Three-dimensional QED using Lattice Regularization

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Systems of interest

Our research focus is on the infrared behavior of three-dimensional gauge theories coupled to N flavors of massless two-component Dirac fermions using non-perturbative lattice regularization. For QED$_3$, the continuum systems we have studied are

- Trivially parity-invariant QED$_3$ with even-valued N fermion flavors:

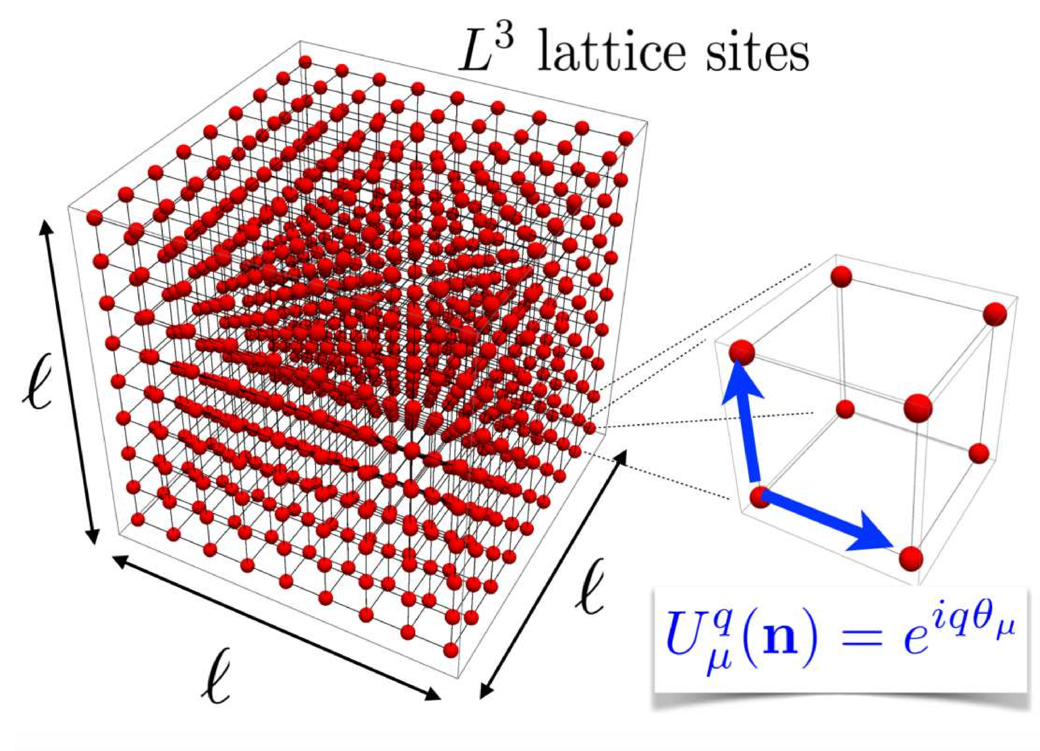
$$\mathcal{L} = \sum_{i=1}^{N/2} \left\{ \overline{\psi}_i \not{C}(a)\psi_i - \overline{\chi}_i \not{C}^\dagger(a)\chi_i \right\} + \frac{1}{4g^2} F_{\mu\nu}(a)F^{\mu\nu}(a) \quad \textbf{Theory-I}$$

- A non-trivially parity-invariant QED$_3$ with $N = 1$ massless fermion with charge q, and two flavors of infinitely massive fermions of charge $q/2$ each inducing a gauge action $(q/2)^2\mathrm{CS}(a)$:

$$\mathcal{L} = \overline{\psi}\not{C}(qa)\psi - i\left(\frac{q}{2}\right)^2 \frac{2}{4\pi}\epsilon_{\mu\nu\varrho}a_\mu\partial_\nu a_\varrho + \frac{1}{4g^2}F_{\mu\nu}(a)F^{\mu\nu}(a) \quad \textbf{Theory-II}$$

In the above equations, $\not{C}(qa)$ is the UV regulated two-component Dirac operator of charge q. We set the scale using $g^2 = 1$.

The lattice setup



- Consider the theory on periodic three-torus of physical volume ℓ^3 discretized to L^3 lattice points.
- **Non-compact gauge action** $\Rightarrow$ No magnetic monopoles in path-integral.

$$S_g = \frac{L}{\ell}\sum_{\mathbf{n}}\sum_{\mu>\nu}^{3}\left(\Delta_\mu\theta_\nu - \Delta_\nu\theta_\mu\right)^2$$

- **Continuum limit:** At fixed physical sizes ℓ of three-torus, keep increasing number of lattice point L and take $L \to \infty$ limit.
- **IR limit:** After taking $L \to \infty$ at different finite ℓ, take the $\ell \to \infty$ limit.

Lattice Dirac operators

Fermions couple to compact gauge fields U^q through lattice Dirac operators which are $2L^3 \times 2L^3$ matrices.

- **Wilson-Dirac operator** $\not{C}_W(U^q)$ uses the naively discretized Dirac operator $\not{C}_n(U^q)$ and Wilson term $B(U^q) \sim \nabla^2$ to avoid doublers:

$$\not{C}_W(U^q) = \not{C}_n(U^q) + B(U^q) - M_W.$$

Drawback $\Rightarrow$ $M_W = 0$ is not exactly massless at finite L, therefore requires tuning. This is rectified by using the Overlap operator.

- **Overlap Dirac operator** is obtained by mapping the 3d fermion determinant $\det \not{C}_o$ to a Slater determinant corresponding to the overlap between the ground-states of two appropriately chosen 3+1d many-body Hamiltonians. The overlap operator $\not{C}_o(M, U^q)$ with fermion mass M in lattice units is

$$\not{C}_o(M, U^q) = \frac{1+M}{2} + \frac{1-M}{2}V_{q\theta},$$

where $V_{q\theta} \equiv \not{C}_W\left(\not{C}_W^\dagger \not{C}_W\right)^{-1/2}$ is a unitary operator. Under parity, $V_{q\theta} \to V_{q\theta}^\dagger$.

Chern-Simons as the induced action $2\mathcal{A}_q$

Consider the following limits:

- **Zero physical mass** $(M = 0)$

$$\Rightarrow \det \not{C}_o(M=0) = \det\left(\frac{1+V_{q\theta}}{2}\right) \equiv \left|\frac{1+V_{q\theta}}{2}\right| e^{iq^2\mathcal{A}_q}$$

with $\mathcal{A}_q \in (-\pi, \pi]$.

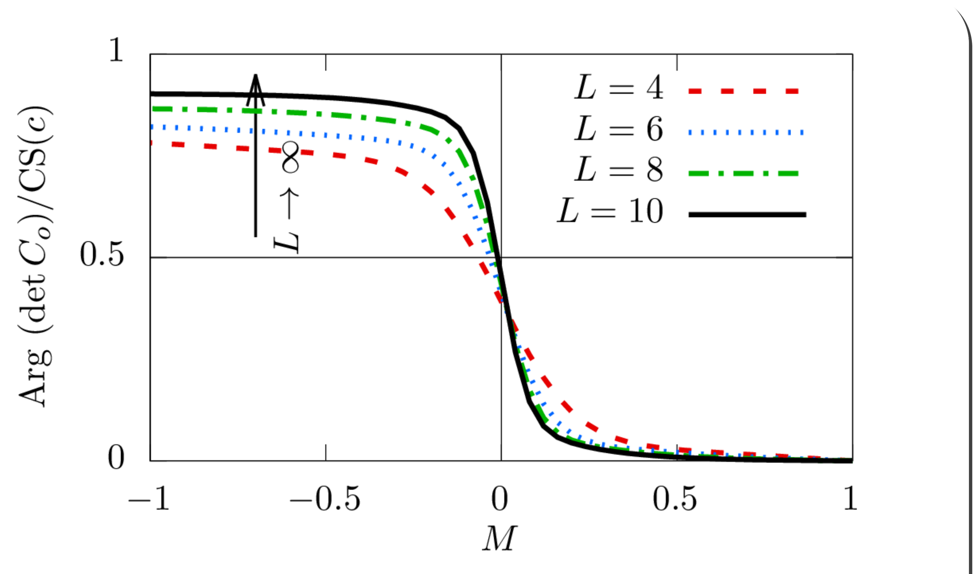
- **Infinite negative physical mass** $(M = -1)$:

$$\Rightarrow \det \not{C}_o(M=-1) = \det V_{q\theta} = e^{2iq^2\mathcal{A}_q}.$$

- **Infinite positive physical mass** $(M = +1)$:

$$\Rightarrow \det \not{C}_o(M=+1) = 1.$$

The phase of $\det \not{C}_o(M, a)$ normalized by $\mathrm{CS}(a)$ is shown as a function of M on the right, for a specific background field. It flows from 0 at $M = +1$ to $2\mathcal{A}_1$ ($\approx \mathrm{CS}(a)$ at finite L) at $M = -1$.



A smooth background field:

$$a_1 = \frac{c}{\ell}\cos\left(\frac{2\pi x_3}{\ell}\right); \quad a_2 = \frac{c}{\ell}\sin\left(\frac{2\pi x_3}{\ell}\right); \quad a_3 = 0$$

which has $\mathrm{CS}(c) = c^2/2$.

Constructing parity-invariant theories on the lattice

Theory-I with $N = 2$

$$Z = \int [d\theta]e^{-S_g(\theta)} \det(1+V_\theta)^2 \det(V_\theta^\dagger) = \int [d\theta]e^{-S_g(\theta)}\left|\det(1+V_\theta)\right|^2.$$

Parity anomaly cancellation is exact even at finite L.

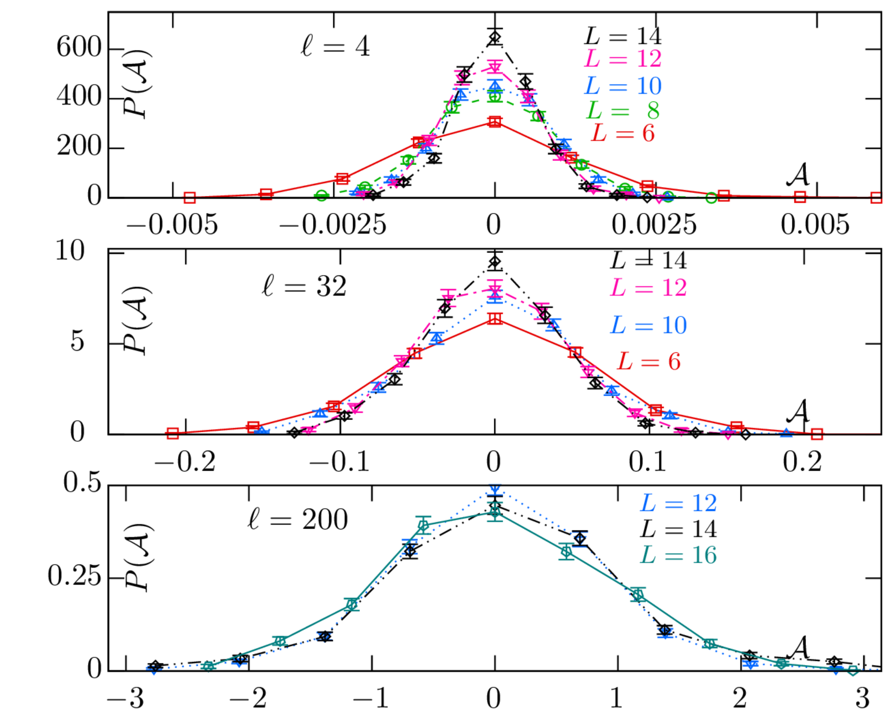
Theory-II with $q = 1$

$$Z = \int [d\theta]e^{-S_g(\theta)} \det(1+V_\theta)\det{}^2\left(V_{\theta/2}^\dagger\right) = \int [d\theta]e^{-S_g(\theta)}\left|\det(1+V_\theta)\right| e^{i(\mathcal{A}_1 - \mathcal{A}_{1/2})}.$$

Anomaly cancellation is inexact since $2\mathcal{A} \equiv 2\mathcal{A}_1 - 2\mathcal{A}_{1/2}$ is not exactly zero (mod 2π) on rough configurations present in the path-integral at any finite L.

Anomaly cancellation in the continuum limit of Theory-II

We simulated theory-II using $p_+(\theta) = e^{-S_g(\theta)}\left|\det(1+V_\theta)\right|$ measure and considered $e^{i\mathcal{A}}$ as an observable. The distributions of $\mathcal{A}$ wrt $p_+(\theta)$ are shown below.

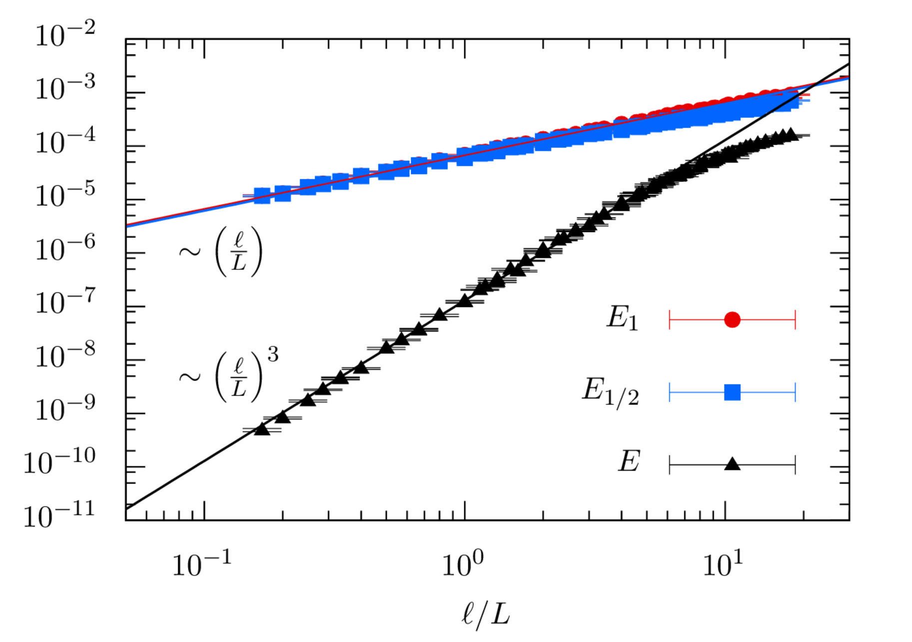


- The distributions get sharper around zero at fixed ℓ in the $L \to \infty$ limit showing anomaly cancellation.
- Even on relatively coarser lattices at $\ell = 200$, no two-peak structure around $\mathcal{A} = 0$ and π. This shows the absence of topological zero modes in three-torus of any size and hence a positive measure.

We computed the fermionic topological current in lattice units,

$$J_i^q(\mathbf{n}) = \frac{\delta}{\delta\theta_i(\mathbf{n})}\mathcal{A}_q.$$

We show $E_q = \langle \mathbf{J}^q.\mathbf{J}^q\rangle$, and their mismatch $E = E_1 - E_{1/2}$ below as a function of lattice spacing ℓ/L.

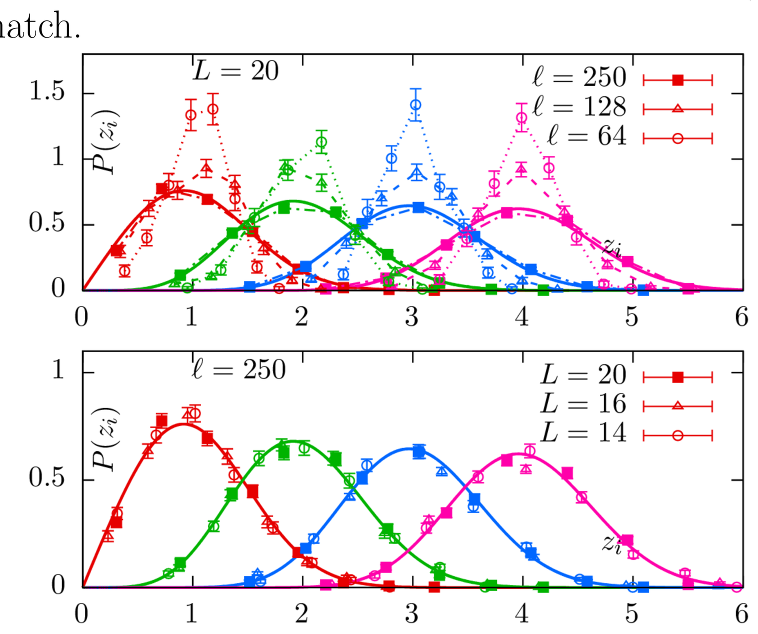


- $E_1 - E_{1/2}$ goes to zero by two-powers of ℓ/L faster than E_1 and $E_{1/2}$, again showing that parity-invariance is restored in the continuum limit.

Spontaneous symmetry breaking of parity in Theory-II

We use low-lying eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ of Dirac operator, to probe the IR. Due to non-zero eigenvalue density of Dirac operator at $\lambda = 0$ in case of SSB

- $\lambda_i \propto \ell^{-3}$
- Universality: Distributions of $\ell^3\lambda_i\Sigma$ (symbols) and z_i from a GUE-type Random Matrix Theory (curves) match.



An exact agreement of the two distributions is seen by using i- and ℓ-dependent values $\Sigma_i(\ell)$ for the matching. In the plot below, we show the extrapolation of $\Sigma_i(\ell)$ to $\ell \to \infty$ leads to same non-zero value $\Sigma = 1.2(2) \times 10^{-5}$. This is the estimate of the condensate $\langle\overline{\psi}\psi\rangle = \Sigma\frac{m}{|m|} + O(m)$.