Symmetry Protected Phases of Fermions

Lukasz Fidkowski

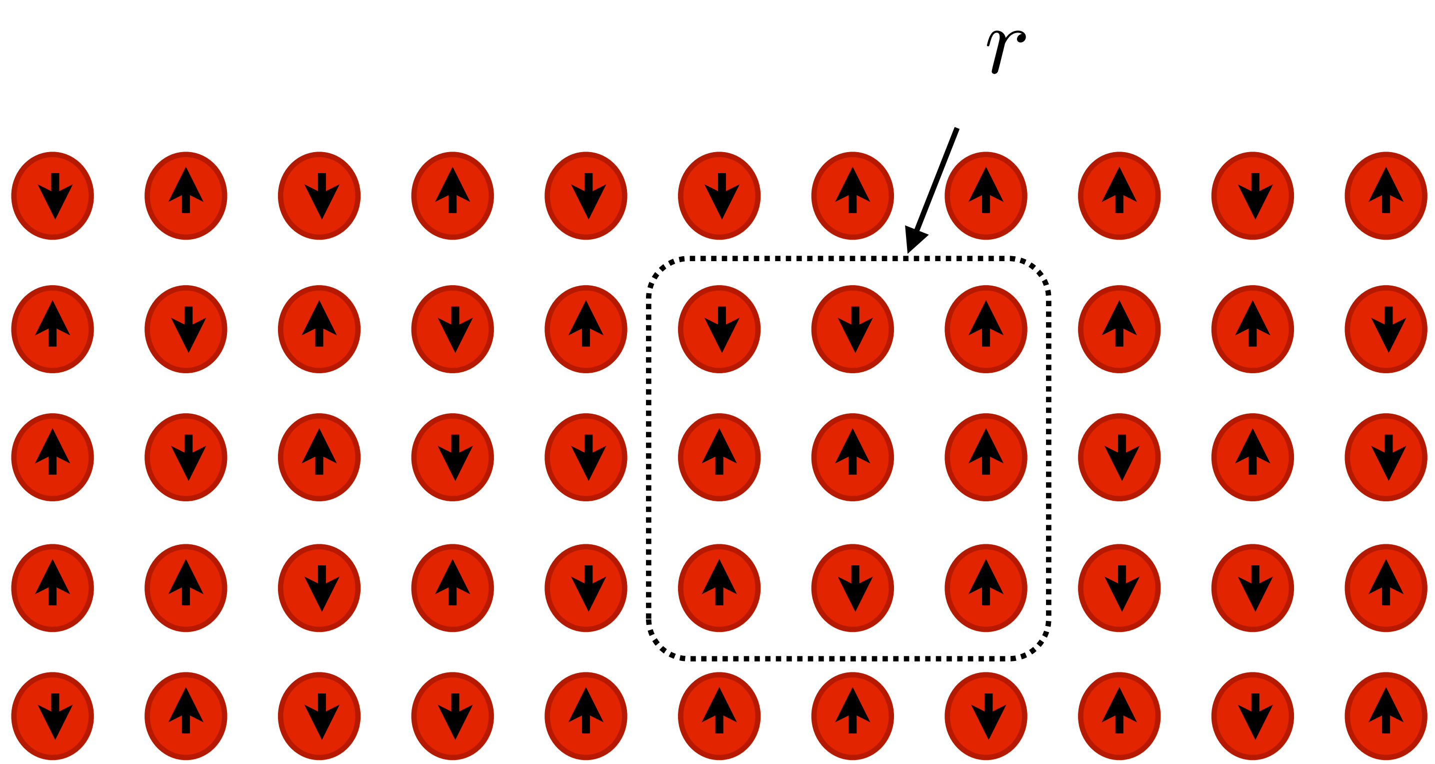
(work with A. Vishwanath and M. Metlitski)

Quantum many body systems:

Hilbert space: $\mathcal{H} = \otimes_{\text{sites i}} \mathcal{H}_i$

Hamiltonian: $H = \sum_r H_r$ $J = \mathrm{max_r} |\mathrm{H_r}|$

Global symmetry: $U^g = \otimes_{\text{sites i}} U_i^g$



SPT phases

`Symmetry protected' topological (SPT) phases

- T=0, gapped phases of lattice models with global symmetry
- can be continuously connected to trivial tensor product state, but only at the expense of breaking symmetry

Interacting SPTs beyond band topological insulators exist!

- at least theoretically

Kapustin, Thorngren arXiv:1701.08264

Wang, Gu arXiv:1703.10937

Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

SPT phases

`Symmetry protected' topological (SPT) phases

- T=0, gapped phases of lattice models with global symmetry
- can be continuously connected to trivial tensor product state, but only at the expense of breaking symmetry

Interacting SPTs beyond band topological insulators exist!

- at least theoretically
- both for bosonic systems

Chen, Wen, Gu;, …

- as well as intrinsically fermionic systems

Kapustin, Thorngren arXiv:1701.08264

Wang, Gu arXiv:1703.10937

Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

SPTs and surface states

- these new intrinsically fermionic SPTs characterized in terms of bulk invariants (3-loop braiding)

Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

- but surface states still mysterious and poorly understood, particularly in 3+1d.

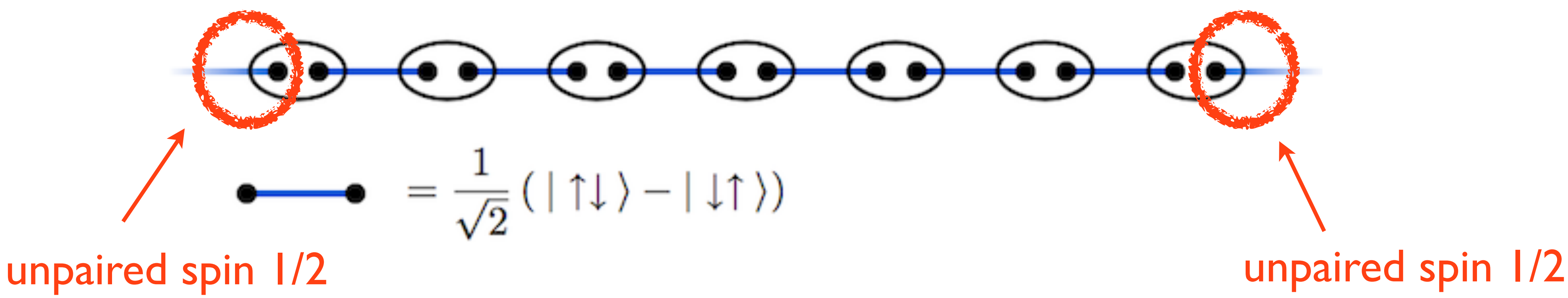
    - physical signature

    - 't Hooft anomalies

- proposal for a topologically ordered fermionic (spin TQFT) surface state of 3+1d in-supercohomolgy fermionic SPTs, and a general bulk / surface 't Hooft anomaly matching condition

1+1 dimensional bosonic SPTs

Haldane phase: $G = SO(3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$



1+1 dimensional bosonic SPTs

General G:

U_g^L U_g^R

$$U_g \approx U_g^L U_g^R \qquad U_{gh}^L = e^{i\phi(g,h)} U_g^L U_h^L$$

1+1 dimensional bosonic SPTs

- Algebraic properties of $\phi(g,h)$:

1) $d\phi(g,h,k) \equiv$
$$\phi(h,k) - \phi(gh,k) + \phi(g,hk) - \phi(g,h) = \ 0 \text{ mod } 2\pi$$

2) $\phi(g,h) \sim \phi(g,h) + \alpha(g) + \alpha(h) - \alpha(gh)$

$$\downarrow$$

$$[\phi] \in H^2(G, U(1))$$

- interpretation as action for 1+1d G-gauge field: bulk/boundary correspondence

(Dijkgraaf, Witten)

Generalization: d+1 dimensional bosonic SPTs

- bulk G-gauge field action in d+1 dimensions:

$$[S] \in H^{d+1}(G, U(1))$$

boundary interpretation?

- Focus: 3+1d bulk, 2+1d boundary surface

    1) gapless surface state

    2) gapped topologically ordered surface state

Generalization: d+1 dimensional bosonic SPTs

- bulk G-gauge field action in d+1 dimensions:

$$[S] \in H^{d+1}(G, U(1))$$

boundary interpretation?

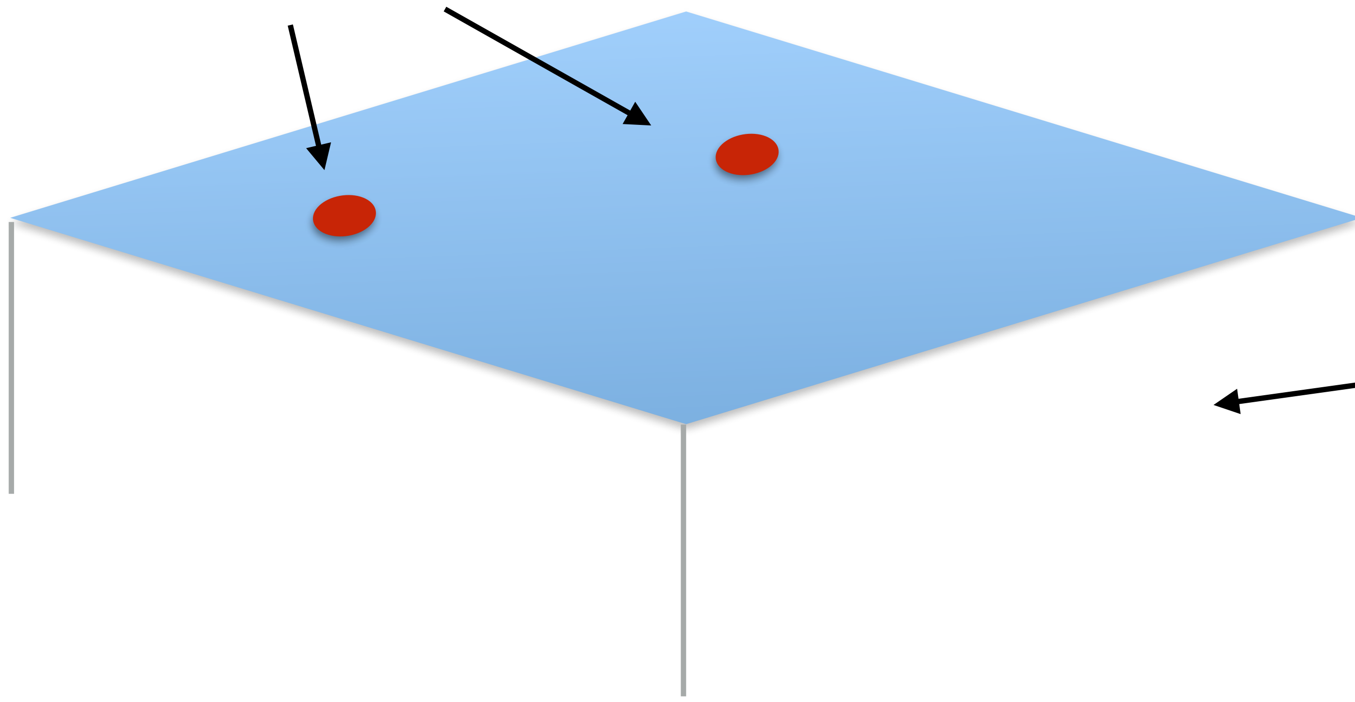
- Focus: 3+1d bulk, 2+1d boundary surface

1) gapless surface state

2) gapped topologically ordered surface state

(Vishwanath Senthil PRX 2013)

3+1 dimensional bosonic SPTs



- anyon fusion rules:

$$a \times b = \sum_c N_c^{a,b} \, c$$

3+1 dimensional bosonic SPTs

- Symmetry fractionalization on surface anyons:

$$U^a_{gh} = e^{i\phi_a(g,h)} U^a_g U^a_h$$

- consistency requirement:

$$\phi_a(g,h) + \phi_b(g,h) = \phi_c(g,h) \mod 2\pi$$

whenever $N^{a,b}_c > 0$

3+1 dimensional bosonic SPTs

- this implies (in a bosonic theory of anyons) that there exists unique abelian anyon $\omega(g,h)$ such that:

$$\phi_a(g,h) = M_{a,\omega(g,h)}$$

- thus fractionalization encoded in

$$\{\phi_a(g,h)\}_a \leftrightarrow \omega(g,h)$$

$$[\omega] \in H^2(G,\mathcal{A})$$

↑ group of abelian anyons

3+1 dimensional bosonic SPTs

- but bulk SPT order encoded in $H^4(G, U(1))$

$$H^2(G, \mathcal{A}) \overset{?}{\leftrightarrow} H^4(G, U(1))$$

3+1 dimensional bosonic SPTs

- but bulk SPT order encoded in $H^4(G, U(1))$

$$H^2(G, \mathcal{A}) \overset{?}{\leftrightarrow} H^4(G, U(1))$$

- Answer: certain fractionalization patterns are anomalous in 2+1d; anomaly exposed when symmetry gauged ('t Hooft anomaly)

$$H^2(G, \mathcal{A}) \to H^4(G, U(1))$$

Chen, Burnell, Vishwanath, Fidkowski PRX 2015

Wang, Lin, Levin PRX 2016

Bi, Rasmussen, Slagle, Xu PRB 2015

Etingof, Niksych, Ostrik;

Barkeshli, Bonderson, Cheng, Wang 2014

3+1 dimensional *fermionic* SPTs

- symmetry $G \times \mathbb{Z}_2^f$
- a lot of recent classification work:

Kapustin, Thorngren arXiv:1701.08264

Wang, Gu arXiv:1703.10937

Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

- invariant: $[\sigma] \in H^2(G, \mathbb{Z}_2)$

3+1 dimensional *fermionic* SPTs

- symmetry $G \times \mathbb{Z}_2^f$
- a lot of recent classification work:

Kapustin, Thorngren arXiv:1701.08264

Wang, Gu arXiv:1703.10937

Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

- invariant: $[\sigma] \in H^2(G, \mathbb{Z}_2)$

- take $[\sigma] = 0$

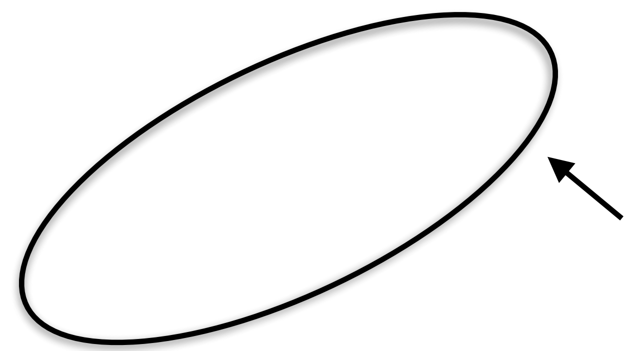
subset of 3+1d fermionic SPTs ("in group-supercohomology" phases) characterized by additional invariant in

$$[\rho] \in H^3(G, \mathbb{Z}_2)$$

3+1 dimensional fermionic SPTs

Physical interpretations of $[\rho]$:

1) symmetry fractionalization on pi flux loops

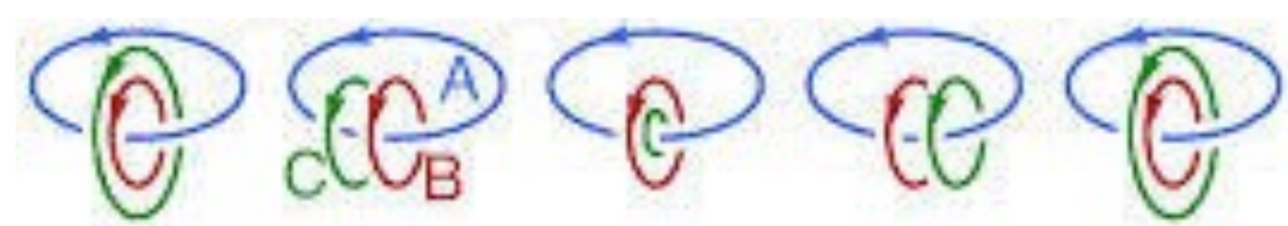


pi flux loop

2) dimensional reduction

$$i_A \rho \in H^2(G, \mathbb{Z}_2)$$

symmetry flux loops:



Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

3+1 dimensional fermionic SPTs

Example: Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

$$G = \mathbb{Z}_2 \times \mathbb{Z}_4$$ (total symmetry $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$)

$$[\rho] = [\alpha][\beta]$$

$$[\alpha] \in H^1(\mathbb{Z}_4, \mathbb{Z}_2)$$
$$[\beta] \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$$

- physical interpretation: Z4 flux tubes bound ν =2 2d Z2 fermionic SPTs

- no free fermion realization

3+1 dimensional fermionic SPT surfaces:

- What kind of surface theories can such SPTs admit?

- analysis of symmetry fractionalization in a fermionic theory of anyons:

$$U^a_{gh} = e^{i\phi_a(g,h)} U^a_g U^a_h$$

with consistency requirement

$$\phi_a(g,h) + \phi_b(g,h) = \phi_c(g,h) \mod 2\pi$$

whenever $N_c^{a,b} > 0$

3+1 dimensional fermionic SPT surfaces:

- this implies that there exists abelian anyon $\omega(g,h)$ such that:

$$\phi_a(g,h) = M_{a,\omega(g,h)}$$

- however, $\omega(g,h)$ *not unique*:

$$\omega(g,h) \to \omega(g,h) + f$$

↑ transparent fundamental fermion

in e.g. a fermionic theory of the form $\mathcal{A} \times \{1, f\}$

↑ bosonic theory

3+1 dimensional fermionic SPT surfaces:

- also, $\omega(g,h)$ is not quite closed:

$$d\omega(g,h,k) \equiv$$
$$g \cdot \omega(h,k) - \omega(gh,k) + \omega(g,hk) - \omega(g,h) \;\in \{1, f\}$$

- key point: it may be *impossible* to gauge $d\omega$ away by modifications of the form

$$\omega(g,h) \to \omega(g,h) + f$$

- $d\omega \in H^3(G, \mathbb{Z}_2)$ is then a possibly non-zero cohomology class.

3+1 dimensional fermionic SPT surfaces:

- nonzero $[d\omega]$ is an obstruction ('t Hooft anomaly) to realizing the theory in a symmetric way in purely 2d:

  Proof: If the theory was realizable in 2d, one could gauge the fermion parity symmetry, and see that the group action on the pi fluxes is not associative; equivalently, it is impossible to write down a consistent fusion product for G-defects.

Claim: a symmetry fractionalization with non-zero $[d\omega]$ can be realized at the surface of an "in group supercohomolgy" fermionic SPT with

$$[\rho] = [d\omega]$$

3+1d fermionic SPT surface: example

Bulk SPT:

(Wang, Gu arXiv:1703.10937)

$G = \mathbb{Z}_2 \times \mathbb{Z}_4$ (total symmetry $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4$)

$$[\rho] = [\alpha][\beta]$$

$$[\alpha] \in H^1(\mathbb{Z}_4, \mathbb{Z}_2)$$
$$[\beta] \in H^2(\mathbb{Z}_2, \mathbb{Z}_2)$$

Surface topological order $= \mathcal{A} \times \{1, f\}$

↑

bosonic $\mathbb{Z}_4$ gauge theory

3+1d fermionic SPT surface: example

Denote surface anyons by $[j, k, \mu] \equiv m^j e^k f^\mu$

(m: $\mathbb{Z}_4$ flux; e: $\mathbb{Z}_4$ charge; f: fundamental fermion)

$$j, k = 0, \ldots, 3\,; \quad \mu = 0, 1$$

Denote group elements by $(\mathbf{g}_1, \mathbf{g}_2)$

($\mathbf{g}_1$: $\text{in } \mathbb{Z}_2$; $\mathbf{g}_2$: $\text{in } \mathbb{Z}_4$)

3+1d fermionic SPT surface: example

Permutation action of G on anyons:

$$[j, k, \mu] \to [j, k + 2\mathbf{g}_2 j, \mu + \mathbf{g}_2 j]$$

Symmetry fractionalization:

$$\omega(\mathbf{g}, \mathbf{h}) = [\mathbf{g}_1 \mathbf{h}_1, \mathbf{g}_2 \mathbf{h}_1, 0]$$

Then $[d\omega] \in H^3(G, \mathbb{Z}_2)$ is non-zero, and equal to $[\rho]$.

Conclusions:

- Connection between anomalous 2+1d symmetry enriched fermionic topological orders and 3+1d SPTs

- Surface topological order for ‘group-supercohomology’ SPTs with no free fermion realization

- Future directions: gapless parent surface state / effective field theory description? beyond supercohomology SPTs?