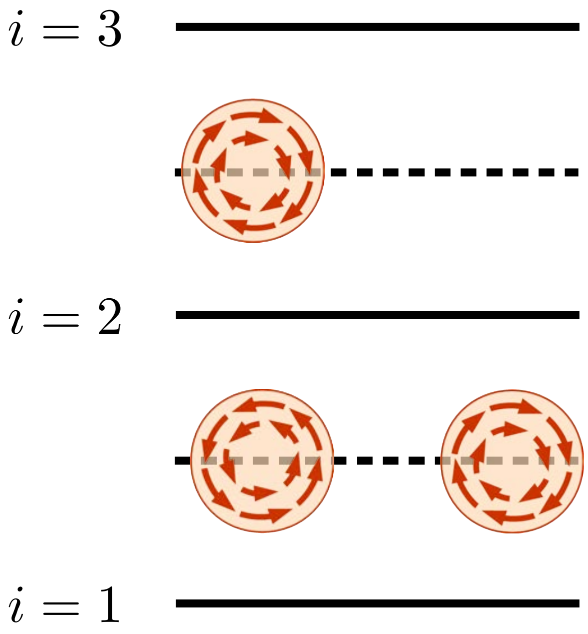


$\langle \tau^z \rangle = 0$: Ferromagnet

$\langle \tilde{b} \rangle = 0$: Superfluid

$\langle \tau^z \rangle \neq 0$: Paramagnet

$\langle \tilde{b} \rangle \neq 0$: Mott Insulator



2+1 dimensions: Particle-vortex duality

- Ising duality: Only spin-exchange and transverse field

$$H_{\text{Ising}} = -J\sigma_i^z\sigma_{i+1}^z - h\sigma_i^x$$

$$\updownarrow \quad \text{duality} \quad \updownarrow$$

$$H_{\text{Dual}} = -J\tau^x_{i+\frac{1}{2}} \quad -h\tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}}$$

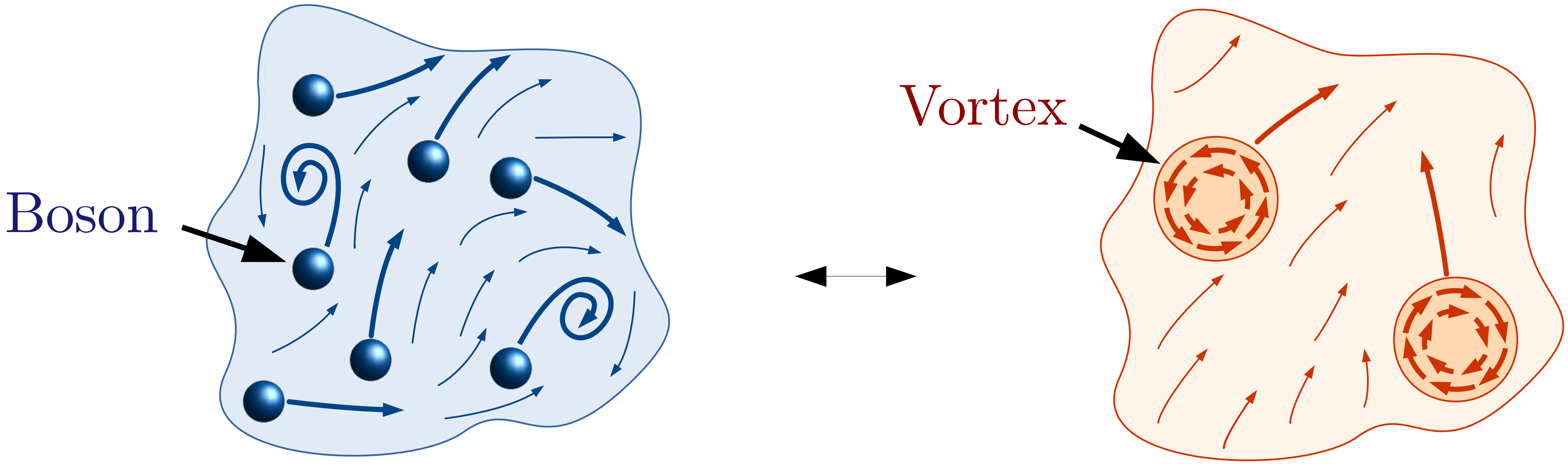
- Boson-vortex duality: Kinetic energy within each wire

$$\mathcal{L}_{\text{Boson}} = \mathcal{L}^{\text{kin.}}_{\text{Boson}} + \sum_i \left[u b_i^\dagger b_{i+1} + v p_i + \text{H.c.}\right]$$

$$\cancel{\text{duality} \updownarrow} \qquad\qquad \updownarrow \quad \text{duality} \quad \updownarrow$$

$$\mathcal{L}_{\text{Vortex}} = \mathcal{L}^{\text{kin.}}_{\text{Vortex}} + \sum_i \left[u\tilde{p}_{i+\frac{1}{2}} + v\tilde{b}^\dagger_{i-\frac{1}{2}}\tilde{b}_{i+\frac{1}{2}} + \text{H.c.}\right]$$

2+1 dimensions: Particle-vortex duality

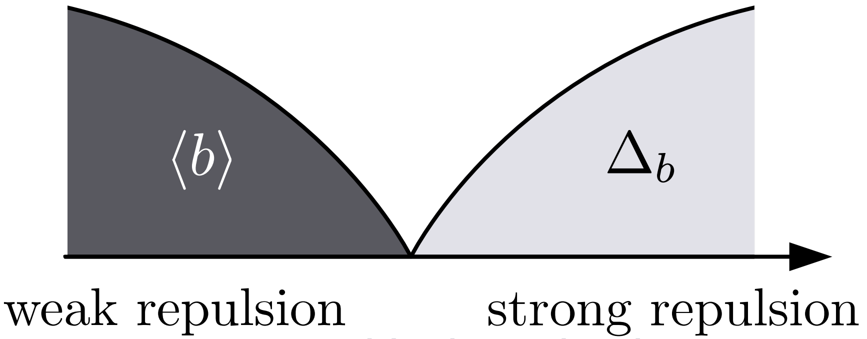


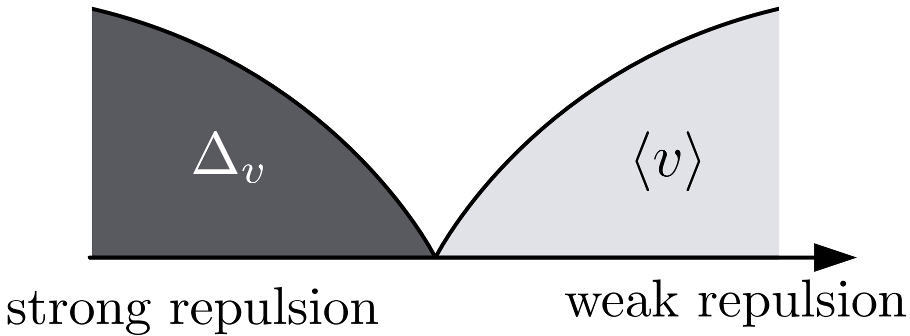
$$\mathcal{L}_{\text{Boson}} = |\partial_\mu \Psi_{\text{Boson}}|^2$$

$$\mathcal{L}_{\text{Vortex}} = |(\partial_\mu - i a_\mu)\Psi_{\text{Vortex}}|^2 + \frac{1}{2\kappa}(\epsilon_{\lambda\mu\nu}\partial_\mu a_\nu)^2$$

Bosons with **short-range** interactions

Vortices with **long-range** interactions





Peskin (1978), Thomas, Stone (1978), Dasgupta, Halperin (1981)

2+1 dimensions: Particle-vortex duality

$$\boxed{\frac{i}{\pi}\partial_x\theta\partial_\tau\varphi + g(\partial_x\varphi)^2 + g^{-1}(\partial_x\theta)^2}$$

$$\boxed{u\cos[\varphi_i - \varphi_{i+1}] + v\cos 2\theta_i}$$

$$\mathcal{L}_{\text{Boson}} = \mathcal{L}^{\text{kin.}}_{\text{Boson}} + \sum_i \left[u b_i^\dagger b_{i+1} + v p_i + \text{H.c.}\right]$$

~~duality~~ $\updownarrow$ $\qquad$ $\updownarrow$ duality $\updownarrow$

$$\mathcal{L}_{\text{Vortex}} = \mathcal{L}^{\text{kin.}}_{\text{Vortex}} + \sum_i \left[u\tilde{p}_{i+\frac{1}{2}} + v\tilde{b}^\dagger_{i-\frac{1}{2}}\tilde{b}_{i+\frac{1}{2}} + \text{H.c.}\right]$$

2+1 dimensions: Particle-vortex duality

$$\mathcal{L}^{\text{kin}}_{\text{Boson}}[b,p] = \mathcal{L}_{\text{non-local}}[\tilde{b},\tilde{p}]$$ (use operator mapping)

$$\mathcal{L}_{\text{gauge}}[\tilde{b},\tilde{p},a] \to \mathcal{L}'_{\text{non-local}}[\tilde{b},\tilde{p}]$$ (integrate out a)

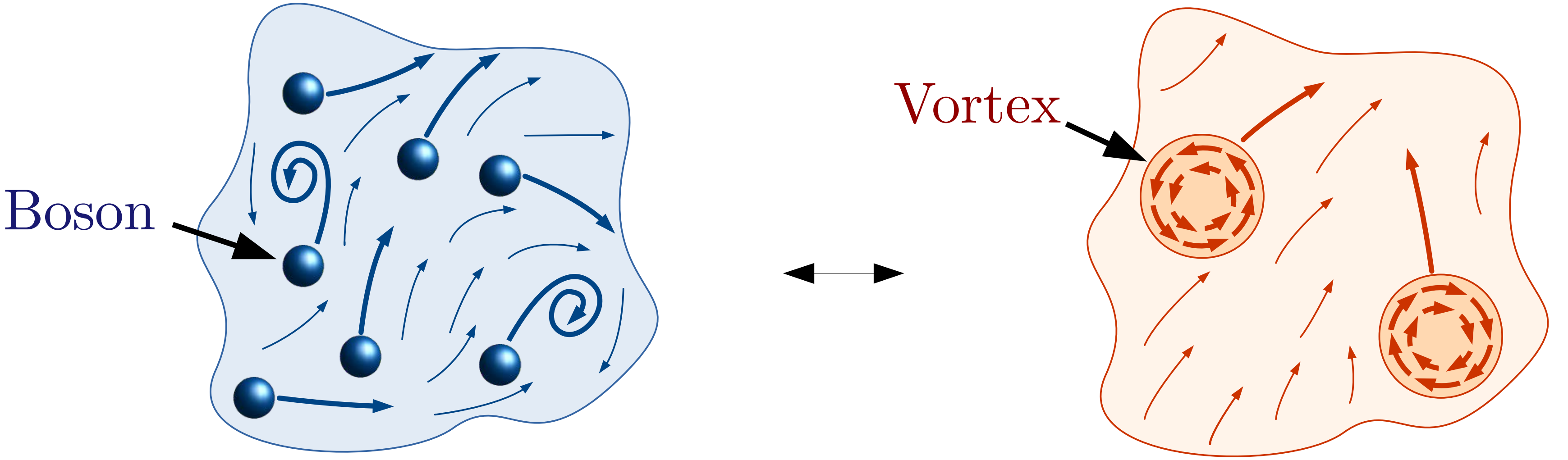
$$\to \text{find } \mathcal{L}_{\text{non-local}} = \mathcal{L}'_{\text{non-local}}$$

$$\mathcal{L}_{\text{Boson}} = \mathcal{L}^{\text{kin.}}_{\text{Boson}} + \sum_i \left[u b_i^\dagger b_{i+1} + v p_i + \text{H.c.} \right]$$

~~duality~~ ↕ ↕ duality ↕

$$\mathcal{L}_{\text{Vortex}} = \mathcal{L}^{\text{kin.}}_{\text{Vortex}} + \sum_i \left[u \tilde{p}_{i+\frac{1}{2}} + v \tilde{b}^\dagger_{i-\frac{1}{2}} \tilde{b}_{i+\frac{1}{2}} + \text{H.c.} \right]$$

2+1 dimensions: Particle-vortex duality

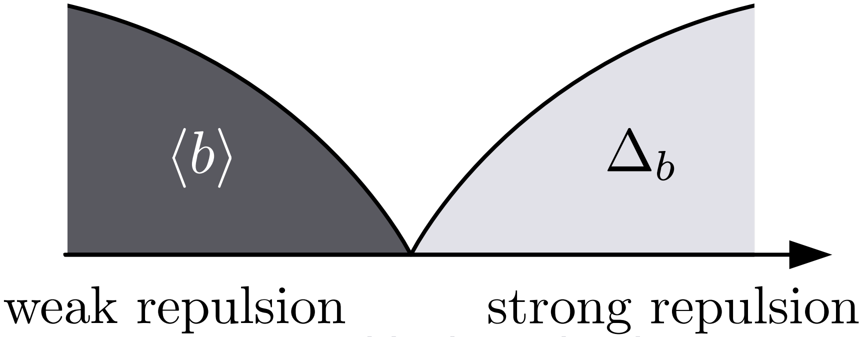


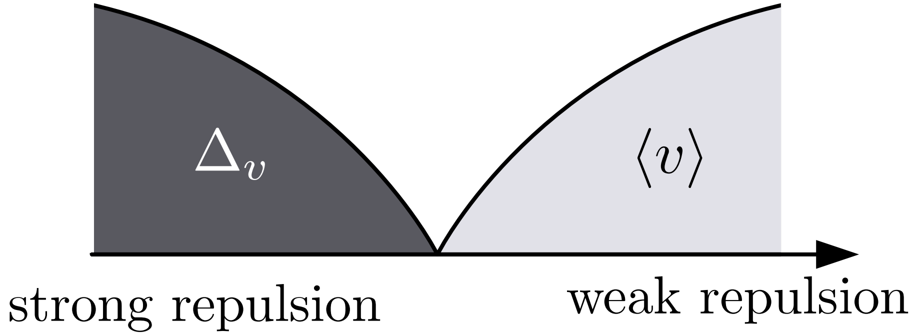
$$\mathcal{L}_{\text{Boson}} = |\partial_\mu \Psi_{\text{Boson}}|^2$$

$$\mathcal{L}_{\text{Vortex}} = |(\partial_\mu - i a_\mu)\Psi_{\text{Vortex}}|^2 + \frac{1}{2\kappa}(\epsilon_{\lambda\mu\nu}\partial_\mu a_\nu)^2$$

Bosons with **short-range** interactions

Vortices with **long-range** interactions





Peskin (1978), Thomas, Stone (1978), Dasgupta, Halperin (1981)

2+1 dimensions: Particle-vortex duality

Boson

$\mathcal{L}_{\text{Bo}}$

$\text{rtex}|^2$

$)^2$

Bo
inte

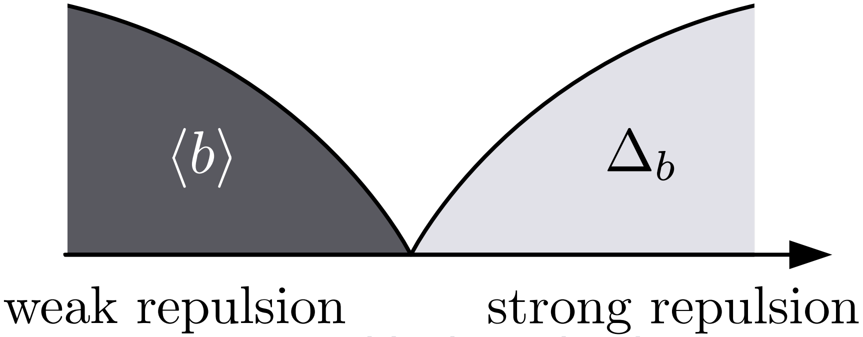
nge

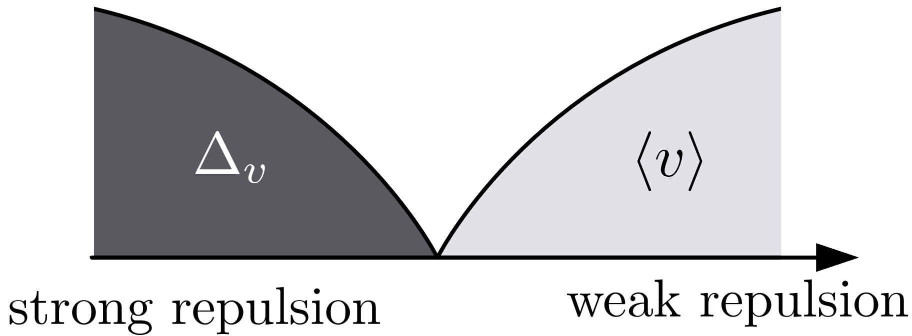
Disclaimer:

1. Use continuum actions to represent wire models

2. For these models, exact mapping

$$\mathcal{Z}_{\text{Boson}} = \text{const.} \times \mathcal{Z}_{\text{Vortex}}$$





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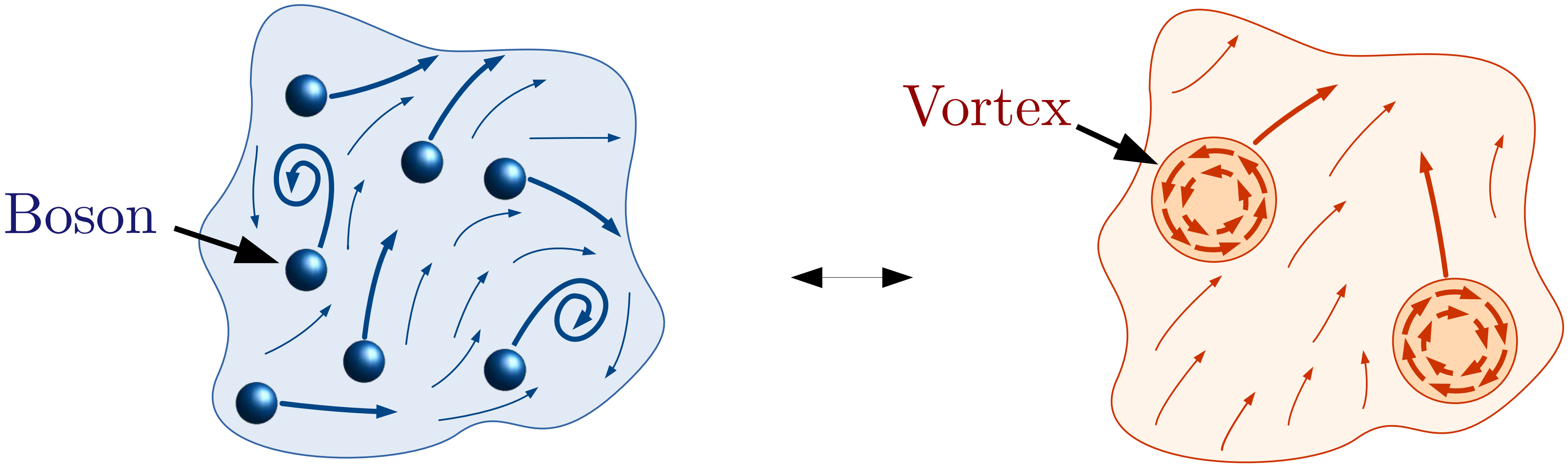
2+1 dimensions: Particle-vortex duality

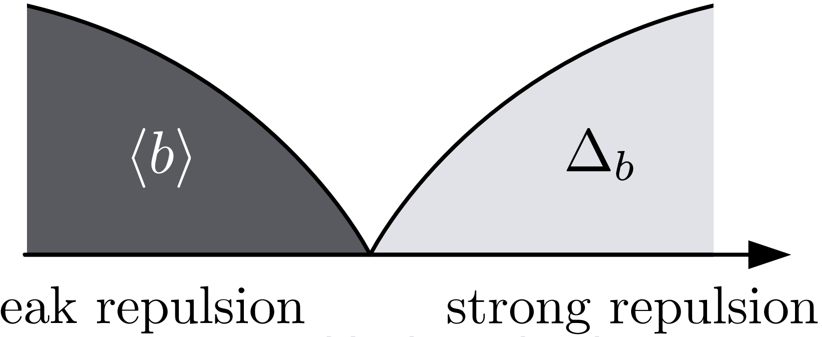


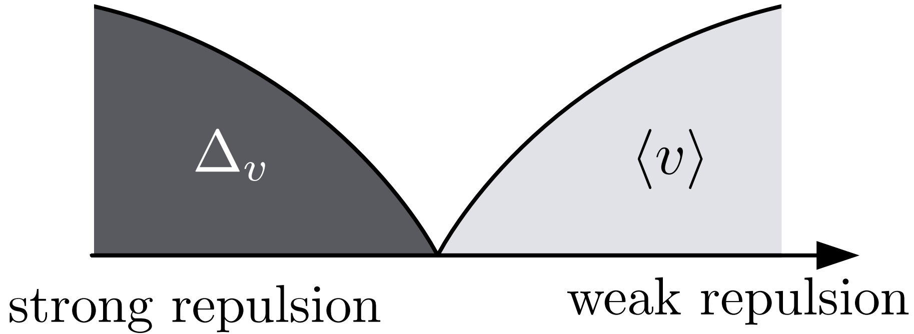
$$\mathcal{L}_{\text{Boson}} = |(\partial_\mu - iA_\mu)\Psi_{\text{Boson}}|^2 + \frac{1}{2K}(\epsilon_{\lambda\mu\nu}\partial_\mu A_\nu)^2$$

$$\mathcal{L}_{\text{Vortex}} = |\partial_\mu \Psi_{\text{Vortex}}|^2$$

Bosons with **long-range** interactions

Vortices with **short-range** interactions





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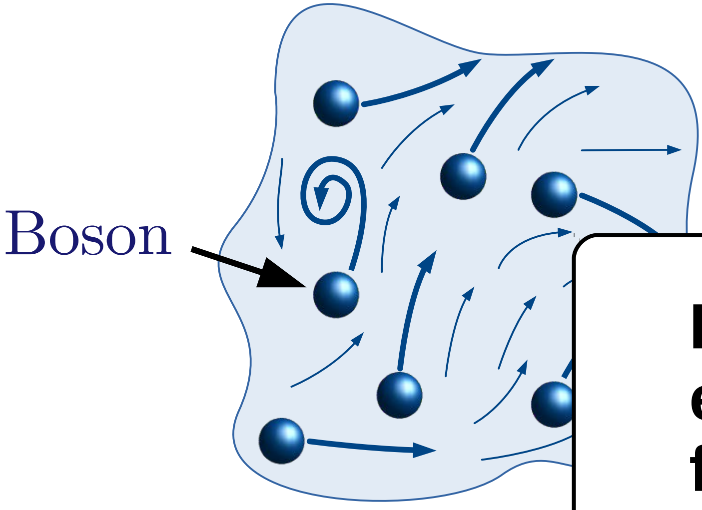
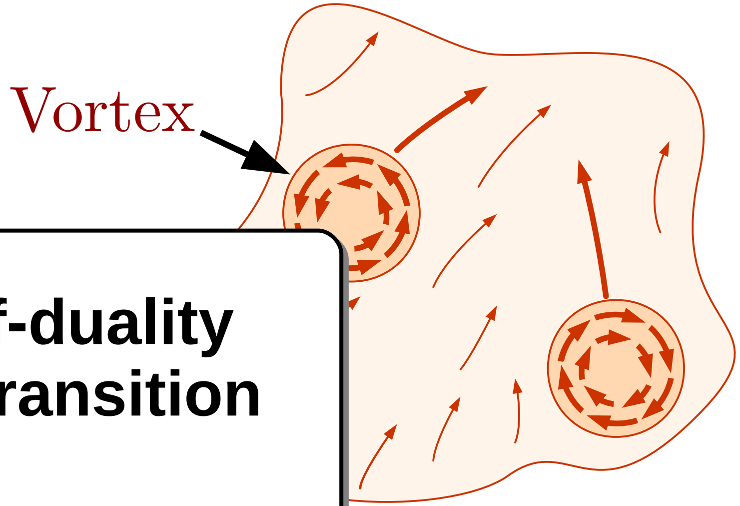
2+1 dimensions: Particle-vortex duality





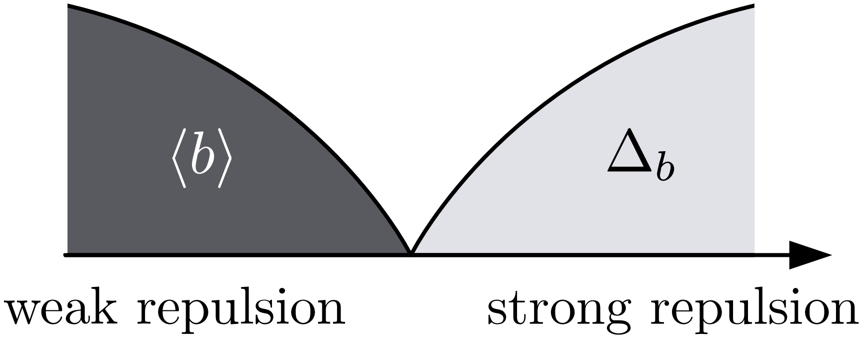
Exact self-duality ensures transition for $\lambda = 1$

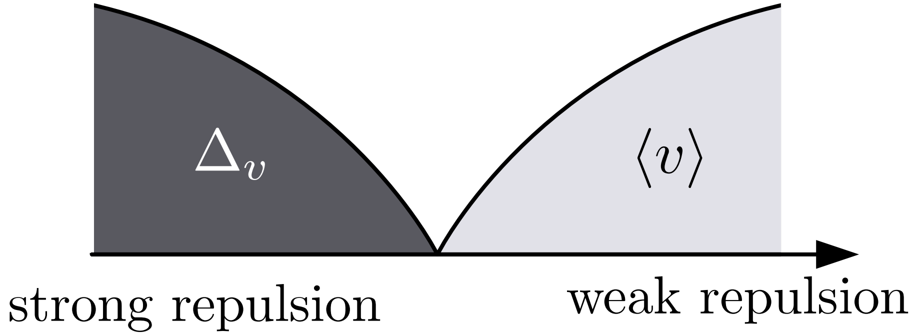
$$\mathcal{L}_{\text{Boson}} = |(\partial_\mu - iA_\mu)\Psi_{\text{Boson}}|^2 + \frac{\lambda}{4\pi|\mathbf{k}|}|\mathbf{k} \times A|^2$$

$$\mathcal{L}_{\text{Vortex}} = |(\partial_\mu - ia_\mu)\Psi_{\text{Vortex}}|^2 + \frac{1}{4\pi\lambda|\mathbf{k}|}|\mathbf{k} \times A|^2$$

Bosons with **intermediate-range** interactions

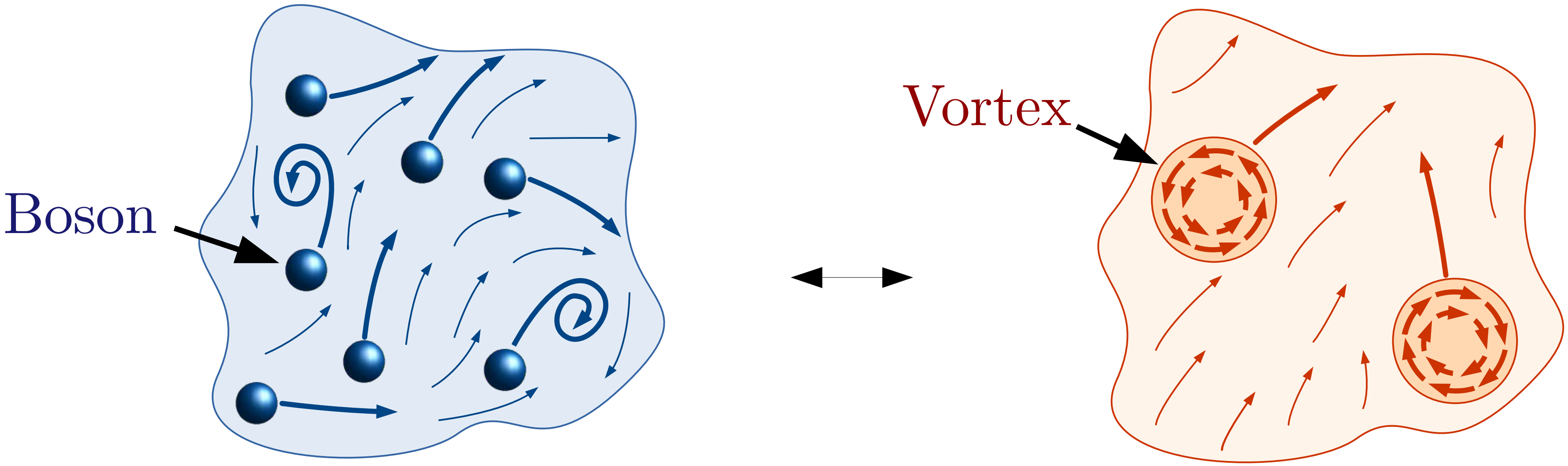
Vortices with **intermediate-range** interactions





Fradkin, Kivelson (1996)

2+1 dimensions: Particle-vortex duality

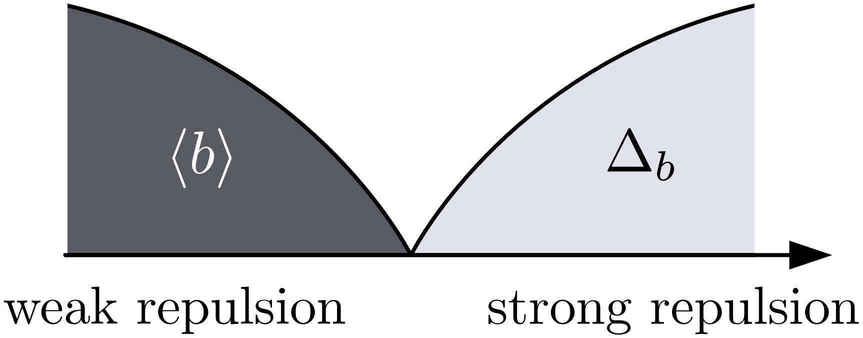


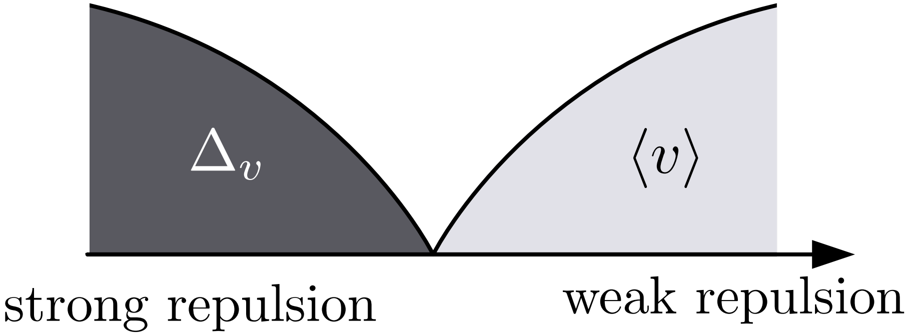
$$\mathcal{L}_{\text{Boson}} = |(\partial_\mu - iA_\mu)\Psi_{\text{Boson}}|^2 + \frac{i}{4\pi}A_\lambda\epsilon_{\lambda\mu\nu}\partial_\mu A_\nu$$

$$\mathcal{L}_{\text{Vortex}} = |(\partial_\mu - ia_\mu)\Psi_{\text{Vortex}}|^2 - \frac{i}{4\pi}a_\lambda\epsilon_{\lambda\mu\nu}\partial_\mu a_\nu$$

Bosons with **intermediate-range** interactions

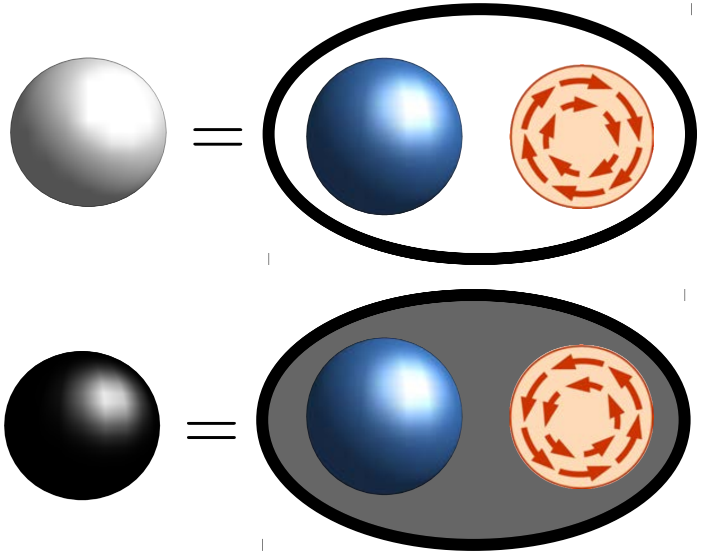
Vortices with **intermediate-range** interactions





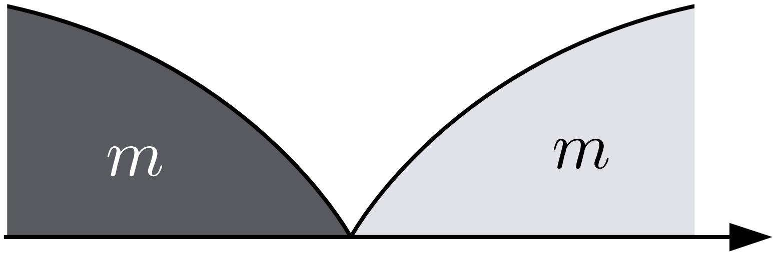
Fradkin, Kivelson (1996)

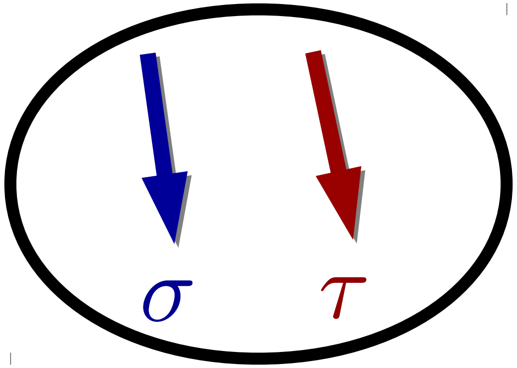
Particle-vortex duality as a symmetry



composite fermion

$$m = \Delta_v + \Delta_b$$

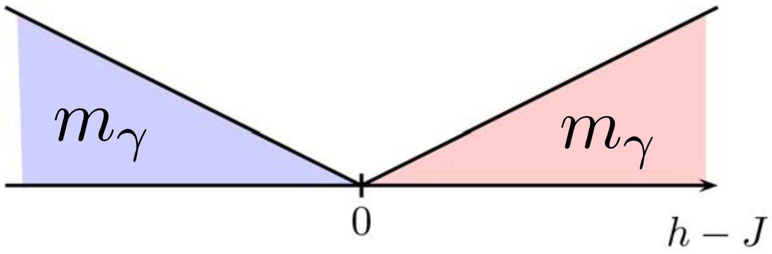




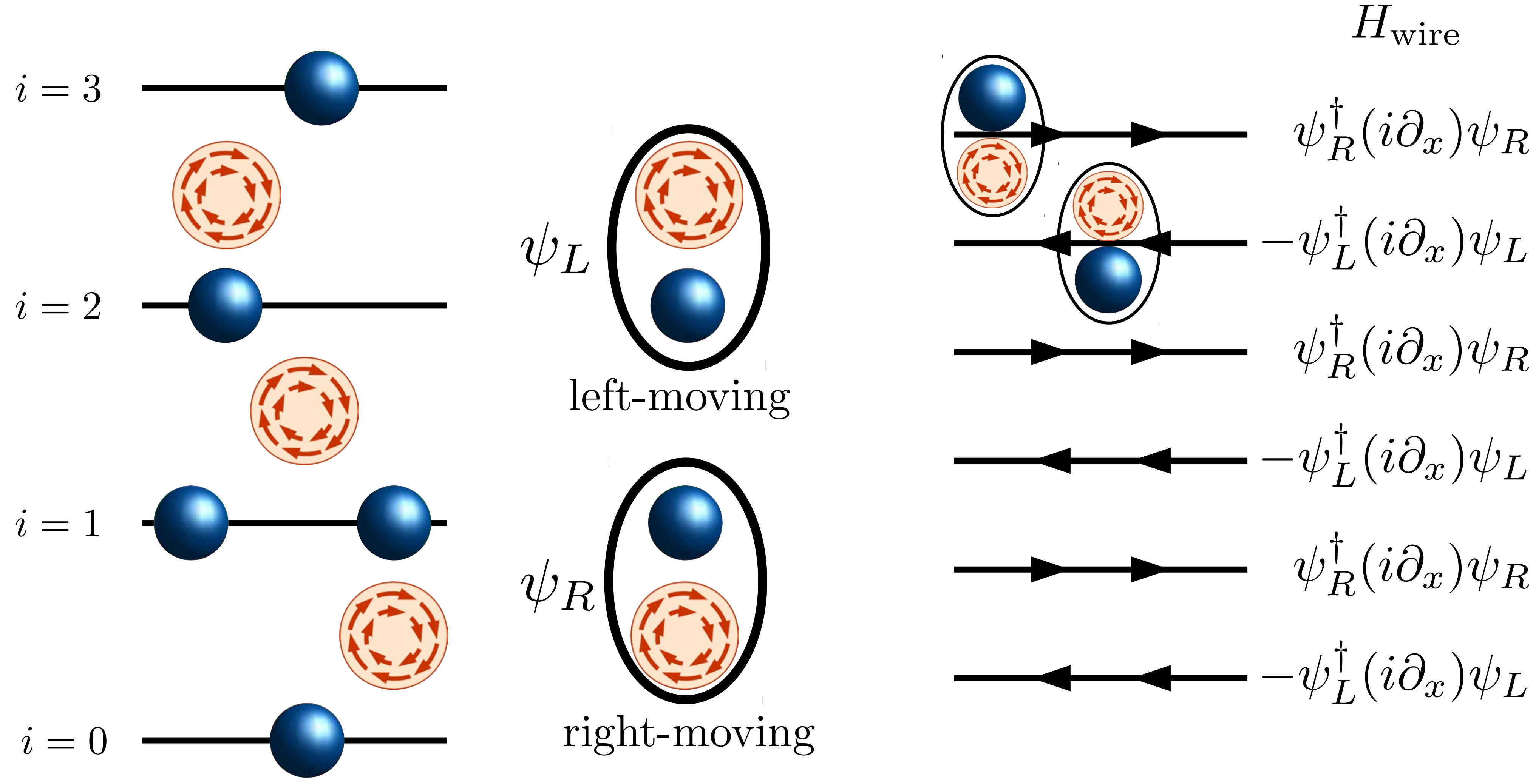
$$\gamma_{i+1/4} = i\sigma_i^z \tau_{i+1/2}^z$$

Jordan-Wigner fermion

$$m_\gamma = \Delta_\sigma + \Delta_\tau$$



Particle-vortex duality as a symmetry



$i = 3$

$i = 2$

$i = 1$

$i = 0$

ψ_L

left-moving

ψ_R

right-moving

H_{wire}

$\psi_R^\dagger (i\partial_x) \psi_R$

$-\psi_L^\dagger (i\partial_x) \psi_L$

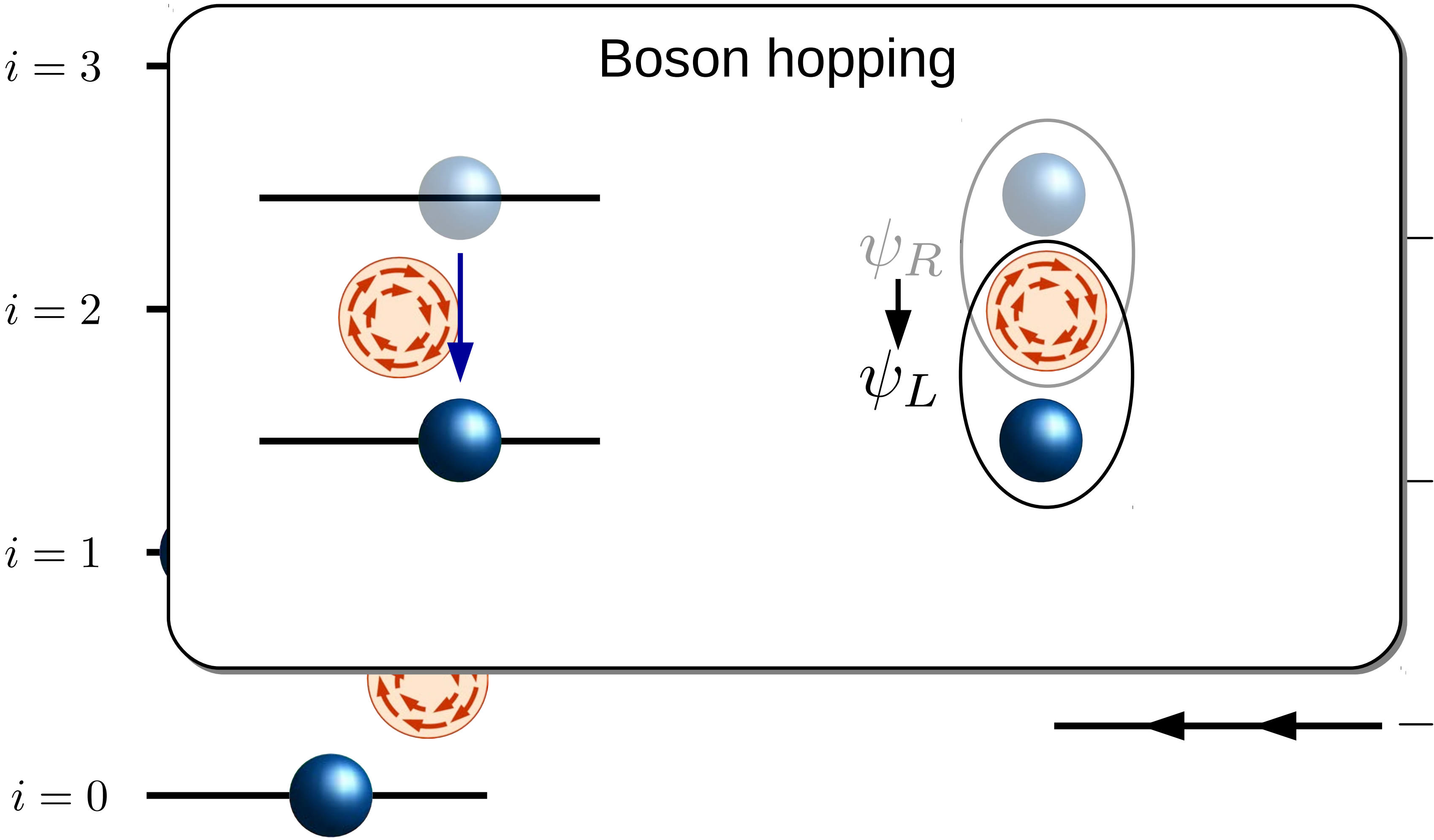
$\psi_R^\dagger (i\partial_x) \psi_R$

$-\psi_L^\dagger (i\partial_x) \psi_L$

$\psi_R^\dagger (i\partial_x) \psi_R$

$-\psi_L^\dagger (i\partial_x) \psi_L$

Particle-vortex duality as a symmetry



Boson hopping

H_{wire}

$\psi_R^\dagger(i\partial_x)\psi_R$

$-\psi_L^\dagger(i\partial_x)\psi_L$

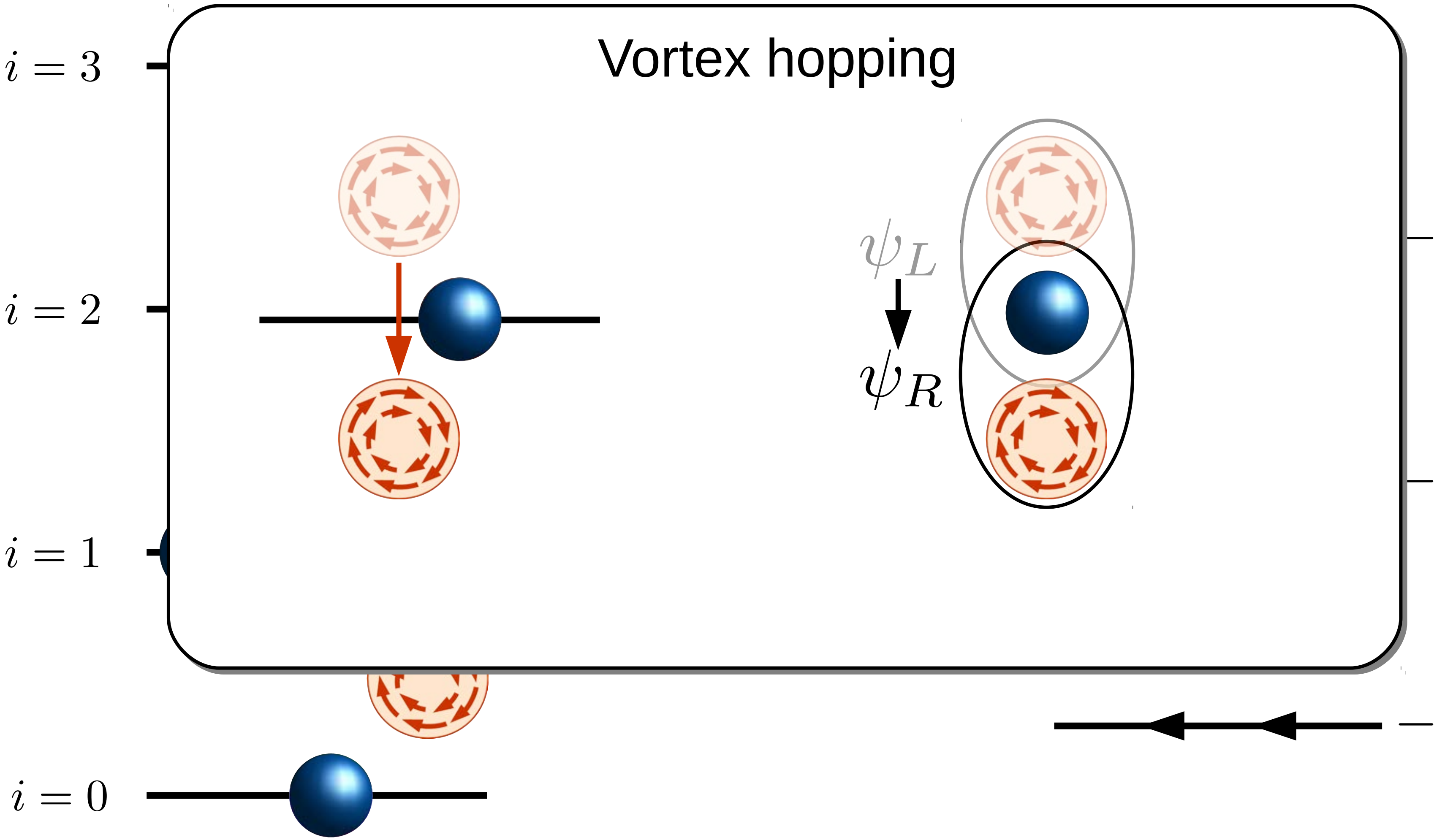
$\psi_R^\dagger(i\partial_x)\psi_R$

$-\psi_L^\dagger(i\partial_x)\psi_L$

$\psi_R^\dagger(i\partial_x)\psi_R$

$-\psi_L^\dagger(i\partial_x)\psi_L$

Particle-vortex duality as a symmetry



H_{wire}

$\psi_R^\dagger(i\partial_x)\psi_R$

$-\psi_L^\dagger(i\partial_x)\psi_L$

$\psi_R^\dagger(i\partial_x)\psi_R$

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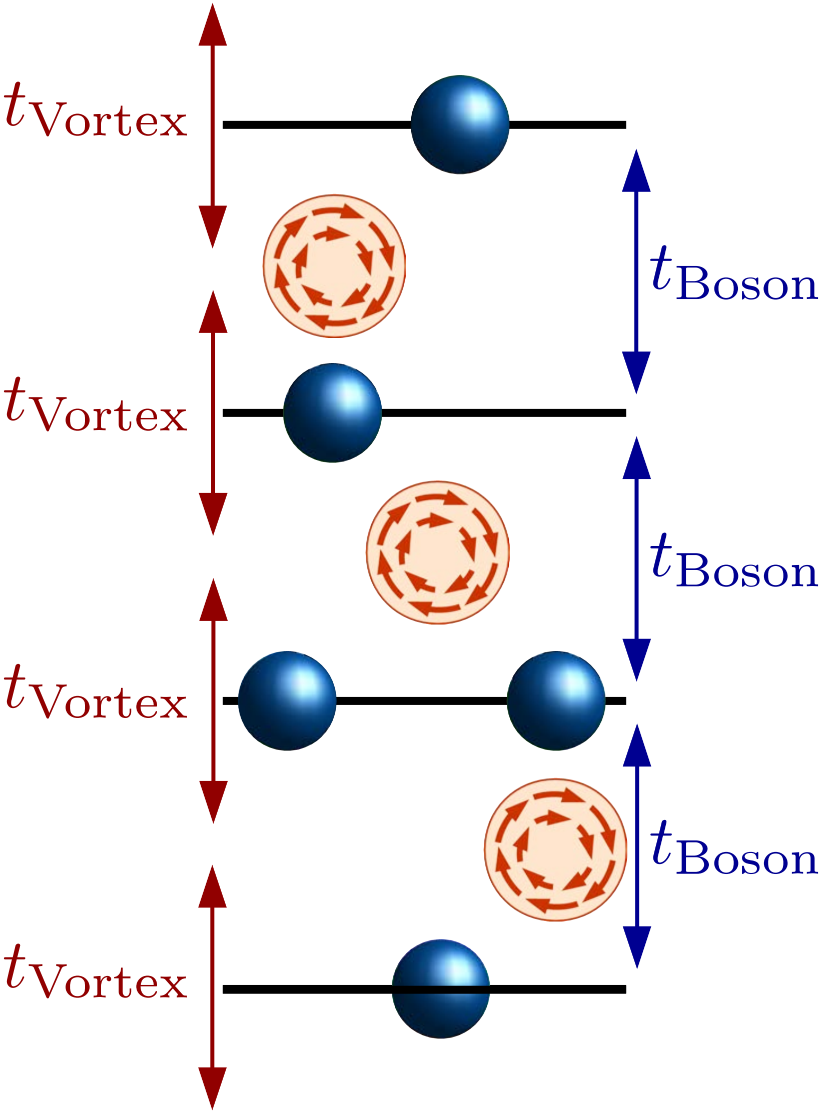
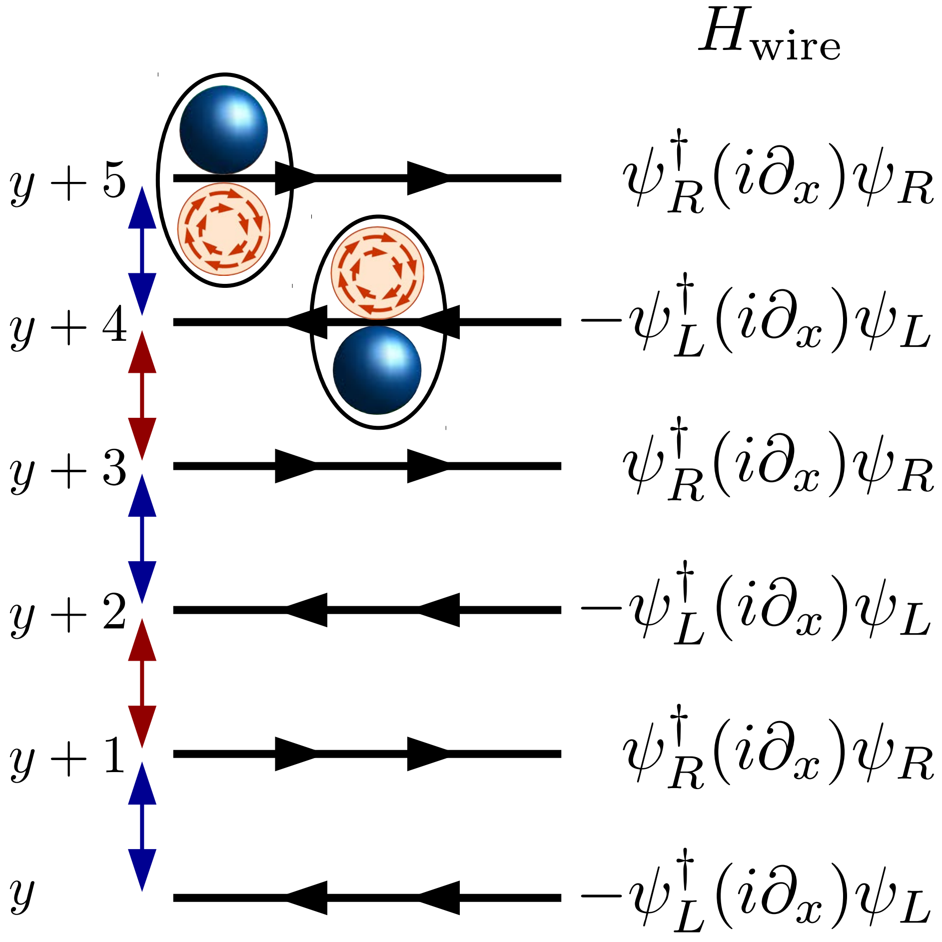
$\psi_R^\dagger(i\partial_x)\psi_R$

$-\psi_L^\dagger(i\partial_x)\psi_L$

Particle-vortex duality as a symmetry





Particle-vortex duality as a symmetry

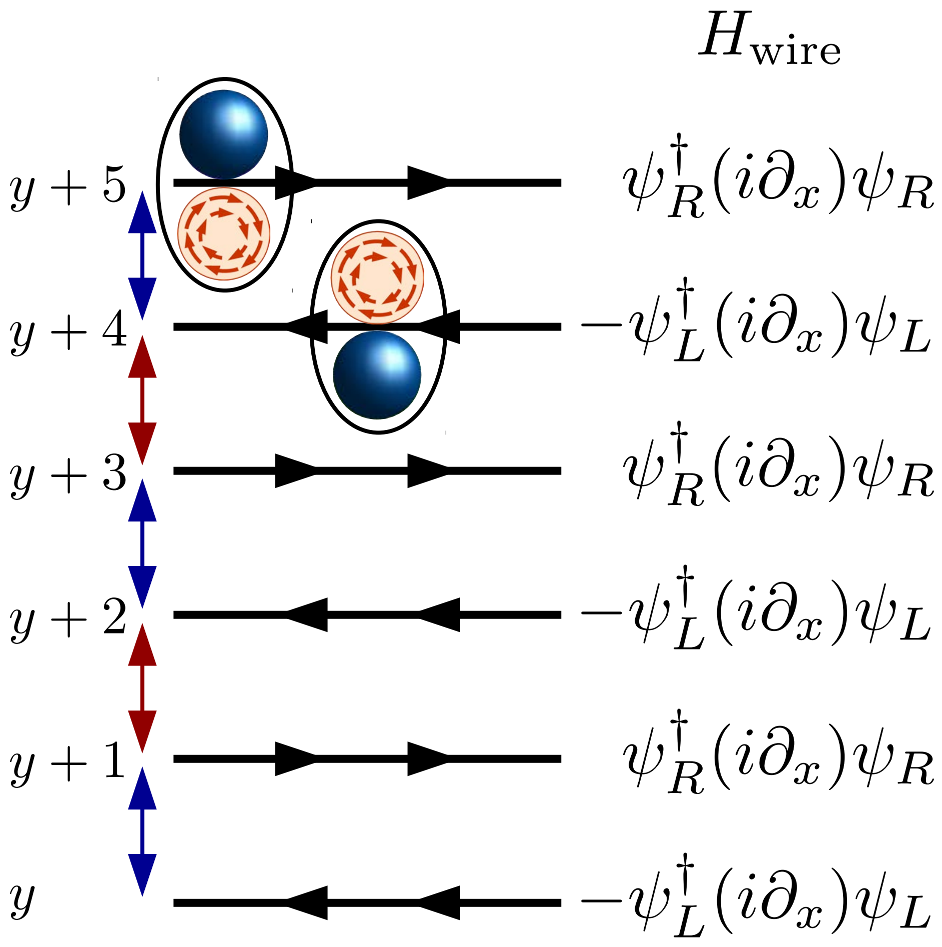
when $t_{\text{Vortex}} = t_{\text{Boson}}$

$$H_{\text{hop}} = -t \sum_y \psi_y^\dagger \psi_{y+1} + \text{H.c.}$$

2-component spinor $\quad \Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$

$$H_{\text{wire}} = \Psi^\dagger k_x \tau_z \Psi$$

$$H_{\text{hop}} = \Psi^\dagger \begin{pmatrix} 0 & k_y \\ k_y & 0 \end{pmatrix} \Psi$$



Particle-vortex duality as a symmetry

when $t_{\text{Vortex}} = t_{\text{Boson}}$

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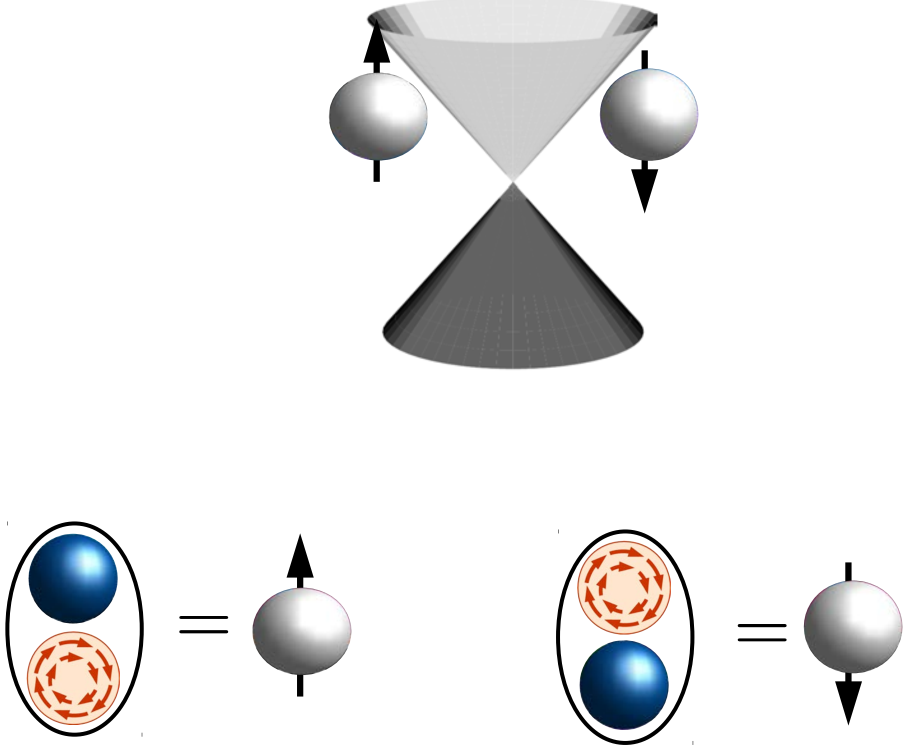
2-component spinor $\Psi = \begin{pmatrix} \psi \\ \psi \end{pmatrix}$

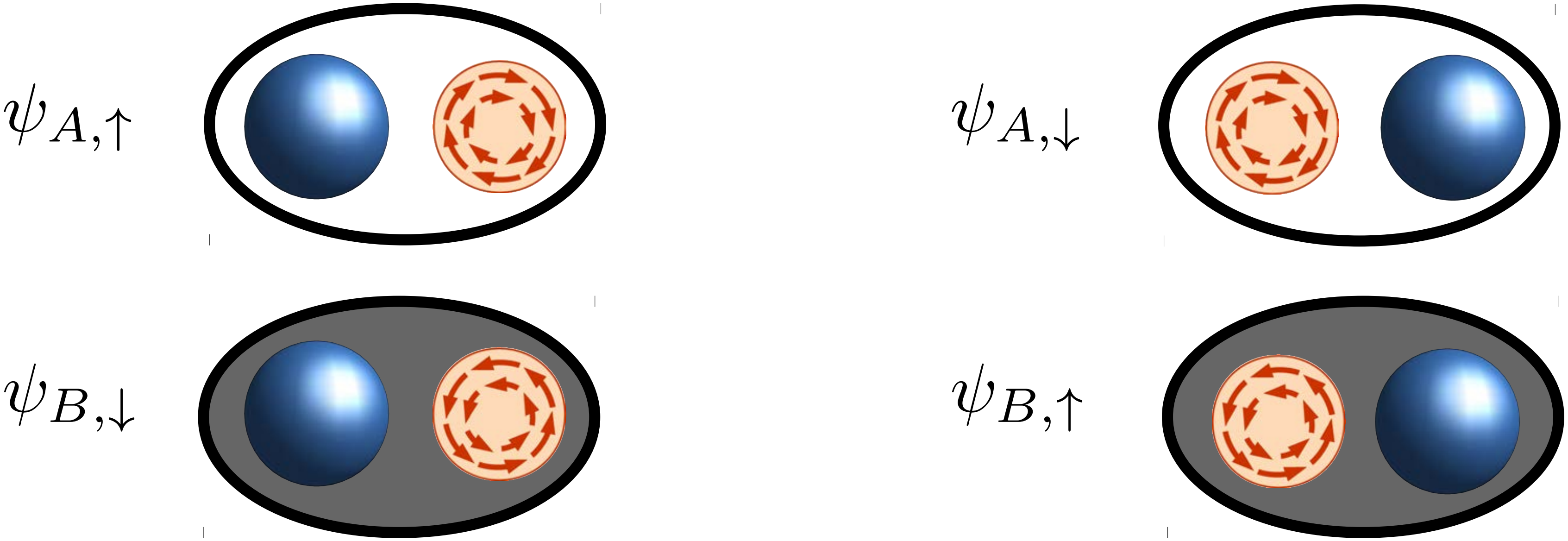
$$H_{\text{wire}} = \Psi^\dagger k_x \tau_z \Psi$$

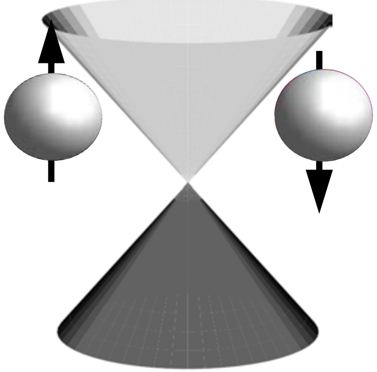
$$H_{\text{hop}} = \Psi^\dagger \begin{pmatrix} 0 & k_y \\ k_y & 0 \end{pmatrix} \Psi$$

$$H_{\text{hop}} + H_{\text{wire}} = H_{\text{Dirac}}$$



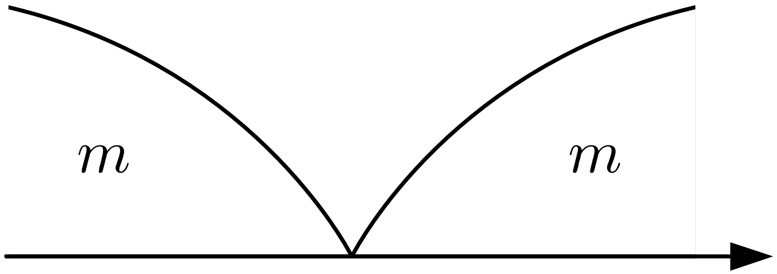
Particle-vortex duality as a symmetry

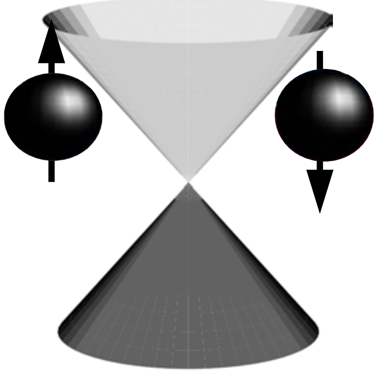




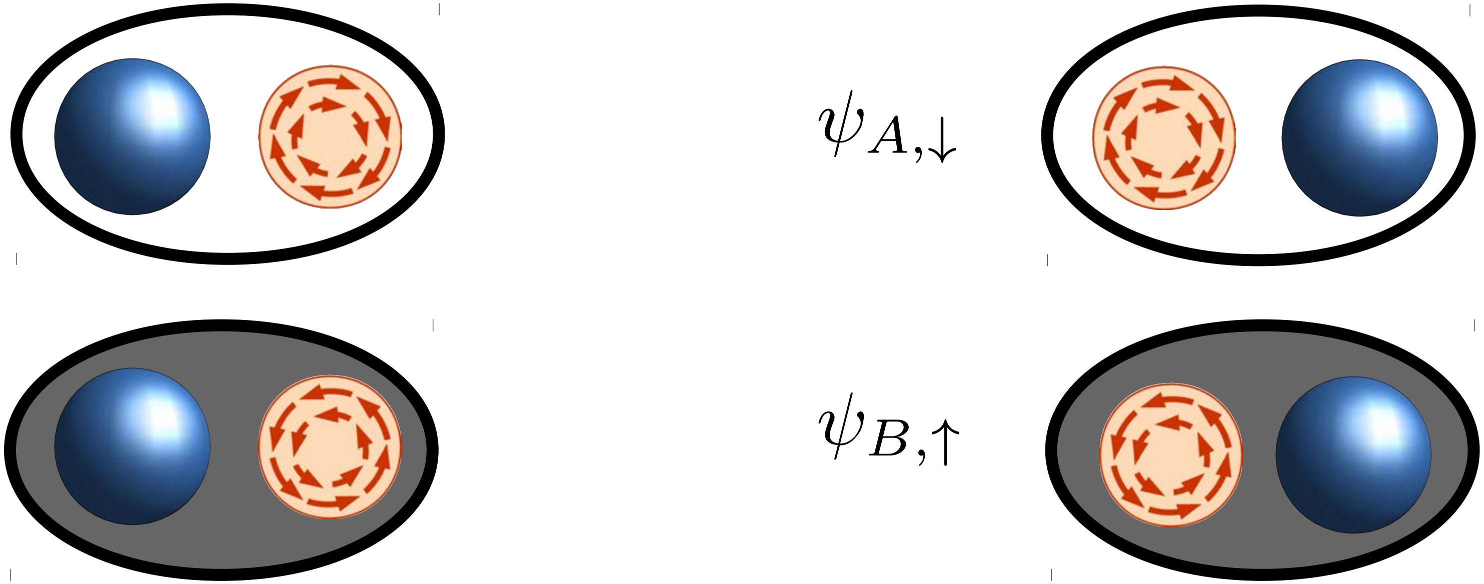
massless Dirac fermions

at critical point





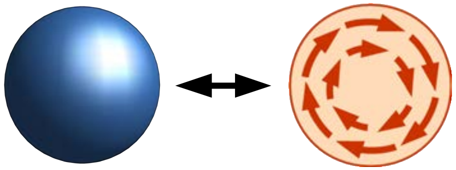
Particle-vortex duality as a symmetry



$\psi_{A,\uparrow}$ $\psi_{A,\downarrow}$

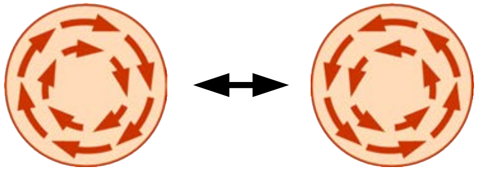
$\psi_{B,\downarrow}$ $\psi_{B,\uparrow}$

Duality



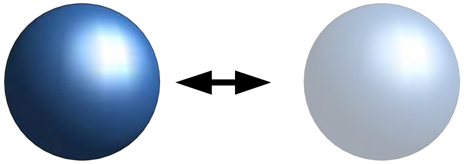
$$\psi_{A,\sigma} \leftrightarrow \psi_{A,-\sigma}$$
$$\psi_{B,\sigma} \leftrightarrow \psi^\dagger_{B,-\sigma}$$

Time reversal



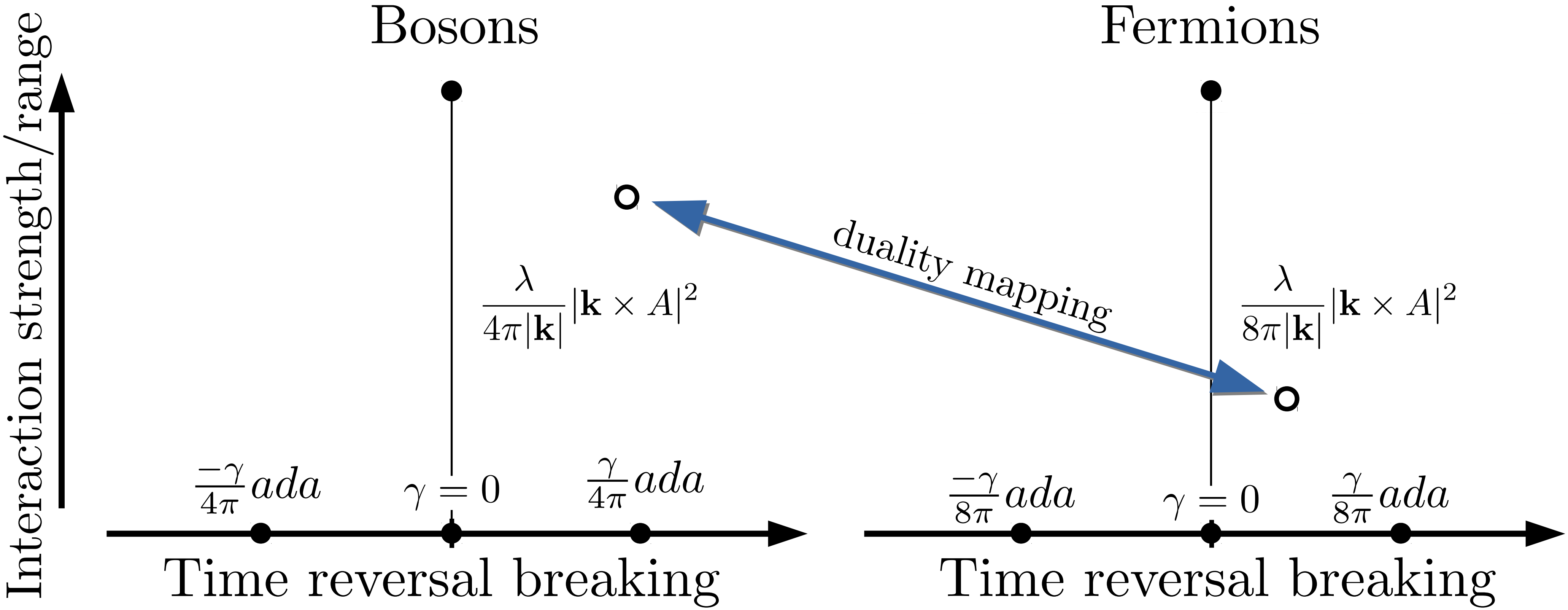
$$\psi_{A,\sigma} \leftrightarrow \psi_{B,\sigma}$$

Charge conjugation



$$\psi_{A,\sigma} \leftrightarrow \psi^\dagger_{B,\sigma}$$

Duality of interactions



Interaction strength/range

Bosons

$$\frac{\lambda}{4\pi|\mathbf{k}|}|\mathbf{k}\times A|^2$$

$\frac{-\gamma}{4\pi}ada$ $\gamma=0$ $\frac{\gamma}{4\pi}ada$

Time reversal breaking

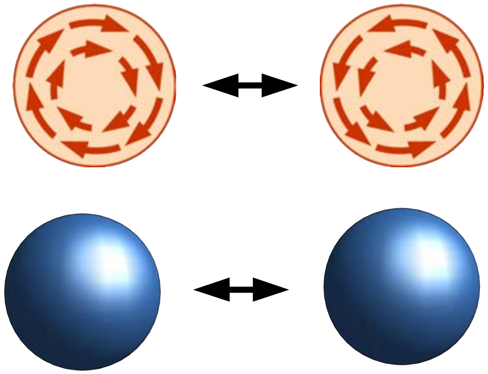
duality mapping

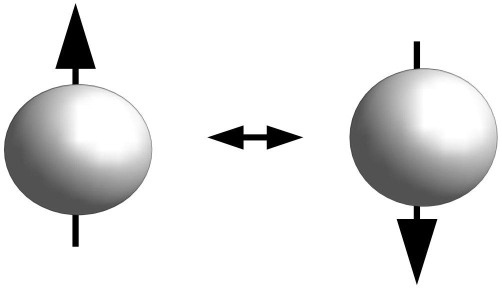
Fermions

$$\frac{\lambda}{8\pi|\mathbf{k}|}|\mathbf{k}\times A|^2$$

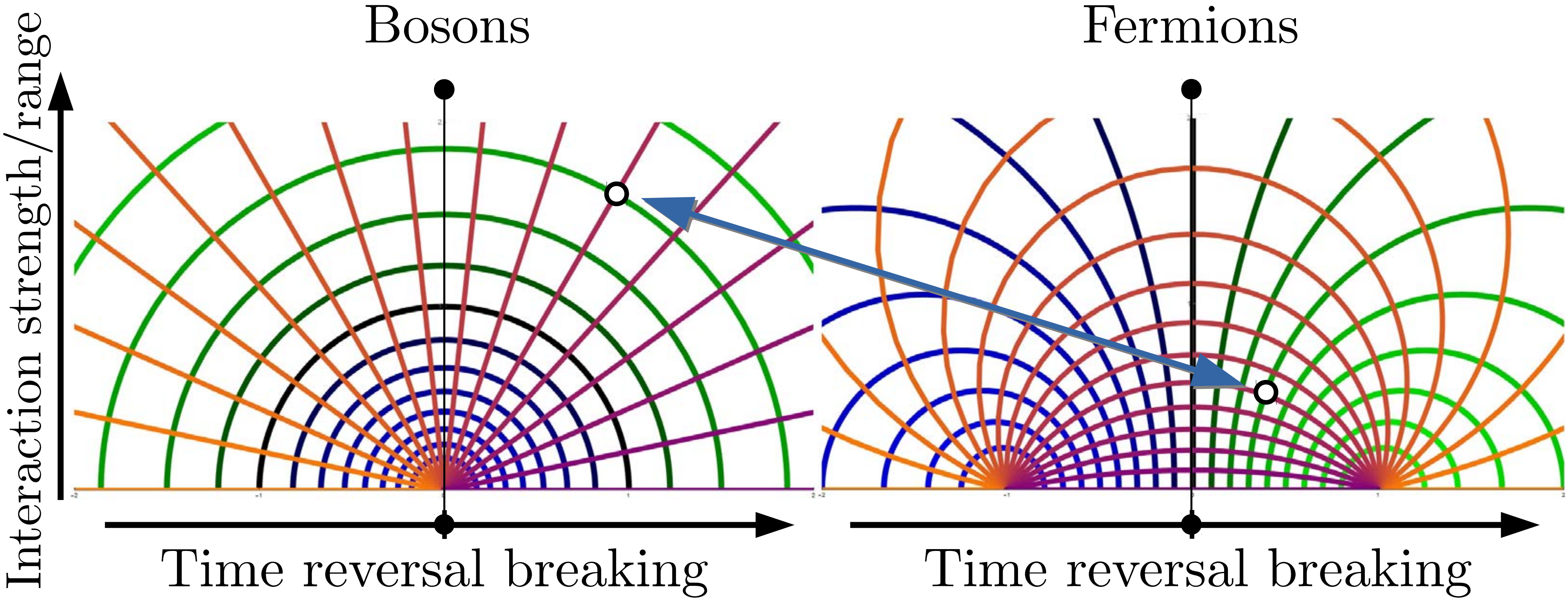
$\frac{-\gamma}{8\pi}ada$ $\gamma=0$ $\frac{\gamma}{8\pi}ada$

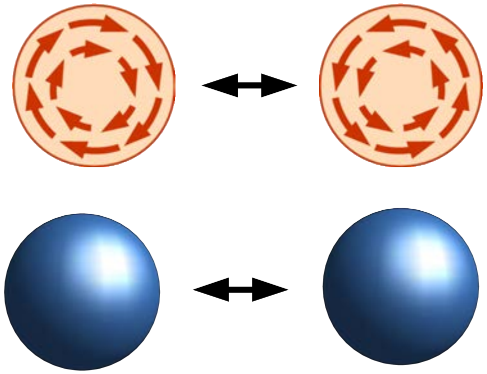
Time reversal breaking



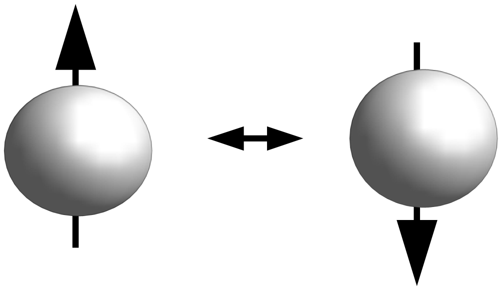


Duality of interactions



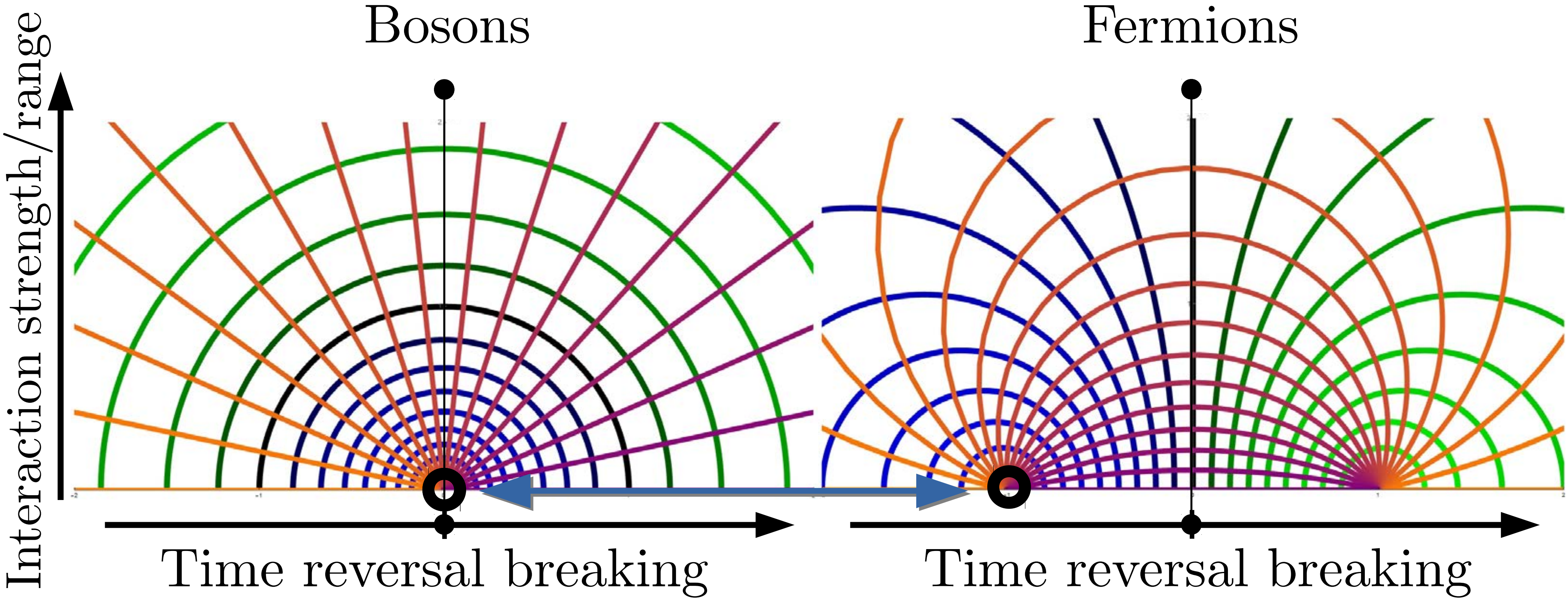


$$f(z) = \frac{z-1}{z+1}$$



DFM, Alicea, Motrunich (2017)

Duality of interactions

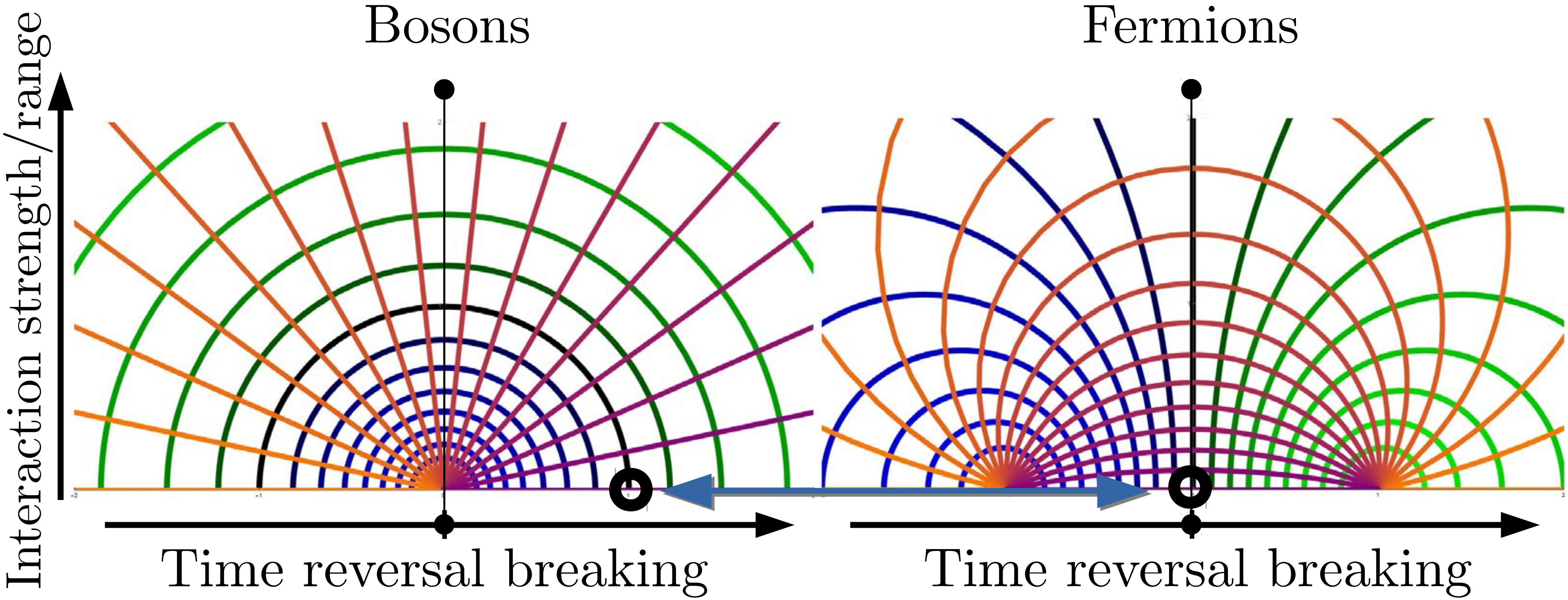


$$\mathcal{L}_{\text{Boson}} = |\partial_\mu \Psi_{\text{Boson}}|^2$$

$$\mathcal{L}_{\text{Fermion}} = i\bar{\Psi}_{\text{F}}(\partial_\mu - iA_\mu)\gamma^\mu \Psi_{\text{F}} + \frac{i}{8\pi} A_\lambda \epsilon_{\lambda\mu\nu} \partial_\mu A_\nu$$

Karch, Tong (2016), Seiberg, Senthil, Wang, Witten (2016)

Duality of interactions



Bosons

Fermions

Interaction strength/range

Time reversal breaking

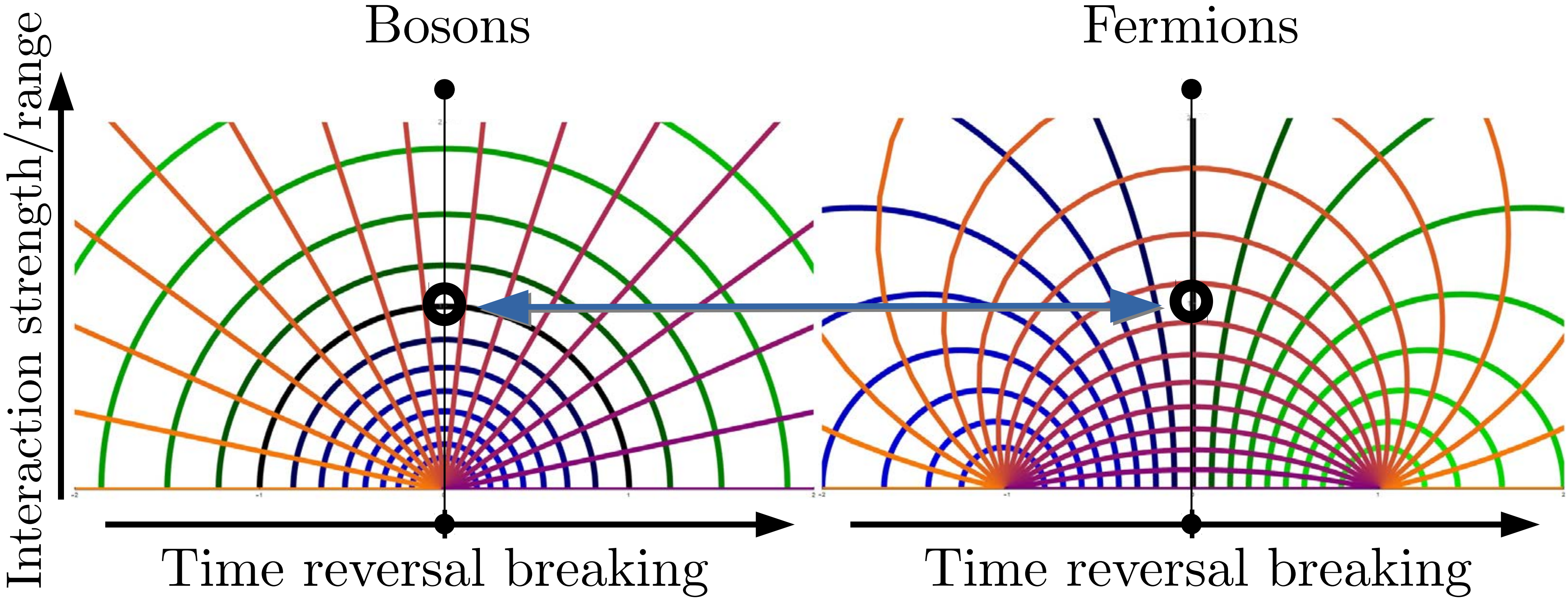
Time reversal breaking

$$\mathcal{L}_{\text{Boson}} = |(\partial_\mu - iA_\mu)\Psi_{\text{Boson}}|^2 + \frac{i}{4\pi} A_\lambda \epsilon_{\lambda\mu\nu} \partial_\mu A_\nu$$

$$\mathcal{L}_{\text{Fermion}} = i\bar{\Psi}_{\text{F}} \partial_\mu \gamma^\mu \Psi_{\text{F}}$$

Karch, Tong (2016), Seiberg, Senthil, Wang, Witten (2016)

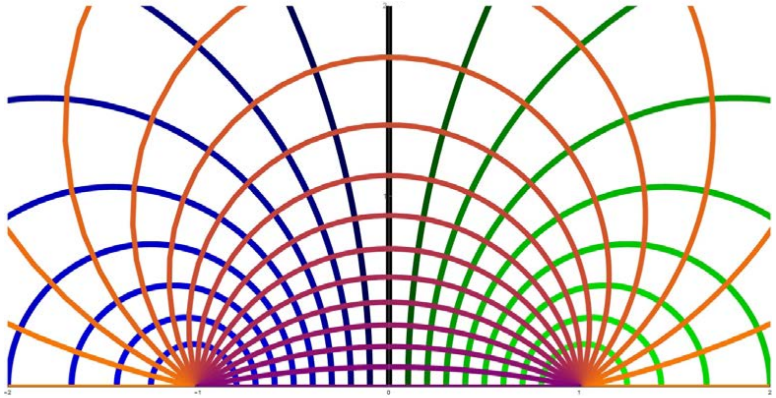
Duality of interactions



$$\mathcal{L}_{\text{Boson}} = |(\partial_\mu - iA_\mu)\Psi_{\text{Boson}}|^2 + \frac{1}{4\pi|\mathbf{k}|}|\mathbf{k}\times A|^2$$

$$\mathcal{L}_{\text{Fermion}} = i\bar{\Psi}_{\text{F}}(\partial_\mu - iA_\mu)\gamma^\mu\Psi_{\text{F}} + \frac{1}{8\pi|\mathbf{k}|}|\mathbf{k}\times A|^2$$

Outlook

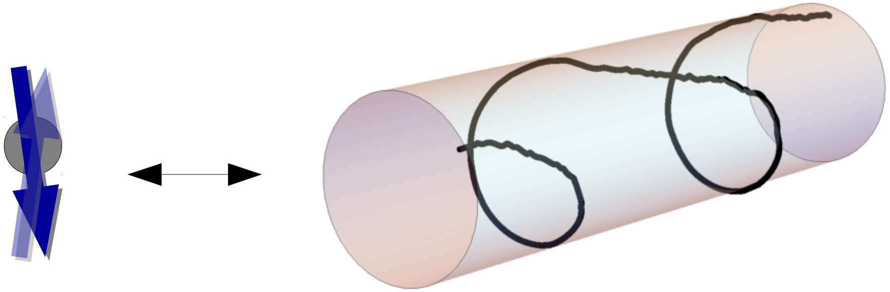


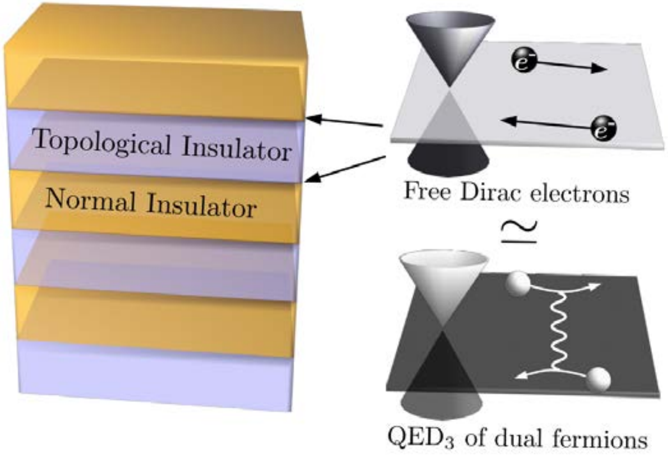
‘Strong’ or ‘weak’ duality

- Need numerical verification

Additional dualities in 2+1d

- Different symmetries
- Use 1d → 2d analogy for other known dualities in 1d





Topological phases in 3+1d

- E.g., stack of 2d systems + interlayer coherence