Loop models, modular symmetry and duality in 2+1 dimensions

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Talk at the Aspen Center for Physics Conference *Field Theory Dualities and Strongly Correlated Matter*, Aspen, CO, March 2018

ArXiv:1801.04936

Motivation

- Recent interest in dualities in CM and QFT
- Particle-Vortex duality and its applications to the Fractional Quantum Hall Effect
- Conjectured web of CFT dualities: bosonization and fermionization

Can this “web of dualities” be “derived”?

- Loop models: flux attachment, duality and periodicity
- Modular Invariance?
- Periodicity and Fractional Spin
- Fractional spin and a web of dualities for loop models

Dualities

- EM duality: E $\Longleftrightarrow$ B, electric charges $\Longleftrightarrow$ magnetic monopoles$\Rightarrow$Dirac quantization
- 2D Ising Model: Kramers-Wannier duality, high T $\Longleftrightarrow$ low T, order $\Longleftrightarrow$ disorder
- Duality of the 3D $\mathbb{Z}_2$ gauge theory $\Longleftrightarrow$ 3D Ising model, order $\Longleftrightarrow$ confinement
- Particle-Vortex duality: electric charge $\Longleftrightarrow$ vortex (magnetic charge)
- Dualities are mappings between phases of matter, most often for different theories
- Conjectured web of dualities in 2+1 dimensions
- These are dualities between fixed point theories (i.e. CFTs)
- What is the relation between these different dualities?

Some General References on Dualities

- M. E. Peskin, Ann. Phys. 113, 122 (1978)
- C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. 47, 1556 (1981)
- A. M. Polyakov, Mod. Phys. Lett. A 3, 325 (1988)
- E. Witten, Comm. Math. Phys. 121, 351 (1989)
- C. P. Burgess and F. Quevedo, Nucl. Phys. B 421, 373 (1993)
- E. Fradkin and F. A. Schaposnik, Phys. Lett. B 338, 253 (1994)
- E. Fradkin and S. A. Kivelson, Nucl. Phys. B 474, 543 (1996)
- D. T. Son, Phys. Rev. X 5, 031027 (2015)
- N. Seiberg, T. Senthil, C. Wang, and E. Witten, Ann. Phys. 374, 395 (2016)
- A. Karch and D. Tong, Phys. Rev. X 6, 031043 (2016)
- O. Aharony, JHEP 02, 093 (2016)
- D. Mross, J. Alicea, and O. Motrunich, Phys. Rev. X 7, 041016 (2017)
- J.-Y. Chen, J.H. Son, C. Wang, and S. Raghu, Rhys. Rev. Lett. 120, 016602 (2018)

Bosonic Particle-Vortex Duality

- Particle-vortex duality of 3D XY model (Peskin, Stone, Halperin-Dasgupta)
- Phases:
    - U(1) broken symmetry phase ($m^2 < 0$); excitations: closed quantized vortex lines with long range interactions
    - U(1) unbroken symmetry phase ($m^2 > 0$); excitations: massive charged bosons, closed worldlines with short-range repulsive interactions
- Duality: particles $\Leftrightarrow$ vortices, high T $\Leftrightarrow$ low T, strong coupling $\Leftrightarrow$ weak coupling
- The 3D XY model is a loop model: lattice partition function sums over configurations of closed loops with short-range repulsive interactions
- Phase diagram for FQH fluids (Kivelson, Lee and Zhang) and plateau transitions
- 2+1-dimensional boson-boson complex scalar field theory mapping

$$|D(A)\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \leftrightarrow |D(a)\varphi|^2 + m^2|\varphi|^2 - \lambda|\varphi|^4 + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda$$

Web of Dualities

Recently conjectured dualities between *fixed points* (relativistic CFTs)

Fermionic particle-vortex duality:

$$i\bar{\psi}\not{D}(A)\psi - \frac{1}{8\pi}AdA \longleftrightarrow i\bar{\chi}\not{D}(a)\chi - \frac{1}{4\pi}adA - \frac{1}{8\pi}AdA$$

bosonization duality:

$$i\bar{\psi}\not{D}(A)\psi - \frac{1}{8\pi}AdA \longleftrightarrow |D(a)\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA$$

Strategy

- “Derive” this web of dualities by looking near criticality, but still sitting in the gapped phase, using quantum loop models.
- These models are related to *modular invariant* models originally introduced by Kivelson and myself, but we will see that modular invariance cannot be kept as the CFT fixed point is approached.
- The “fractional spin” responsible for breaking modular invariance will give rise to Dirac fermions, leading us to loop model “proofs” of the CFT duality web.

Quantum Loop Models and Duality

(EF and Kivelson 1996)

- Non-intersecting linked loops [J_μ] in Euclidean space-time (with no spin) with exact particle-hole symmetry
- flux attachment with fractional statistics θ, long ranged interactions with coupling g, and short-range repulsion ("hard core")
- The imaginary part of the action is given in terms of the linking number

$$Z[g,\theta] = \sum_{\{J_\mu\}\in\mathbb{Z}} \delta(\Delta_\mu J^\mu) e^{-S[J_\mu]}$$

$$S[J_\mu] = \frac{g^2}{2}\sum_{x,y} J_\mu(x) G_{\mu\nu}(x-y) J_\nu(y) + i\theta \sum_{x,y} J_\mu(x) K_{\mu\nu}(x-y) J_\nu(y)$$

long ranged interactions

linking number = $\theta\Phi[J_\mu]$

$$G_{\mu\nu}(p) = \frac{1}{\sqrt{p^2}}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \; K_{\mu\nu}(p) = i\epsilon_{\mu\nu\lambda}\frac{p_\lambda}{p^2}$$

Field theory picture $\quad \mathcal{L} = |D_\mu\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4g^2} f_{\mu\nu}\frac{1}{\sqrt{-\partial^2}} f_{\mu\nu} + \frac{k}{4\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$

Self-Duality and Modular Invariance

Modular parameter: $\tau = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}$

- The partition functions of loop models regularized without fractional spin the partition have the symmetries
- $\mathcal{S}$: duality: Z[τ]=Z[-1/τ], and $\mathcal{T}$: Periodicity: Z[τ]=Z[τ+1]
- $\mathcal{S}$ and $\mathcal{T}$ generate the modular group PSL(2,$\mathbb{Z}$)
- The partition function is self dual at the fixed points of the modular group
- Two types of PSL(2,$\mathbb{Z}$) fixed points: “bosonic” and “fermionic”
- FK showed that the finite modular fixed points are quantum critical points with $\sigma_{xx} \neq 0$ and $\sigma_{xy} = 0$
- Problem: the predicted conductivities are different in the FK loop models and the web of dualities

The Role of Fractional Spin

- The linking number of two separate loops l_1 and l_2 is

$$\Phi[J = \ell_1 + \ell_2] = 2 \times (\text{Linking number of } \ell_1 \text{ with } \ell_2) + W[\ell_1] + W[\ell_2]$$

"Writhe." Associated with self linking. **Not necessarily a topological invariant.**

- Witten: point-split the loops into ribbons so that the writhe is a frame-dependent topological invariant $W[l] = SL[l] =$ integer. **Only consistent deep in the topological phase, not as the critical point is approached.**

- Polyakov: no-point splitting and $W[l] = SL[l] - T[l]$ (writhe = self-linking - twist)

$$T[\ell] = \frac{1}{2\pi} \int_0^L ds \int_0^1 du \boldsymbol{e} \cdot \partial_s \boldsymbol{e} \times \partial_u \boldsymbol{e}$$

$T[l]$ is a Berry phase, referred to as **fractional spin**, and ***e*** is the tangent vector to the loop. $T[l]$ **is not quantized**.

Fractional Spin: Periodicity Lost, 3D Bosonization Recovered

- *T*[l] is **not** quantized. Means Duality S remains a symmetry, **but not** periodicity $\mathcal{T}$.

- Polyakov's construction: fractional spin leads to the (IR) duality between a complex massive scalar with CS at k = 1 and a massive Dirac *spinor* (with a parity anomaly)

$$Z_{\text{fermion}} = \det\left[i\partial\!\!\!/ + M\right] = \int \mathcal{D}J\delta(\partial_\mu J^\mu)e^{-|m|L[J]+i\pi\,\mathrm{sgn}(M)\Phi[J]}$$

L[J]: length of loop, *Φ*[J]: linking number

- ***Can we use this to "derive" the web of dualities? Yes!***
- First step: We introduce background fields to the boson side of Polyakov's duality, find the parity anomaly on the fermionic side

$$i\Psi \not{D}_A \Psi + M\bar{\Psi}\Psi - \frac{1}{8\pi} AdA \qquad\qquad |D_{a-A}\phi|^2 - m^2|\phi|^2 - |\phi|^4 - \frac{\text{sgn}(M)}{4\pi} ada + \frac{\text{sgn}(M)-1}{8\pi} AdA$$

exact rewriting as loop model

$$\det[i\not{D}_A + M] e^{-iCS[A]/2} = \int \mathcal{D}J \delta(\partial_\mu J^\mu) e^{-|m|L[J] + iS_{\text{fermion}}[J,A]} e^{-iCS[A]/2}$$

$$S_{\text{fermion}}[J,A] = \int d^3x\, J_\mu A^\mu + \text{sgn}(M)\pi\, \Phi[J] + \frac{\text{sgn}(M)}{2} CS[A] + \cdots$$

short-ranged interactions ("hard core")

Loop Models: Tools for Deriving Dualities

How to derive a duality:

1. Start with a proposed duality and write down boson loop models for each theory using Polyakov's duality.

2. Use path integral manipulations to equate the two loop model partition functions.

3. Match both sides of the critical point by invoking bosonic particle-vortex duality. Relates superfluid of particles to insulator of vortices.

Example: fermionic particle-vortex duality

We proceed to derive the duality between free Dirac fermion and QED3 (with properly quantized CS levels). We show the derivation for the trivial phase explicitly

$$i\bar{\Psi}\not{D}_A\Psi - \frac{1}{8\pi}AdA \longleftrightarrow i\bar{\psi}\not{D}_a\psi + \frac{1}{8\pi}ada - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA$$

$$+M\bar{\Psi}\Psi \qquad\qquad -M'\bar{\psi}\psi$$

$$\int d^3x\, J_\mu A^\mu + \pi\Phi[J] \longleftarrow -\pi\Phi[J] + \int d^3x \left[J_\mu a^\mu - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA\right]$$

Integrate out a, b.

- Case for opposite mass signs (QH phase) follows from the same logic

- Current mapping also natural upon integrating out b:

$$\bar{\Psi}\gamma^\mu\Psi \longleftrightarrow \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda$$

General Bosonization Dualities

Loop models can also be used to fermionize theories of the form

$$\mathcal{L} = |D_a\phi|^2 - |\phi|^4 + \frac{k_\phi}{4\pi} ada + \frac{1}{2\pi} adA$$

One obtains, consistently with the duality web

$$|D_a\phi|^2 - |\phi|^4 + \frac{k_\phi}{4\pi} ada + \frac{1}{2\pi} adA \leftrightarrow i\bar{\psi}\not{D}_b\psi + \frac{1}{8\pi} bdb + \frac{1}{2\pi} ad(b+A) + \frac{k_\phi + 1}{4\pi} ada$$

No periodicity in flux attachment $\quad k_\psi = \frac{1}{2}\frac{k_\phi - 1}{k_\phi + 1}$

Modular parameter for the self-dual non-local models $\quad \tau_\psi = \frac{\tau_\phi - 1}{\tau_\phi + 1}$

Conclusions

- Loop models offer a bridge to the “derivation” of the web of dualities for relativistic theories
- The loop models are always interacting
- The scalar fields are never free
- Fractional spin plays a key role
- It is always possible to find a dual theory
- Periodicity of flux attachment, and SL(2,$\mathbb{Z}$), is not a symmetry for relativistic theories