RANKING AGENDAS FOR NEGOTIATIONS

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Abstract. Consider a negotiation in which agents will make costly concessions to benefit others—e.g., by implementing tariff reductions, environmental regulations, or disarmament policies. An agenda specifies which issue or dimension each agent will make concessions on; after an agenda is chosen, the negotiation comes down to the magnitudes of agents' contributions. We seek a ranking of agendas based on the marginal costs and benefits generated at the status quo, which are captured in a Jacobian matrix for each agenda. In a transferable utility (TU) setting, there is a simple ranking based on the best available social return per unit of cost (measured in the numeraire). Without transferable utility, the problem of ranking agendas is more difficult, and we take an axiomatic approach. First, we require the ranking to depend only on trade-offs between costs and benefits that an issue induces, not on choices of units. Second, we require that the ranking be consistent with the TU ranking on problems that are equivalent to TU problems in a suitable sense. The unique ranking satisfying these axioms is represented by the spectral radius (Frobenius root) of a normalized Jacobian—a statistic we interpret in terms of a network whose links represent agents' abilities to help each other.

1. Introduction

The leaders of several countries have an upcoming summit and their aides are hashing out the agenda. There are many dimensions on which agreements could benefit all countries—e.g., disarmament, pollution, intellectual property, trade—but there is not time for discussions on everything.[1] On each dimension, each leader can make costly concessions that benefit the other countries. The choice the aides face is what single issue to discuss. Are the aides' incentives aligned, in some sense, regarding what to put on the agenda?

Once an issue is selected, we model each agent's concessions as a one-dimensional continuous variable—how many weapons to destroy, how much to cut pollution—and assume negotiators know the marginal costs and benefits (evaluated at the status quo) of concessions they can make. However, they may not know preferences far from the status quo. Formally, the data of the problem is, for each potential dimension, a Jacobian matrix that describes the derivative of each player's payoff in each player's concession. Our goal

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[1]For example, the agenda for the 2014 G20 summit in Brisbane focused on coordinating a response to support the global economy in the wake of the financial crisis (G20, 2014).

is to develop a theory for comparing dimensions "at the margin," i.e., based on these Jacobians, imposing only minimal assumptions on the subsequent negotiations.[2]

It is useful at this stage to break the problem into two cases, depending on whether agents can costlessly transfer some numeraire good in addition to putting forth effort in the selected dimension. If transfers are available, we are in what we call the transferable utility (TU) case. In this case, we claim there is a natural Coasian way to rank dimensions at the margin. We say dimension A is dominated by dimension B if, given any local deviation from the status quo under dimension A, there is another improvement under dimension B that Pareto dominates the first and whose total cost is no greater (in terms of the numeraire). Since surplus can be redistributed, one dimension dominates another as long as it can generate more total surplus per unit of overall cost. The first step of our formal analysis is to flesh out this account, which we do in Section 3.

In practice, contingent transfers of a numeraire are not always a part of negotiations among countries. Therefore, it is interesting to examine the problem where countries can do favors, but not make transfers—the non-transferable utility (NTU) case. In this case, the problem is more difficult (Section 4). Transferable utility allowed us to value everyone's marginal costs and benefits in the numeraire and compare them, but without this there are no units that are as compelling for comparing agents' payoffs. Thus, there is no natural analogue of the marginal social surplus of a given deviation from the status quo. Indeed, quantities akin to the utilitarian total surplus are unappealing because they are not invariant to transformations of the problem such as changing the numerical representation of one agent's preferences.

In view of these difficulties, we ask whether there is any reasonable way to compare dimensions in the NTU case. We take an axiomatic approach, seeking an index that assigns a value to each dimension based on the available data (the Jacobian), which can then be used to compare dimensions. We first treat the case of two agents. We require that such a ranking satisfy two sorts of properties. First, we require *invariance* to transformations of units: changing the numerical representations of preferences that give rise to the Jacobian, or the units in which we measure actions. Second, we require *monotonicity* of the ranking in unambiguous improvements (i.e., lower marginal costs and higher marginal benefits from all actions for all agents). We show in Section 5 that in the case of two players there is only one ranking satisfying the axioms. It is characterized by a simple formula. The ranking has the following simple interpretation: if dimension A is ranked above dimension B, then for any local deviation from the status quo under dimension A, we can find a deviation under dimension B for which the marginal benefit received by each agent per unit of own marginal cost is greater.

Next, we observe a connection between the TU problem and the two-agent NTU problem. There is a small set of TU problems where transfers are not necessary to move around the Pareto frontier given a certain budget. The effect of any transfer can be replicated by agents' taking different actions. Such TU problems can be thought of as equivalent to NTU problems. Reassuringly, for these problems, our TU and two-agent NTU ranking of dimensions are the same (Section 6).

Continuing our analysis of the NTU case, we are able to generalize the index of social returns to investment described above to more than two agents, and to extend the axiomatization we have presented for the two-agent case. The general index is the *spectral*

[2]We do not posit that the changes considered are necessarily small, but only that they are evaluated based on knowledge of marginal costs and benefits.

radius of a matrix that captures agents' returns on each others' investments. The axioms that characterize it are closely related to those used in the two-agent case. The key difference is that the monotonicity axiom is replaced by one that *requires* consistency with the TU analysis on NTU problems that are, in the sense described above, equivalent to TU problems (Section 7). Section 8 discusses subtleties of the axioms and some alternative approaches.

Beyond comparing various dimensions for a given set of agents to negotiate about, our approach also allows us to consider the scope of negotiations more generally. What are the benefits from extending the negotiations to include additional agents? Are there large gains from permitting negotiators to make concessions on additional dimensions? These questions are practically important and discussed in Section 9.

International negotiations are crucial for establishing cooperation and overcoming free-riding on a variety of important issues, and the questions we address are central to these efforts. Other work considers different aspects of the problem by taking different approaches. Prominent examples include Harstad (2012), who models a dynamic non-cooperative game in which players make contributions to a public good and invest in technologies, and Bagwell and Staiger (1999), who analyze the design of the General Agreement on Tariffs and Trade (the predecessor to the World Trade Organization).

The paper most closely related to the approach we take here is Elliott and Golub (2014). That work focuses on a negotiation where the dimension of each agent's concession is fixed, and then establishes connections between Pareto efficiency, Lindahl outcomes, and network statistics (eigenvalues and eigenvectors). Our analysis here is different in that we work with *marginal* data rather than payoffs known on the entire domain. Nevertheless, as in Elliott and Golub (2014), find that focusing on certain network invariants yields crucial insights into the economics of the problem.

A key conceptual contribution of our analysis relative to that paper to take a step back and focusing on the question of *which negotiation to have* instead of fixing the dimension and asking *what outcome to choose*. In doing so, we move from an environment with a fundamental conflict of interest to one in which basic considerations of consistency and efficiency can deliver unambiguous advice.[3]

A second conceptual contribution is a connection between an axiomatic theory of choosing which negotiation to have and invariants of a *network* of economic interactions. As we discuss in Section 10, the spectral radius mentioned above can be seen as a network statistic. The network is one in which a link from i to j means that i can help j, and the strength of this link is defined by the marginal benefit that i can confer on j. The index we axiomatize—the spectral radius of this network—is an invariant of a type that comes up in a wide variety of network analyses. It measures the total strength of cycles in this network (a cycle is a sequence such as: i can help j who can help k who can help i). Our axioms select this invariant as the right one for ranking agendas, and the analysis therefore opens a new connection between the theory of negotiations and the theory of networks.

[3]There is also an extensive political economy literature on agenda-setting, but the approach taken in this literature is quite different from our analysis. Legislative bargaining is modeled as a non-cooperative and typically dynamic game, and the analysis focuses on how the structure of the game affects the equilibrium outcomes. In contrast, we aim for an approach that abstracts from strategic details, and seek a ranking of agendas that all agents agree on in a certain sense.

2. Model

2.1. **Primitives and notation.** The set of players is $\mathcal{N} = \{1, \ldots, N\}$ (an important special case will be $N = 2$). A dimension (also called an issue or agenda) in the set $\mathcal{K} = \{1, \ldots, K\}$ must be selected in which the players will make concessions.

2.1.1. *Payoffs.* The utility function of agent i is $U_i : \mathcal{K} \times \mathbb{R}_+^N \times \mathbb{R} \to \mathbb{R}$, with a typical payoff written $U_i^k(\mathbf{a}, m_i)$. The function's arguments are a *dimension* $k \in \mathcal{K}$ selected for the negotiation (which we write in the superscript), the *action profile* $\mathbf{a} \in \mathbb{R}^N$, a vector of real numbers which captures agents' contributions in the chosen dimension, and a net transfer m_i of a numeraire.

A triple $(k, \mathbf{a}, m_i) \in \mathcal{K} \times \mathbb{R}_+^N \times \mathbb{R}$ is called an (individual's) *outcome*. The outcomes $\{(k, \mathbf{0}, 0)\}_{k \in \mathcal{K}}$ are called the *status quo*, and we set $U_i^k(\mathbf{0}, 0) = 0$ for all k.[4] A single dimension is selected for negotiation, reducing each agent's contribution to a one-dimensional variable. We show in Section 9 that our analysis extends to permitting concessions in multiple dimensions and limiting negotiations to include only a subset of the agents simply by expanding the set $\mathcal{K}$.

Letting the vector $\mathbf{m}$ stack all the transfers, we write $\mathbf{U}^k(\mathbf{a}, \mathbf{m})$ for the vector containing all the payoffs. We assume that a representation of i's preferences can be chosen so that each U_i^k is differentiable at the status quo[5] and so that, for each i and k:

(i) $\frac{\partial U_i^k}{\partial a_i}(\mathbf{0}, 0) < 0$;
(ii) $\frac{\partial U_i^k}{\partial a_j}(\mathbf{0}, 0) > 0$;
(iii) $\frac{\partial U_i^k}{\partial m_i}(\mathbf{0}, 0) = \mu \in \{0, 1\}$.

The case in which utilities are transferable is modeled by setting $\mu = 1$. This amounts to scaling utility representations for each agent so that everyone's marginal utility of the numeraire at the status quo is equal. The NTU case is modeled by setting $\mu = 0$. Since there are many actual negotiations where monetary transfers are not on the table, we view this as a relevant case to model. We do so by assuming that agents do not value any transferable numeraire—though in reality they may value some that are not available to transfer. In the NTU case, we will suppress the argument m_i of U_i; when we do not write the numeraire component, it is set to zero.

2.1.2. *The Jacobian.* Next, we introduce the Jacobian matrix. For any profile $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ of functions $u_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, let

$$J_{ij}(\mathbf{u}) = \frac{\partial u}{\partial a_j}(\mathbf{0}, 0).$$

We will be particularly interested in the Jacobians $\mathbf{J}^k := \mathbf{J}(\mathbf{U}^k)$ corresponding to the preferences in dimension k, keeping in mind that $\mathbf{J}^k$ is not a matrix power.

[4]That is, if a dimension is selected to negotiate about and then nothing is changed in that dimension, then the same utility must obtain regardless of which dimension was selected. We normalize this payoff to zero.

[5]Although we take differentiable utility functions as our primitive rather than preferences, we only do so to aid our focus on *marginal* comparisons between issues. We will not use the utility function representations to make interpersonal comparisons.

When the profile of utility functions is clear from context, we will suppress the argument $\mathbf{u}$. Let $\mathcal{J}$ be the set of N-by-N matrices such that $J_{ij} > 0$ for all $i \neq j$ and $J_{ii} < 0$ for all i—that is, the set of Jacobians arising from preferences satisfying the assumptions laid out in the previous section.

2.1.3. *Orderings.* Lastly, we introduce some notation for orderings for vectors. (We use the same notation with matrices.) Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, for any n, we define (i) $\mathbf{v} \geq \mathbf{w}$ to mean $v_i \geq w_i$ for every i; (ii) $\mathbf{v} \gneq \mathbf{w}$ to mean $v_i \geq w_i$ for every i with strict inequality for some i; and (iii) $\mathbf{v} > \mathbf{w}$ to mean $v_i > w_i$ for every i. We define $\mathbb{R}^n_+ = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0}\}$ to be the nonnegative orthant.

2.1.4. *An Index.* We seek an ordinal ranking—a planner's preference ordering—over Jacobians. This then yields a ranking over dimensions based on the Jacobian for each dimension. Equivalently, we can look for an index[6] $f : \mathcal{J} \to \mathbb{R}$, to represent the ordinal ranking which assigns to a Jacobian $\mathbf{J}$ a value $f(\mathbf{J})$ based on the derivatives of agents' utilities. We call such a function a *marginal efficiency index* or an index of marginal social returns, since a Jacobian with a higher value of f will be interpreted as corresponding to a more efficient negotiation.

3. Transferable Utility

We begin by looking for an index to rank dimensions in the benchmark case of transferable utility, where $\mu = 1$. This TU benchmark is worth working out in its own right, and will provide an important point of reference when we deal with the harder problem of ranking dimensions in the NTU case.

More precisely, the problem addressed in this section is that of a planner who knows the Jacobian $\mathbf{J}^k$ for every k, knows that $\mu = 1$, and must make a recommendation for a dimension to use based on these data and nothing else. Before making the formal definitions of the criteria the planner uses to make the needed comparisons, we discuss the planner's problem informally.

Loosely, we would like to consider dimension k superior to dimension k' when any feasible outcome in dimension k' can be Pareto dominated by some feasible outcome on dimension k. Carrying through this standard idea in our setting takes some extra care for two reasons. First, the data of the problem do not fully describe agents' preferences, only their marginal utilities. Second, the data of the problem do not include constraints that would give the "feasibility" restriction enough bite. Without imposing some constraints, we will see that it can be too easy for any dimension to beat any other, rendering the ranking too coarse to be useful.

We address the first problem—the planner's ignorance of utilities away from the margin—by using the approximation $\mathbf{U}^k(\mathbf{a}, \mathbf{m}) \approx \mathbf{J}^k\mathbf{a} + \mathbf{m}$, and taking this linearization of $\mathbf{U}^k$ as the payoffs the planner works with.[7] This addresses the first problem, but the second—formulating a suitable notion of "feasibility"—becomes, if anything, more difficult. One obvious feasibility restriction is that $\sum_i m_i = 0$: no money is created or destroyed. But if this is the only constraint, it is clear that, given linearity, payoffs can often made

[6]That is, a numerical representation of the planner's ranking.

[7]This approximation is simplest to interpret when the actions are small relative to the curvature of the functions u_i^k—i.e., when the linear approximation is known to be close to the actual utilities—but may be used for other reasons as well.

arbitrarily large or small by scaling actions up or down. Thus, the remaining challenge is to formulate appropriate restrictions to avoid this "blowing up" problem, and make dimensions comparable.

We now introduce the key definition we make for this purpose.

Definition 1. Given any Jacobian $\mathbf{J} \in \mathcal{J}$ and real numbers $b > 0$ and $c \geq 0$, define the *feasible linearized payoffs given* b *and* c as the set of payoff vectors

$$\mathrm{FLP}(\mathbf{J}; b, c) = \left\{ \mathbf{J}\mathbf{a} + \mathbf{m} : \sum_i m_i = 0, \ \sum_i (-J_{ii}) a_i = b, \text{ and } \forall i, \ (-J_{ii}) a_i \geq -c \right\} \cap \mathbb{R}^N_+.$$

In words, this is the set of nonnegative linearized payoff vectors that can be achieved under a Jacobian $\mathbf{J}$ subject to four constraints: money is conserved; the total expenditure on increasing actions is b units of the numeraire[8] (an aggregate budget constraint); no individual reduces investment by more than c units of the numeraire[9] (a backsliding constraint); and "participation constraints" ensuring that each player's payoff is at least the status quo payoff of 0.

We now discuss in more detail the formulation of this definition. Both the budget and backsliding constraints are necessary to avoid the danger of the set of linearized payoffs becoming too large to be useful. If an aggregate budget constraint were not imposed, then for some Jacobians $\mathbf{J} \in \mathcal{J}$, the set could achieve arbitrarily high payoffs for everyone. The same would be the case if c were infinite; in that case, by reducing some of the a_i toward $-\infty$, some of the a_i could be made arbitrarily large without violating $\sum_i (-J_{ii}) a_i = b$. This would allow the creation of an arbitrary amount of utility for someone, which could then be redistributed via transfers (m_i).

With the constraints, the sets $\mathrm{FLP}(\mathbf{J}; b, c)$ become sufficiently well-behaved that we can use them rank dimensions along the lines we suggested at the start of the section. After presenting this ranking and characterizing it, we return to some further discussion of the interpretation of these constraints.

Definition 2. The Jacobian $\mathbf{J}$ *TU-dominates* $\mathbf{J}'$ (written $\mathbf{J} \succ^T \mathbf{J}'$) if for any real numbers $b > 0$, $c \geq 0$ and any $\mathbf{v}' \in \mathrm{FLP}(\mathbf{J}'; b, c)$, there is a $\mathbf{v} \in \mathrm{FLP}(\mathbf{J}; b, c)$ such that $\mathbf{v} \gneq \mathbf{v}'$.

By our assumption on the planner's knowledge, the comparison of dimensions boils down to the comparison of Jacobians. So we can consider dimension k at least as good as k' exactly when $\mathbf{J}^k \succeq^T \mathbf{J}^{k'}$. Given this, Definition 2 says that dimension k is better than k' when any feasible linearized payoff vector in the latter can be Pareto-dominated by some feasible linearized payoff vector in the former, under the same b and c.

We can now characterize the relation $\succeq^T$ through a simple index:

$$h(\mathbf{J}) = \max_j \sum_{i:i \neq j} \frac{J_{ij}}{-J_{jj}}.$$

Proposition 1. For any $\mathbf{J}, \mathbf{J}' \in \mathcal{J}$, we have $\mathbf{J} \succeq^T \mathbf{J}'$ if and only if $h(\mathbf{J}) \geq h(\mathbf{J}')$.

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[8]We multiply actions by $(-J_{ii})$ so that the costs of actions are measured in terms of the numeraire and comparable across agents. Thus, the budget constraint is on total expenditure of the numeraire. It is imposed with equality for technical convenience, as we discuss further below.

[9]A special case of the backsliding constraint occurs when $c = 0$, implying that actions can only be increased.

Proof of Proposition 1. Define the constraint set

$$C(\mathbf{J}; b, c) = \{\mathbf{a} \in \mathbb{R}^N : \sum_i (-J_{ii}) a_i = b, \text{ and } \forall i,\ (-J_{ii}) a_i \geq -c\}.$$

Define

$$s(\mathbf{J}; b, c) = \max_{\mathbf{a} \in C(\mathbf{J}; b, c)} \mathbf{1}^\mathsf{T} \mathbf{J} \mathbf{a}. \tag{1}$$

This is the largest total surplus that can be generated by the permissible $\mathbf{a}$. The following lemma is proved by observing that, because surplus can be redistributed via transfers, the condition for domination in Definition 2 is equivalent to having greater total surplus.

Lemma 1. The following statements are equivalent:

(1) For any $\mathbf{v}' \in \text{FLP}(\mathbf{J}'; b, c)$, there is a $\mathbf{v} \in \text{FLP}(\mathbf{J}; b, c)$ such that $\mathbf{v} \gneq \mathbf{v}'$.
(2) $s(\mathbf{J}; b, c) > s(\mathbf{J}'; b, c)$.

Now we prove the proposition. Fix b and c. Let us define $\widetilde{a}_i = (-J_{ii}) a_i$. Then $s(\mathbf{J}; b, c)$ is the maximum of $\sum_{i,j} \frac{J_{ij}}{-J_{jj}} \widetilde{a}_j$ over the set of $\widetilde{\mathbf{a}}$ satisfying (i) $\sum_i \widetilde{a}_i = b$ and, (ii) for all i, $\widetilde{a}_i \geq -c$. For an $\widetilde{\mathbf{a}}$ satisfying these constraints, we have:

$$\sum_{i,j} \frac{J_{ij}}{-J_{jj}} \widetilde{a}_j = \sum_j \widetilde{a}_j \sum_{i: i \neq j} \frac{J_{ij}}{-J_{jj}} - \sum_j \widetilde{a}_j = \sum_j \widetilde{a}_j \sum_{i: i \neq j} \frac{J_{ij}}{-J_{jj}} - b.$$

Let k be any element of $\text{argmax}_j \sum_{i: i \neq j} \frac{J_{ij}}{-J_{jj}}$. The summation on the right-hand side is maximized by setting $\widetilde{a}_k = b + (n-1)c$ and $\widetilde{a}_i = -c$ for all $i \neq k$. Thus $s(\mathbf{J}; b, c) = h(\mathbf{J})(b + (n-1)c) - b$. Applying Lemma 1 concludes the proof. ∎

Proposition 1 establishes that if we want our index to rank a dimension that TU dominates another higher, then we must choose an index that ranks Jacobians identically to ρ.

In Definition 1, the set $\text{FLP}(\mathbf{J}; b, c)$ is defined so that the budget constraint holds with equality and exactly b units of effort, in aggregate, are invested (measured in the numeraire). Using the constraint $\sum_i (-J_{ii}) a_i \leq b$ instead may seem more natural. Unfortunately, no analog of Proposition 1 would then hold in general. To see this, suppose we had defined the FLP set with a budget *in*equality and consider any $\mathbf{J}$ with $h(\mathbf{J}) < 1$. In this case, more free-riding than the status quo is Pareto efficient and for any b, one action profile that would not be dominated by any other is the one in which $a_i = -c$ for all i. For the environments we have in mind, the condition $h(\mathbf{J}) > 1$ is natural; it means that there are Pareto improvements to be obtained through (some) agents making costly concessions. In this case the budget constraint can be relaxed to an inequality and the analysis goes through unchanged.

Why are the constraints in the definition of the set FLP chosen the way they are? Our discussion has already shown why one must, loosely speaking, constrain $\mathbf{a}$ both from above and from below to avoid the blowing up of payoffs. We now offer one simple story to go with our definition and aid in its interpretation. The constraint from above, $\sum_i (-J_{ii}) a_i = b$ (or $\leq b$), can be interpreted as a limit on appropriations made to invest in the collective project, which is a realistic feature of negotiations. Each country, say, sets aside a certain amount it is willing to spend on the project. Should such a constraint be imposed agent-by-agent or in total? Since transfers are possible, we view the latter as more appropriate: for instance, one country can pay for clean technology to be built in another. As for the constraint from below, a natural case is one in which past investment is sunk, and so one can only increase investment relative to the status quo (this is the case $c = 0$). However, it may also be the case that some current investment is required to maintain the status quo. If these resources can be reallocated, there may then be scope for "backsliding"—reducing investment relative to the current level and repurposing these resources. If transfers are limited, this can be used as an additional way to compensate some agents, an issue we return to in Section 6.

4. Nontransferable Utility: Challenges

Suppose now that $\mu = 0$, so that there is no valuable numeraire to transfer. We would again like to define a Pareto relation over dimensions based only on marginal benefits and marginal costs. However, a number of difficulties arise.

In Proposition 1 we saw that the relation $\succeq^T$ is represented by the function h on the domain of Jacobians. It is tempting to try to use this or a similar index to rank dimensions in the NTU setting. The problem is that the ranking implied by this index is not invariant to some economically irrelevant transformations of the problem. Suppose, for instance, that we are seeking to compare two dimensions with Jacobians $\mathbf{J}$ and $\mathbf{J}'$. Suppose we fix an agent ℓ and choose a different numerical representation of agent ℓ's preferences, simply multiplying his utility function by a constant $\beta > 0$. This yields new Jacobians $\widetilde{\mathbf{J}}$ and $\widetilde{\mathbf{J}}'$, which coincide with the old ones except that $\widetilde{J}_{\ell j} = \beta J_{\ell j}$ and $\widetilde{J}'_{\ell j} = \beta J_{\ell j}$ for the particular agent ℓ and all j. All agents make the same trade-offs among dimensions as before under these new Jacobians; but how h ranks dimensions can be altered by such a change. Indeed, as β grows very large, h essentially cares only about agent ℓ and ranks dimensions according to the value of $\max_j \frac{J_{\ell j}}{-J_{jj}}$. As β changes, the ranking changes.

This argument would not have made sense in the TU case of $\mu = 1$, because there we had fixed a non-arbitrary choice of scales for each utility function by requiring $\partial u_i/\partial m_i = 1$ for each i. No such canonical choice is available in the NTU case.

Rather than trying to guess a modification of the ranking $\succeq^T$, we will now take an axiomatic approach to finding a ranking of dimensions. That is, we will see whether there is any ranking in the NTU case that respects some reasonable invariances and other essential properties.

5. An Index of Marginal Efficiency: Axioms for the Two-Agent Case

We now take an axiomatic approach to finding an index $f : \mathcal{J} \to \mathbb{R}$ for ranking dimensions in NTU environments.

5.1. The axioms.

There are some properties we would like our marginal efficiency index to satisfy. First, we want the index $f(\mathbf{J})$ to depend only on the ordinal information contained in the utility representation giving rise to $\mathbf{J}$. If two different utility functions can be used to represent the same preference over action profiles in a given dimension, then we will require that our index for that dimension be the same for the Jacobians induced by the two representations. (For further discussion of this requirement, see Section 8.1.)

It is affine transformations of a utility profile $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ that induce exactly the same preferences over lotteries as $\mathbf{u}$. Affine transformations of utility functions correspond to transformations of the Jacobian whereby agent i's marginal costs, J_{ii}, and marginal benefits, $(J_{ij})_{j:j\neq i}$, are scaled by some constant S_{ii}. Let $\mathcal{S}$ denote the space of all N-by-N diagonal matrices with $S_{ii} > 0$ for each i.

P1: Invariance to utility representation

> We say f satisfies invariance to utility representation if $f(\mathbf{SJ}) = f(\mathbf{J})$ for all $\mathbf{J} \in \mathcal{J}$ and all $\mathbf{S} \in \mathcal{S}$.[10]

A second invariance we seek is an invariance to the units in which actions are measured. Such changes are economically meaningless and should not affect our marginal efficiency index. For example, whether pollution reductions are measured in tons or grams of the pollutant should not affect the ranking of dimensions. Suppose we reparameterize a given agent's action axis by the transformation $\widetilde{a}_i = s_i a_i$ (such that one unit before corresponds to s_i units now). The marginal costs (per unit) that i incurs (J_{ii}) are then multiplied by $1/s_i$, but so are the marginal benefits that i provides to each other agent, $(J_{ji})_{j:j\neq i}$. Let $\mathbf{S} \in \mathcal{S}$ be a diagonal matrix with $S_{ii} = s_i$. To obtain the transformed Jacobian, we postmultiply the original Jacobian by $\mathbf{S}^{-1}$.[11]

P2: Invariance to action units

> We say f is invariant to action units if $f(\mathbf{JS}^{-1}) = f(\mathbf{J})$ for all $\mathbf{J} \in \mathcal{J}$ and all $\mathbf{S} \in \mathcal{S}$.

The next axiom imposes a ranking across Jacobians in cases where one uniformly dominates another.[12] If a dimension k generates higher marginal benefits than dimension k' without any increase in anyone's marginal costs, then our marginal efficiency index should rank the Jacobian $\mathbf{J}^k$ above $\mathbf{J}^{k'}$.

P3: Monotonicity

> We say f is monotonic if $f(\mathbf{J}) > f(\mathbf{J}')$ whenever (i) $J_{ij} > J'_{ij}$ for all pairs i, j such that $i \neq j$ and (ii) $J_{ii} \geq J'_{ii}$ for each i.[13]

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[10]Equivalently, let $\widetilde{\mathbf{u}}$ be defined for each i by $\widetilde{u}_i = s_i u_i$, for some positive scalars s_i. Then f satisfies invariance to affine transformations if we have $f(\mathbf{J}(\widetilde{\mathbf{u}})) = f(\mathbf{J}(\mathbf{u}))$ for any choices of $\mathbf{u}$ and $(s_i)_{i\in\mathcal{N}}$. We are not formally considering adding constants to the representations because all utilities are relative to the status quo. Note that since we are working at the margin, this assumption is equivalent to assuming invariance to any transformation of utility functions that takes u_i to $f_i \circ u_i$, where f is some strictly increasing function. Then, by the chain rule, the role of s_i is played by the derivative of f_i at the status quo payoff.

[11]To see more formally why postmultiplying the Jacobian by $\mathbf{S}^{-1}$ corresponds to scaling the units, suppose that we changed the units of the ith action as follows: instead of considering actions a_j, we consider actions $\tilde{a}_j = s_j a_j$, for some $s_j > 0$. The marginal benefits received by agent i from agent j before this change of units was $\frac{\partial u_i}{\partial a_j}$. After the change these marginal benefits become

$$\frac{\partial u_i}{\partial \tilde{a}_j} = \frac{\partial u_i}{\partial a_j}\frac{\partial a_j}{\partial \tilde{a}_j} = \frac{\partial u_i}{\partial a_j}\frac{1}{s_j}.$$

[12]Section 8.2 contains some further discussion of this axiom.

[13]Under P1, this is equivalent to saying $f(\mathbf{J}) > f(\mathbf{J}')$ whenever $\mathbf{J} > \mathbf{J}'$ entrywise. By P1, scaling each row of $\mathbf{J}$ cannot change $f(\mathbf{J})$. As $J_{ij} > J'_{ij}$ for all $i \neq j$ there then exists a factor strictly less than one by which each row of $\mathbf{J}$ can be scaled without violating these inequalities. After such a scaling, the diagonal entries of $\mathbf{J}$ will be strictly greater (less negative) than the corresponding entries of $\mathbf{J}'$.

5.2. **A marginal efficiency index.** Throughout this subsection, we fix $N = 2$. Define the marginal efficiency index of a 2×2 Jacobian matrix $\mathbf{J} \in \mathcal{J}$ as

$$\rho(\mathbf{J}) := \left[\frac{J_{12}J_{21}}{J_{11}J_{22}}\right]^{1/2}.$$

Theorem 1. For $N = 2$, a function $f : \mathcal{J} \to \mathbb{R}$ satisfies P1, P2, and P3 if and only if f orders Jacobians identically to ρ: For every $\mathbf{J}, \mathbf{J}' \in \mathcal{J}$

$$f(\mathbf{J}) \geq f(\mathbf{J}') \qquad \text{if and only if} \qquad \rho(\mathbf{J}) \geq \rho(\mathbf{J}').$$

Proof of Theorem 1. We first show the *if* direction. Take $f = g \circ \rho$ for any strictly increasing function g. Note that for any $\mathbf{J} \in \mathcal{J}$ and $\mathbf{S} \in \mathcal{S}$ it holds that

$$\rho(\mathbf{SJ}) = \rho\left(\begin{bmatrix} S_{11}J_{11} & S_{11}J_{12} \\ S_{22}J_{21} & S_{22}J_{22} \end{bmatrix}\right) = \left[\frac{(S_{11}J_{12})(S_{22}J_{21})}{(S_{11}J_{11})(S_{22}J_{22})}\right]^{1/2} = \left[\frac{J_{12}J_{21}}{J_{11}J_{22}}\right]^{1/2} = \rho(\mathbf{J}).$$

A very similar calculation establishes that $\rho(\mathbf{JS}^{-1}) = \rho(\mathbf{J})$. Thus P1 and P2 hold. Finally, it follows immediately by inspection that over the domain $\mathcal{J}$ of matrices with positive off-diagonal and negative diagonal entries, ρ is strictly increasing in each entry of $\mathbf{J}$. From this it follows (recalling that g is strictly increasing) that that $f(\mathbf{J}) > f(\mathbf{J}')$ whenever $\mathbf{J} > \mathbf{J}'$.

We now show the *only if* direction: that if f satisfies P1, P2, and P3, then we can find a strictly increasing function g so that $f = g \circ \rho$.

Define $\bar{\mathcal{J}} = \{\mathbf{J} \in \mathcal{J} : \ J_{11} = J_{22} = -1\}$ and $\widehat{\mathcal{J}} = \{\mathbf{J} \in \bar{\mathcal{J}} : \ J_{12} = J_{21}\}$. The latter set collects all matrices with diagonal entries equal to -1 and off-diagonal entries equal to each other. Define $y : \mathcal{J} \to \bar{\mathcal{J}}$ by $y(\mathbf{J}) = \mathbf{SJ}$, where $\mathbf{S}$ is a diagonal matrix with $S_{ii} = 1/J_{ii}$ for each i. This function divides each row of $\mathbf{J}$ by the diagonal entry in that row. Next, define $z : \bar{\mathcal{J}} \to \widehat{\mathcal{J}}$ by $z(\mathbf{J}) = \mathbf{SJS}^{-1}$ where $\mathbf{S}$ is again a diagonal matrix, this time with $S_{ii} = \sqrt{J_{ji}}$ for each i (and j stands for the index not equal to i). It is easily checked that z does indeed take values in $\widehat{\mathcal{J}}$: that is, applying z to a matrix in $\bar{\mathcal{J}}$ makes the off-diagonal entries equal.

Since f satisfies both P1 and P2 (by hypothesis) and so does ρ (by the "if" direction above), it follows that $f(\mathbf{J}) = f(z(y(\mathbf{J}))$ and $\rho(\mathbf{J}) = \rho(z(y(\mathbf{J}))$ for each $\mathbf{J} \in \mathcal{J}$. It therefore suffices to show that $f = g \circ \rho$ holds on $\widehat{\mathcal{J}}$ for some strictly increasing g. To this end, take

$$\widehat{\mathbf{J}} = \begin{bmatrix} -1 & \widehat{J} \\ \widehat{J} & -1 \end{bmatrix} \qquad \text{and} \qquad \widehat{\mathbf{J}}' = \begin{bmatrix} -1 & \widehat{J}' \\ \widehat{J}' & -1 \end{bmatrix}.$$

Since f satisfies P3, $\widehat{J} > \widehat{J}'$ implies $f(\widehat{\mathbf{J}}) > f(\widehat{\mathbf{J}}')$ and $\widehat{J} < \widehat{J}'$ implies $f(\widehat{\mathbf{J}}) < f(\widehat{\mathbf{J}}')$; thus $\widehat{J} = \widehat{J}'$ implies $f(\widehat{\mathbf{J}}) = f(\widehat{\mathbf{J}}')$. The same sequence of statements holds if f is replaced with ρ, since ρ also satisfies P3. Thus f and ρ represent the same linear order on $\widehat{\mathcal{J}}$, from which the conclusion follows. ∎

Theorem 1 motivates the use of ρ as a marginal efficiency index; insofar as such an index should satisfy P1, P2, and P3, the index given by ρ is essentially the only option.

5.3. **A Pareto ranking.** Is there a sense in which the ranking of dimensions that ρ gives is unanimous? With transferable utility there was a complete Pareto ordering of the dimensions, in the sense made precise by Section 3. However, as the proof of Proposition 1 shows, this complete ordering relied on transfers reducing the dimensionality of the problem—the goal was simply to maximize the marginal social surplus and use transfers to redistribute. Despite such transfers not being possible in the NTU case, we will show that ρ generates a similar complete ordering of dimensions when there are two agents.

In the transferable utility case we measured the marginal benefits in terms of the numeraire good. Although this option is no longer available, we can measure an agent's marginal benefits relative to his own marginal costs. Let $\Delta_N := \{\mathbf{d} \in \mathbb{R}^n : \sum_i d_i = 1, d_i > 0 \text{ for all } i\}$ be the set of *directions*—vectors with positive coordinates adding up to 1. We will now define a preorder over Jacobians (a reflexive, transitive, but not necessarily complete relation) called $\succeq^P$, which is a sort of Pareto ranking.

Definition 3. Given a direction $\mathbf{d} \in \Delta_N$, agent i's marginal private return from Jacobian $\mathbf{J}$ is

$$p_i(\mathbf{d}; \mathbf{J}) := \frac{\sum_{j:j\neq i} J_{ij} d_j}{-J_{ii} d_i}.$$

Define $\succeq^P$ as follows: $\mathbf{J} \succeq^P \mathbf{J}'$ if for any $\mathbf{d}' \in \Delta_N$ there exists a $\mathbf{d} \in \Delta_N$ such that for all i, we have $p_i(\mathbf{d}; \mathbf{J}) \geq p_i(\mathbf{d}'; \mathbf{J}')$.

The relation $\mathbf{J} \succeq^P \mathbf{J}'$ holds if and only if for any direction $\mathbf{d} \in \Delta_N$ in which actions could be increased under a Jacobian $\mathbf{J}$, there is a direction $\mathbf{d}' \in \Delta_N$ in which actions could be increased under a Jacobian $\mathbf{J}'$ to generate a weakly higher return (per unit of cost) for all i and a strictly higher return for some i.

We will now show that $\succeq^P$ is not just a preorder but a complete order on Jacobians, and moreover is represented by the ρ defined above.

Proposition 2. Suppose $N = 2$. Then $\mathbf{J} \succeq^P \mathbf{J}'$ if and only if $\rho(\mathbf{J}) \geq \rho(\mathbf{J}')$.

The proof of Proposition 2 is relegated to Appendix A. As a consequence of it, given any finite set of dimensions[14] we have that for any dimension k', any agreement that could be reached is dominated at the margin by some other agreement in an optimal dimension—the dimension k^* maximizing the value of $\rho(\mathbf{J}^k)$ over all values of k. In this sense the incentives of the parties are aligned when selecting a dimension.

6. Comparing the TU and NTU Environments

The TU and NTU environments differ in a crucial way. In the NTU environment, transfers cannot be used to redistribute surplus. However, there is a small class of environments in which the NTU and TU problems coincide in certain sense. These are TU environments in which any optimal allocation that can be achieved using transfers is just as good (from the perspective of each agent) as some allocation in which no transfers are used. As transfers are not required, the TU and NTU problems are then essentially the same. As we will describe in this section, these environments turn out to be the ones where each agent generates the same marginal social surplus (in terms of the numeraire) per unit of effort.

[14]Or more generally, a compact set of Jacboians.

Formally, recall Definition 1 from Section 3, in which we introduced feasible linear payoffs for a given Jacobian. Imitating that definition, but excluding the possibility of transfers, we make the following definition:

Definition 4. Given any Jacobian $\mathbf{J} \in \mathcal{J}$ and real numbers $b > 0$ and $c \geq 0$, define the *feasible linearized payoffs with no transfers given* b *and* c as the set of payoff vectors

$$\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c) = \left\{ \mathbf{Ja} : \sum_i (-J_{ii}) a_i = b, \text{ and } \forall i,\ (-J_{ii}) a_i \geq -c \right\} \cap \mathbb{R}^N_+.$$

Definition 5. A Jacobian $\mathbf{J} \in \mathcal{J}$ is called *TU-equivalent* if, for any $b > 0, c \geq 0$, there is a $\underline{c} \geq 0$ such that $\mathrm{FLP}(\mathbf{J}; b, c) = \mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c')$ for all $c' \geq \underline{c}$. Define $\mathcal{E}$ as the set of TU-equivalent Jacobians.

In other words, a Jacobian is TU-equivalent if any feasible linearized payoffs achievable with transfers are also achievable without transfers, perhaps with an adjustment of the backsliding constraint c.

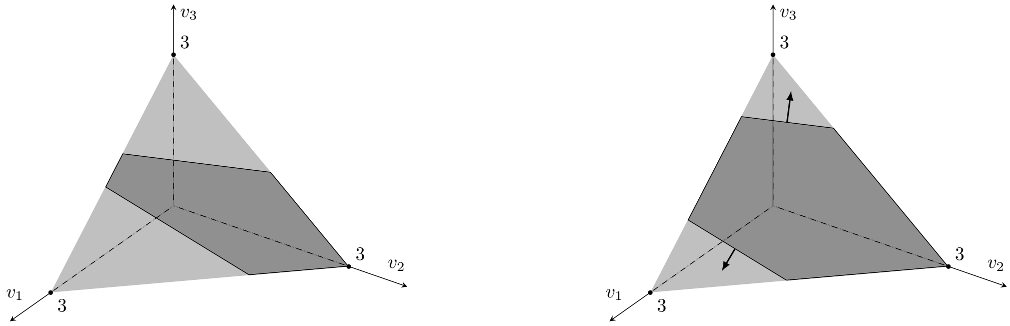
This section does two things. First, it gives a simple characterization of TU-equivalence directly in terms of matrix entries. Second, it shows that for TU-equivalent Jacobians, the NTU index ρ that we have derived for the two-agent problem yields the same ranking as the TU index h. This consistency gives independent confirmation that the way we have defined TU-equivalence is reasonable. In Section 7, where we consider the many-agent NTU problem, rather than observing agreement between TU and NTU indices we have already derived, we will impose it as an axiom on our index.

To give our characterization, define

$$\mathcal{H} = \left\{ \mathbf{J} \in \mathcal{J} : \sum_{i:i \neq j} \frac{J_{ij}}{-J_{jj}} = \sum_{i:i \neq k} \frac{J_{ik}}{-J_{kk}} \text{ for all } j, k \right\}.$$

We will think of these matrices, for the moment, as Jacobians in the TU setting, where $\mu = 1$ and so comparisons of payoffs are meaningful because everything is measured in terms of the numeraire. This is then the set of Jacobians where each agent's effort generates the same total surplus per unit of cost paid by the contributing agent. In terms of Definition 1, this is the set where any j achieves the maximum involved in defining h, which explains the notation $\mathcal{H}$. It will turn out that these matrices are the ones that are TU-equivalent according to Definition 5.

Why is this? For Jacobians outside $\mathcal{H}$, not all actions generate the same surplus per unit of cost. In the proof of Proposition 1, we saw that the greatest total surplus is achieved by having certain players (with the most efficient actions) be the "contributors," taking the greatest possible actions consistent with the constraints. When transfers are permitted, the surplus generated can then be redistributed arbitrarily. However, the set $\mathrm{FLP}^{\mathrm{NT}}$ (which involves no transfers) will typically be quite different from FLP with the same budget, because without transfers there may be no way to achieve the maximum total surplus *and* distribute it arbitrarily. The tension is that maximizing total surplus requires the efficient contributors to exhaust the budget constraint; giving these contributors a high payoff without transfers entails making them contribute less, while others contribute for their benefit. The set $\mathcal{H}$ is exactly the one in which there is no such conflict, because everyone's action is equally efficient and (generically) changes in actions can be used to move around surplus without any transfers. The only limitation to moving surplus around in this way is the backsliding constraint. So, for a sufficiently loose backsliding



(A) $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};1,0)$, shaded darker, a subset of $\mathrm{FLP}(\mathbf{J};1,0)$, the entire simplex.

(B) $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};1,c)$ growing as c increases.

FIGURE 1. Panel (a) illustrates that with no backsliding, the no-transfer feasible linearized payoffs $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};1,0)$ constitute a strict subset of $\mathrm{FLP}(\mathbf{J};1,0)$. Panel (b) shows that allowing negative actions to be taken ($c>0$) enlarges $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};1,c)$ until it encompasses all of $\mathrm{FLP}(\mathbf{J};1,0)$.

constraint c', it holds that $\mathrm{FLP}(\mathbf{J};b,c') = \mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};b,c')$. Moreover, in contrast to other Jacobians, for $\mathbf{J}\in\mathcal{H}$, relaxing the backsliding constraint does not increase the surplus that can be obtained. No overall gain in surplus can be realized by some agents taking lower actions to permit another to take a higher action, because all actions are equally efficient. Thus $\mathrm{FLP}(\mathbf{J};b,c') = \mathrm{FLP}(\mathbf{J};b,c)$ for all $c\in\mathbb{R}_+$, and combining equalities

$$\mathrm{FLP}(\mathbf{J};b,c) = \mathrm{FLP}(\mathbf{J};b,c') = \mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};b,c')$$

and $\mathbf{J}\in\mathcal{E}$. Before showing this characterization of TU-equivalent Jacobians formally, we illustrate it with the following example. Take a Jacobian $\mathbf{J}\in\mathcal{H}$,

$$\mathbf{J} = \begin{bmatrix} -1 & 1 & 0.5 \\ 3 & -0.5 & 3.5 \\ 1 & 1 & -1 \end{bmatrix}.$$

Let us consider $\mathrm{FLP}(\mathbf{J};b,0)$—the feasible linearized payoff vectors when actions can only be increased. No matter what action profile is taken, if we obey the constraint

$$\sum_i (-J_{ii})a_i = b$$

on total contributions, then the total of players' linearized payoffs will be $3b$. This is because by inspection of the columns of $\mathbf{J}$ each unit of cost generates 4 total units of benefit to others, and so there is a net increase of 3 in total surplus. Thus, if we write $\mathbf{v} = \mathbf{J}\mathbf{a}+\mathbf{m}$, it follows that $\sum v_i = 3b$, given that transfers sum to 0 by assumption. Moreover, given the freedom to redistribute through transfers, *any* nonnegative $\mathbf{v}$ satisfying this equation is achievable; in other words,

$$\mathrm{FLP}(\mathbf{J};b,0) = \{\mathbf{v}\in\mathbb{R}^N_+ : \sum v_i = 3b\}.$$

This is illustrated by the whole simplex in Figure 1a (the light and the dark region together), where we set $b=1$.

Suppose we now remove the possibility of transfers and consider the payoff profiles in $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J};b,c)$. Actions must still jointly satisfy the aggregate budget constraint so, as before, $\sum v_i = 3b$. This must hold regardless of the backsliding constraint (the value of c) being considered. However, because of the absence of transfers, not all nonnegative payoff profiles satisfying $\sum v_i = 3b$ are necessarily in $\text{FLP}^{\text{NT}}(\mathbf{J};b,c)$. This is illustrated by the darkly shaded regions in Figure 1. As we relax the backsliding constraint by increasing c, some players can take more negative actions while others take higher actions without violating the budget constraint $\sum_i(-J_{ii})a_i = b$. This results in an expansion of $\text{FLP}^{\text{NT}}(\mathbf{J};b,c)$ as depicted in Figure 1b,[15] which eventually includes all of $\text{FLP}(\mathbf{J};b,0)$. Note too that, as $\mathbf{J} \in \mathcal{H}$, relaxing the backsliding constraint does not permit any more surplus to be generated and so $\text{FLP}(\mathbf{J};b,c) = \text{FLP}(\mathbf{J};b,0)$ for all $c \geq 0$.

This example illustrates how (full-rank) matrices in $\mathcal{H}$ are TU-equivalent and satisfy Definition 5. The following lemma, proved in the Appendix, generalizes this.

Lemma 2. Let $\mathcal{F}$ be the set of full-rank Jacobians in $\mathcal{J}$. Then $\mathcal{E} \cap \mathcal{F} = \mathcal{H} \cap \mathcal{F}$. That is, for full-rank Jacobians, TU-equivalence is characterized by being in $\mathcal{H}$.

Since, in the sense just described, the absence of transfers does not make a major difference to the set of payoff profiles achievable on $\mathcal{E}$, it would be reasonable to expect that matrices in this set are ranked the same by the TU and NTU measures we have introduced:

$$h(\mathbf{J}) \geq h(\mathbf{J}') \qquad \text{if and only if} \qquad \rho(\mathbf{J}) \geq \rho(\mathbf{J}').$$

Reassuringly, this condition holds for the function ρ we have presented in the $N = 2$ case. Indeed, we will show that for $\mathbf{J} \in \mathcal{H}$, it holds that $h(\mathbf{J}) = \rho(\mathbf{J})$. Taking any $\mathbf{J} \in \mathcal{E}$, there is some $\gamma > 0$ so that $\frac{J_{12}^k}{-J_{22}^k} = \frac{J_{21}^k}{-J_{11}^k} = \gamma$, and so

$$h(\mathbf{J}) = \max_j \sum_{i:i \neq j} \frac{J_{ij}^k}{-J_{jj}^k} = \gamma.$$

At the same time,

$$\rho(\mathbf{J}) = \left[\frac{J_{12}^k J_{21}^k}{J_{11}^k J_{22}^k}\right]^{1/2} = \gamma.$$

Here we observe this consistency in rankings. In the next section, we will *require*, as an axiom, equivalence of the NTU and TU ranking for TU-equivalent Jacobians (ones in $\mathcal{E}$). That will offer a key foothold that we use to make progress with the many-agent problem.

7. An Index for Many Agents in the NTU Case

Our next objective is to extend the analysis to the case of more than two agents ($N > 2$). There are several challenges in doing this. Axioms P1, P2, and P3 no longer identify a unique ranking.[16] As an alternative, we could use the Pareto relation $\succeq^P$ defined in the previous section, but, in contrast to the case of two agents, this ordering is not complete.[17] Nevertheless, we could require consistency with the Pareto relation. As an index should rank any two Jacobians (possibly equally), this still leaves open the question of how to complete the Pareto relation. In this section we consider replacing P3 by an alternative that imposes enough structure to yield a complete ordering. The ranking we identify completes the Pareto relation.

[15] For example, consider the point $\mathbf{v}' = (3,0,0)^{\mathsf{T}}$. The only way to reach these payoffs without transfers is with the action profile $\mathbf{a}' = (-12/15, 13/15, 14/15)$. So $\mathbf{v}' \in \text{FLP}^{\text{NT}}(\mathbf{J};b,c)$ if and only if $c \geq 4/5$.

[16] See Appendix B.

[17] See Example 1 in Section 7.5.

7.1. **The index.**

To present this ranking, we will first define a useful normalization of the Jacobian. The *benefits matrix* corresponding to a Jacobian $\mathbf{J}$ is denoted by $\mathbf{B}(\mathbf{J})$. It is defined by:

$$B_{ij}(\mathbf{J}) = \begin{cases} \frac{J_{ij}}{-J_{ii}} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

An off-diagonal entry B_{ij} captures the marginal benefit that i receives when j (different from i) increases his action, per unit of marginal cost i incurs. Said differently, this is i's marginal rate of substitution between j's contribution and i's own effort.

The content of P1 is precisely that $f(\mathbf{J})$ can depend on the Jacobian $\mathbf{J}$ only through this matrix:

Remark 1. Let $\mathbf{S}$ be a diagonal matrix with diagonal elements $S_{ii} = -1/J_{ii}$. By P1 we then have $f(\mathbf{J}) = f(\mathbf{SJ}) = f(\mathbf{B}(\mathbf{J}) - \mathbf{I})$. If two Jacobians have the same benefits matrix, then any f must have the same value at both of them. In other words, it is only the ratios of marginal benefits relative to marginal costs that matter, and not their absolute levels.

We now introduce a statistic for matrices that we will use to rank Jacobians.

Definition (Spectral Radius)**.** The spectral radius $r(\mathbf{M})$ of a matrix $\mathbf{M}$ is the magnitude of the largest eigenvalue of $\mathbf{M}$.

The ranking that will be selected by our axioms is the following:

Definition 6 (Spectral Radius Index)**.** The *spectral radius index* is given by $f(\mathbf{J}) = r(\mathbf{B}(\mathbf{J}))$.

For public goods problems with concave preferences, the spectral radius index can be used to characterize Pareto efficiency, and more generally its magnitude provides information on what sorts of Pareto improvements are available (Elliott and Golub, 2014). So it is a natural candidate statistic for comparing two benefits matrices.[18] Moreover, in the special case of two agents, the spectral radius of the benefits matrix is the index that axioms P1, P2, and P3 identify and so $r(\mathbf{B}(\cdot))$ generalizes the function ρ; when $N = 2$ it holds that $\rho(\mathbf{J}) = r(\mathbf{B}(\mathbf{J}))$.

7.2. **The new axioms.** We need to replace P3 with a stronger axiom. In order to do this we draw inspiration from our comparison of the TU and NTU analysis in Section 6. In that section, we identified a special class, $\mathcal{E}$, of TU-equivalent Jacobians in which utility was effectively transferable without requiring any exchanges of a numeraire good. We then verified that our TU and (two-agent) NTU rankings are equal. More generally, for Jacobians in $\mathcal{E}$, the optimal payoffs (around the status quo) that a planner could implement with transfers can also be achieved without transfers. So the equality between TU and NTU rankings should also hold more generally on this class. Any NTU index should rank two Jacobians that are both in $\mathcal{E}$ in the same way as the TU ranking.

P4: Consistency with TU rankings

> We say f is consistent with TU rankings if $\mathbf{J} \succeq^T \mathbf{J}'$ is equivalent to $f(\mathbf{J}) \geq f(\mathbf{J}')$, whenever $\mathbf{J}$ and $\mathbf{J}'$ are both full-rank and in $\mathcal{E}$ (i.e., are TU-equivalent).

[18] We discuss the interpretation of the spectral radius in more detail in Section 10.

Finally, we require our index to be continuous. This ensures that small changes in the data do not cause large changes in the marginal efficiency index.

P5: Continuity

> $f : \mathcal{J} \to \mathbb{R}$ is a continuous function (when $\mathcal{J}$ is endowed with the supremum norm).

7.3. **Main result.** We now uniquely identify the ordering of Jacobians implied by P1, P2, P4 and P5, which is the same as the ordering given by the index $r(\mathbf{B}(\cdot))$.

Theorem 2. A function $f : \mathcal{J} \to \mathbb{R}$ satisfies P1, P2, P4 and P5 if and only if f orders Jacobians identically to the spectral radius index $r(\mathbf{B}(\cdot))$: For every $\mathbf{J}, \mathbf{J}' \in \mathcal{J}$

$$f(\mathbf{J}) \geq f(\mathbf{J}') \qquad \text{if and only if} \qquad r(\mathbf{B}(\mathbf{J})) \geq r(\mathbf{B}(\mathbf{J}')).$$

Theorem 2 shows that there is a unique ranking of Jacobians that is consistent with P1, P2, P4 and P5. The numerical representation of this ranking is computed by taking the spectral radius of the benefits matrix corresponding to the Jacobian. Before presenting a formal proof of Theorem 2 we discuss some intuition.

To show that any index f satisfying the axioms must order Jacobians identically to the spectral radius index, we proceed as follows. First, we show that, by suitably scaling the numerical representations of agents' preferences and the units that actions are measured in, any Jacobian $\mathbf{J}$ can be transformed to some $\mathbf{J}' \in \mathcal{H}$. By properties P1 and P2, the index must then give the same value to $\mathbf{J}$ and $\mathbf{J}'$. So it suffices to rank the Jacobians in $\mathcal{H}$. Property P5 (continuity) implies that ranking the ones in $\mathcal{H} \cap \mathcal{F}$ (i.e., $\mathcal{H}$ minus a negligible set) is sufficient for the index to assign a value to all Jacobians in $\mathcal{H}$. We can now apply Lemma 2, which says that $\mathcal{H} \cap \mathcal{F} = \mathcal{E} \cap \mathcal{F}$; that is, f is determined by its values on full-rank Jacobians that are TU-equivalent (in the class $\mathcal{E} \cap \mathcal{F}$). For these Jacobians, P4 implies we should use the transferable utility ranking $\succeq^T$. Because the spectral radius index agrees with $\succeq^T$ on $\mathcal{E} \cap \mathcal{F}$, it follows that it and f give *all* Jacobians the same ranking. Thus, in short, we obtain our complete ordering by using properties P1, P2, and P5 to extend the TU ordering we have on $\mathcal{E} \cap \mathcal{F}$ to all Jacobians in $\mathcal{J}$.

7.4. **Proving Theorem 2.** We begin by establishing two lemmas.

Lemma 3. For any $\mathbf{J} \in \mathcal{J}$, there is a Jacobian $\widetilde{\mathbf{J}} \in \mathcal{H}$ such that, for any f satisfying P1 and P2, we have $f(\mathbf{J}) = f(\widetilde{\mathbf{J}})$.

Proof of Lemma 3. Fix a $\mathbf{J} \in \mathcal{J}$ and write $\mathbf{B} = \mathbf{B}(\mathbf{J})$. As $\mathbf{B}$ is a non-negative, irreducible, square matrix we can apply the Perron–Frobenius theorem to find a positive eigenvector $\mathbf{v} > \mathbf{0}$ of $\mathbf{B}$ such that

$$\mathbf{v}^{\mathsf{T}}\mathbf{B} = r(\mathbf{B})\mathbf{v}^{\mathsf{T}} \tag{2}$$

Let $\mathbf{V}$ be the diagonal matrix with $V_{ii} = v_i$. Define $\widetilde{\mathbf{J}} = \mathbf{V}(\mathbf{B} - \mathbf{I})\mathbf{V}^{-1} = \mathbf{V}\mathbf{B}\mathbf{V}^{-1} - \mathbf{I}$.

The content of P1 is that $f(\mathbf{J})$ depends only on $\mathbf{B}(\mathbf{J})$. Indeed, by Remark 1,

$$f(\mathbf{J}) = f(\mathbf{B} - \mathbf{I}). \tag{3}$$

By P1 and P2, we have

$$f(\widetilde{\mathbf{J}}) = f(\mathbf{V}(\mathbf{B} - \mathbf{I})\mathbf{V}^{-1}) = f(\mathbf{B} - \mathbf{I}),$$

and by (3), this is equal to $f(\mathbf{J})$.

It remains only to verify that $\widetilde{\mathbf{J}} \in \mathcal{H}$. Note that (2) implies $\mathbf{1}^\mathsf{T}\mathbf{VB} = r(\mathbf{B})\mathbf{1}^\mathsf{T}\mathbf{V}$. As $\mathbf{V}$ is a diagonal matrix with all its diagonal entries strictly positive, it is invertible, so we can conclude that $\mathbf{1}^\mathsf{T}\mathbf{VBV}^{-1} = r(\mathbf{B})\mathbf{1}^\mathsf{T}$. In other words, every column of $\mathbf{B}(\widetilde{\mathbf{J}}) = \mathbf{VBV}^{-1}$ sums to $r(\mathbf{B})$. By definition of $\mathcal{H}$, this implies $\widetilde{\mathbf{J}} \in \mathcal{H}$. ∎

Lemma 4. For all full rank $\mathbf{J}, \mathbf{J}' \in \mathcal{H}$, the statement $\mathbf{J} \succeq^T \mathbf{J}'$ holds if and only if $r(\mathbf{B}(\mathbf{J})) \geq r(\mathbf{B}(\mathbf{J}'))$.

Proof of Lemma 4. By Proposition 1, the ranking $\succeq^T$ is represented by the function h defined by

$$h(\mathbf{J}) = \max_j \sum_{i:i\neq j} \frac{J_{ij}}{-J_{jj}} = \max_j \sum_{i:i\neq j} B_{ij}, \tag{4}$$

where $\mathbf{B} = \mathbf{B}(\mathbf{J})$. Therefore, to establish the lemma it suffices to show that for any $\mathbf{J} \in \mathcal{H}$, we have $h(\mathbf{J}) = r(\mathbf{B}(\mathbf{J}))$, which we now do. Fix a $\mathbf{J} \in \mathcal{H}$ and note that by definition of $\mathcal{H}$ all columns of $\mathbf{B}(\mathbf{J})$ have the *same* sum, $h(\mathbf{J})$. In other words, $\mathbf{1}^\mathsf{T}\mathbf{B}(\mathbf{J}) = h(\mathbf{J})\mathbf{1}^\mathsf{T}$. This equation says that $h(\mathbf{J})$ is an eigenvalue of $\mathbf{B}(\mathbf{J})$ with the *positive* (left-hand) eigenvector $\mathbf{1}^\mathsf{T}$. By the Perron–Frobenius Theorem, this entails that $h(\mathbf{J})$ is actually equal to the spectral radius.[19] ∎

With the help of these two lemmas, we now prove Theorem 2.

Proof of Theorem 2. To verify the "if" direction, define $f(\mathbf{J}) = r(\mathbf{B}(\mathbf{J}))$. To check P1, take any $\mathbf{S} \in \mathcal{S}$ and let $\mathbf{J}' = \mathbf{SJ}$. Then note that $f(\mathbf{J}') = r(\mathbf{B}(\mathbf{J}')) = r(\mathbf{B}(\mathbf{J})) = f(\mathbf{J})$, where the middle equality holds by Remark 1.

To verify P2, note that $\mathbf{B}(\mathbf{JS}^{-1}) = \mathbf{SB}(\mathbf{J})\mathbf{S}^{-1}$. Conjugation by an invertible matrix does not change eigenvalues, and in particular does not change the spectral radius: $r(\mathbf{SB}(\mathbf{J})\mathbf{S}^{-1}) = r(\mathbf{B}(\mathbf{J}))$. Therefore,

$$f(\mathbf{JS}^{-1}) = r(\mathbf{B}(\mathbf{JS}^{-1})) = r(\mathbf{SB}(\mathbf{J})\mathbf{S}^{-1}) = r(\mathbf{B}(\mathbf{J})) = f(\mathbf{J}).$$

Combining Lemmas 2 and 4 implies that f satisfies P4 and as the spectral radius is a continuous function of matrix entries (Wilkinson, 1965), P5 is satisfied.

Conversely, we will show that if $f : \mathcal{J} \to \mathbb{R}$ is any function satisfying P1, P2, and P4, and P5, then f represents the same ordering on $\mathcal{J}$ as the spectral radius index $\widehat{f} : \mathbf{J} \mapsto r(\mathbf{B}(\mathbf{J}))$.

Fix some $\mathbf{J}$ and $\mathbf{J}'$. In the "if" part of the proof, we saw that the function $\widehat{f}$ satisfies P1 and P2. And by hypothesis, so does f. Thus, by Lemma 3, there are $\widetilde{\mathbf{J}}, \widetilde{\mathbf{J}}' \in \mathcal{H}$ such that $\widehat{f}(\mathbf{J}) = \widehat{f}(\widetilde{\mathbf{J}})$ and $f(\mathbf{J}) = f(\widetilde{\mathbf{J}})$. By Lemma 2, $\mathcal{H} \cap \mathcal{F} = \mathcal{E} \cap \mathcal{F}$, where $\mathcal{F} \subset \mathcal{J}$ is the set of full-rank Jacobians. Moreover, as the set of full rank Jacobians is dense in $\mathcal{E}$, by continuity (P5) showing that f and $\widehat{f}$ order Jacobians identically on $\mathcal{E} \cap \mathcal{F}$ implies that they also do on $\mathcal{E}$. So, to finish the proof of the theorem, it suffices to show that $\widehat{f}$ and f induce the same ranking on $\mathcal{E} \cap \mathcal{F}$. P4 yields that f induces the same order on $\mathcal{E}$ for full rank Jacobians as $\succeq^T$. And Lemma 4 says that for full rank Jacobians, $\succeq^T$ is represented by $\widehat{f}$ on $\mathcal{E}$. Thus, f represents the same ordering on as $\widehat{f} : \mathbf{J} \mapsto r(\mathbf{B}(\mathbf{J}))$ for any Jacobians in $\mathcal{E} \cap \mathcal{F}$, and this completes the proof. ∎

[19]See (Meyer, 2000, Section 8.3). Any eigenvalue that is *not* the largest cannot have a positive eigenvector.

7.5. **Consistency with the Pareto ordering.** We show now that the index $r(\mathbf{B}(\mathbf{J}))$ is consistent with NTU Pareto ordering $\succeq^P$.

Proposition 3. If $\mathbf{J} \succeq^P \mathbf{J}'$ then $r(\mathbf{B}(\mathbf{J})) \geq r(\mathbf{B}(\mathbf{J}'))$.

The proof is relegated to Appendix A. In contrast to the case of two agents, Proposition 3 does not state that if $r(\mathbf{B}(\mathbf{J})) \geq r(\mathbf{B}(\mathbf{J}'))$ then $\mathbf{J} \succeq^P \mathbf{J}'$. This is because the NTU Pareto ordering $\succeq^P$ is no longer complete. To make this point we provide a counterexample.

Example 1. Assume, toward a contradiction, that the Pareto relation $\succeq^P$ is complete. We will then show that this implies an intransitivity among the following three Jacobians.[20]

$$\mathbf{J} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \qquad \mathbf{J}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \qquad \mathbf{J}'' = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

In all three Jacobians, one agent can provide benefits to both others (while the other agents can provide benefits to only one agent) and one agent can receive benefits from both others (while the other agents can receive benefits from only one agent). The difference between the three matrices lies only in the identities of these agents. It is efficient for the agent who can provide more benefits to take relatively high actions. The other agents will then receive relatively high returns on their investments, although this agent will receive a relatively low return. Different dimension are therefore best for achieving high returns for the different agents. We now show this more formally.

We will show that if $\succeq^P$ were complete on $\{\mathbf{J}, \mathbf{J}', \mathbf{J}''\}$ then it would be intransitive. Consider first the direction $\mathbf{d}'' = (0.25, 0.5, 0.25)$. For this direction, $\mathbf{p}(\mathbf{d}''; \mathbf{J}'') = (2, 0.5, 3)$. If $\mathbf{J} \succeq^P \mathbf{J}''$, then there must exist a $\mathbf{d}$ such that $\mathbf{p}(\mathbf{d}; \mathbf{J}) \geq \mathbf{p}(\mathbf{d}''; \mathbf{J}'')$. This requires the following three inequalities to hold: $(d_2 + d_3)/d_1 \geq 2$, $d_3/d_2 \geq 0.5$ and $d_1/d_3 \geq 3$. The last two inequalities imply that $d_1 \geq 3d_3 \geq d_2 + d_3$, while the first inequality implies that $d_1 < d_2 + d_3$. Thus, there does not exist a $\mathbf{d}$ such that $\mathbf{p}(\mathbf{d}; \mathbf{J}) \geq \mathbf{p}(\mathbf{d}''; \mathbf{J}'')$. Indeed, as the ordering is complete, and as there does not even exist $\mathbf{d}$ such that $\mathbf{p}(\mathbf{d}) = \mathbf{p}(\mathbf{d}'')$, we can conclude that $\mathbf{J}'' \succeq^P \mathbf{J}$. Symmetric arguments for the directions $\mathbf{d}' = (0.5, 0.25, 0.25)$ and $\mathbf{d} = (0.25, 0.25, 0.5)$ establish that $\mathbf{J}' \succeq^P \mathbf{J}''$ and $\mathbf{J} \succeq^P \mathbf{J}'$. This contradicts the assumption that $\succeq^P$ is transitive.

8. Discussion

8.1. **Invariance to utility representation.** Our theory is ultimately about ranking pairs of Jacobians. The axiomatic exercise proceeds by imposing restrictions on this ranking, and in particular requiring that some pairs of Jacobians be treated the same. However, there are multiple ways to interpret this exercise, depending on how the Jacobians arise.

Most simply, the two utility profiles, $\mathbf{u}, \widetilde{\mathbf{u}} : \mathbb{R}^N \to \mathbb{R}^N$, giving rise to two N-by-N Jacobians could correspond to different representations of the same agents' fixed preferences. In this case, our index should assign the same value to the corresponding Jacobians—i.e.,

[20] It is also possible to construct examples where directly comparing just two dimensions is sufficient to demonstrate that they cannot be ordered. We present the intransitivity counterexample because the matrices that generate it are simple.

it should be insensitive to the choice of representation. If $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ induce the same preferences over lotteries, then $\widetilde{u}_i = c_i u_i$ for each i.[21] Therefore, we should treat the Jacobian $\mathbf{J}$ the same as $\widetilde{\mathbf{J}} = \mathbf{SJ}$, where $\mathbf{S}$ is the diagonal matrix whose (i,i) entry is c_i. This is the content of P1.

A second interpretation of $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ is that they represent the preferences of different agents involved in different negotiations. (We discuss in more detail how to use our theory to compare such negotiations in Section 9.2.) In particular, suppose $\widetilde{u}_i = c_i u_i$ so that $\widetilde{\mathbf{J}} = \mathbf{SJ}$, where $\mathbf{S}$ is again a the diagonal matrix whose (i,i) entry is c_i. Even though the agents are no longer the same, these two sets of agents would make all choices identically, and so there is no revealed-preference basis for treating them differently. Thus, we should again impose that our index ranks the two Jacobians the same: $f(\widetilde{\mathbf{J}}) = f(\mathbf{J})$, as required by P1.

A final interpretation of $\mathbf{u}$ and $\widetilde{\mathbf{u}}$—the one we focused on in Section 2.1.1—is that they are the same agents' preferences restricted to different dimensions: $u_i(\mathbf{a}) = U_i^k(\mathbf{a}, \mathbf{0})$ and $\widetilde{u}_i(\mathbf{a}) = U_i^{k'}(\mathbf{a}, \mathbf{0})$, where the $\mathbf{0}$ corresponds to no transfers. In this case, the statement that $\widetilde{\mathbf{J}} = \mathbf{SJ}$ need *not* imply that we should treat $\mathbf{J}$ and $\widetilde{\mathbf{J}}$ the same. If the Jacobians in different dimensions are generated by the same utility representation of preferences, U, then there is information in the relative magnitudes of the Jacobians' entries. In particular, the quantity $J_{ij}/\widetilde{J}_{ij}$ measures agent i's marginal rate of substitution between concessions made by agent j in dimension k and concessions made in dimension k'.

In this case, requiring invariance to the numerical representation of preferences yields a weaker axiom. Suppose, for example, we were to change the representation of each agent i's preferences by rescaling, to get a new representation V_i. In this case, we would have $V_i^k = c_i U_i^k$, simultaneously for *every* dimension k. Thus, if we were interested in comparing dimensions k and k', and the Jacobians before the transformation were $\mathbf{J}^k$ and $\mathbf{J}^{k'}$, then the Jacobians after the transformation would be $\mathbf{SJ}^k$ and $\mathbf{SJ}^{k'}$, where $\mathbf{S}$ is defined as before. The appropriate axiom is then that the outcome of comparing $\mathbf{J}^k$ and $\mathbf{J}^{k'}$ is the same as that of comparing $\mathbf{SJ}^k$ and $\mathbf{SJ}^{k'}$. This is the content of the following axiom, P1$'$.

P1$'$: Limited invariance to utility representation

> We say f satisfies limited invariance to utility representation if for all $\mathbf{S} \in \mathcal{S}$ and all $\mathbf{J}, \mathbf{J}' \in \mathcal{J}$, the inequality $f(\mathbf{SJ}) > f(\mathbf{SJ}')$ holds if and only if $f(\mathbf{J}) > f(\mathbf{J}')$.

Property P1 is strictly stronger than P1$'$.[22] There are four reasons for focusing on P1 instead of P1$'$. First, we want our index to be able to compare negotiations involving different agents, as discussed above. Second, the analyst may only be able to elicit information about choices between action profiles in a given dimension, giving no basis on which to establish a relative scale between $\mathbf{J}^k$ and $\mathbf{J}^{k'}$. Third, and relatedly, we may not be sure that the Jacobian in dimension k comes from the same numerical representation of preferences as the Jacobian in dimension k'. Finally, suppose an agent represents many people (as with international negotiations). Then even if the same negotiators are involved in all negotiations being compared, the gains and losses in dimension k may accrue to different people from those who experience the gains and losses in dimension

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[21] See Section 5.1, and particularly footnote 10.

[22] To see that P1 implies P1$'$ note that by P1, $f(\mathbf{SJ}) = f(\mathbf{J})$ and $f(\mathbf{SJ}') = f(\mathbf{J}')$. It then follows immediately that $f(\mathbf{SJ}) > f(\mathbf{SJ}')$ if and only if $f(\mathbf{J}) > f(\mathbf{J}')$.

k'. By demanding that we ignore the scale of matrix entries, and restricting attention to the tradeoffs *within* a dimension, P1 rules out interpersonal comparisons in such cases.

As P1′ is weaker than P1, the index $r(\mathbf{B}(\cdot))$, which satisfies P1, also satisfies P1′. So the part of Theorem 2 that says our index satisfies the axioms continues to hold; it is the converse which needs more than P1′. The sharpness of the characterization we obtain in Theorem 2 stems from asking our index to satisfy P1 instead of just P1′.

8.2. **Interpreting the monotonicity axiom when different agents are involved.** Next, we discuss how the interpretation of the monotonicity axiom (P3) changes depending on which of the above-discussed interpretations we focus on. The Jacobian $\mathbf{J}$ exceeding $\mathbf{J}'$ entrywise means that, for any $\mathbf{d}$, the linearized payoffs $\mathbf{Jd}$ are all at least as large as the linearized payoffs $\mathbf{J}'\mathbf{d}$. If $\mathbf{J}$ and $\mathbf{J}'$ represent the marginal utilities of the same agents on different dimensions—as discussed in Section 2.1.1—then these numerical comparisons yield information about these agents' preferences: the agents all weakly prefer to implement a given action profile with $\mathbf{J}'$ rather than $\mathbf{J}$. In this case, P3 states (loosely) that dimensions which are preferred by all agents in this sense should be ranked higher. But when different agents are involved, the payoffs are numerical representations of different agents' preferences and cannot be directly compared.

What *can* be compared across different agents are their marginal rates of substitution, the marginal benefits they receive per unit of own marginal cost. We therefore weaken the monotonicity axiom to the following.

P3′: Weak Monotonicity

> We say f is weakly monotonic if $f(\mathbf{J}) > f(\mathbf{J}')$ whenever there exists a permutation $\sigma : \mathcal{N} \to \mathcal{N}$ such that, for all i, j with $i \neq j$, we have $J_{ij}/(-J_{ii}) > J'_{\sigma(i)\sigma(j)}/(-J'_{\sigma(i)\sigma(i)})$.

The final inequality can be stated equivalently as $B_{ij} > B_{\sigma(i)\sigma(j)}$, where $\mathbf{B} = \mathbf{B}(\mathbf{J})$ is the benefits matrix defined in Section 7.1. Weak monotonicity requires that a Jacobian $\mathbf{J}$ is ranked above $\mathbf{J}'$ if the following holds. Agents can be matched across the two negotiations (i.e., a permutation of the set of agents can be found) so that any two agents in the $\mathbf{J}$ negotiation value each other's contributions more than the corresponding agents in the $\mathbf{J}'$ negotiation. The amount that i values j's contribution is defined in the sense of marginal rates of substitution: how many units of his own effort i is willing to put in to receive a unit of j's effort. It is straightforward to verify that the proof of Theorem 1 goes through unchanged with property P3′ replacing property P3.

8.3. **An alternative formulation of the NTU problem.** We model the NTU problem by setting agents' values of numeraire good to zero ($\mu = 0$). This prevents the numeraire from being transferred to provide compensation, but also prevents the agents' preferences for it from being used to measure their payoffs in some common scale. An alternative would have been to use the exchange rate afforded by preferences over the numeraire. That is, we would set $\mu = 1$, which gives rows of the Jacobian a non-arbitrary scale. But we would still rule out actual transfers. This can be done by working in the TU setting and using the set $\mathrm{FLP}^{\mathrm{NT}}$ of Definition 4 in Section 6:

$$\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c) = \left\{ \mathbf{Ja} : \sum_i (-J_{ii}) a_i = b, \text{ and } \forall i,\ (-J_{ii}) a_i \geq -c \right\} \cap \mathbb{R}^N_+.$$

Proceeding in this way we could obtain a partial Pareto ordering for the NTU problem, by saying that $\mathbf{J}$ is better than $\mathbf{J}'$ when any element of $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c)$ is Pareto-dominated by some element of $\mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c)$. This alternative approach is not equivalent to the one we take.

Relative to the approach just described, setting $\mu = 0$ implies that transfers cannot be made *and* that, in effect, the social planner either: (i) does not know agents' marginal values for the numeraire good; or (ii) cannot justify using these values as a means of making interpersonal comparisons.[23] This motivates the inclusion of Property P1 in our axiomatization, which plays a crucial role in identifying a unique efficiency measure.

9. Extensions and Applications

In this section we discuss various ways in which our analysis can be extended to address additional questions.

9.1. **Extending the scope of negotiations.** While we have focused on selecting a single dimension for negotiations, it is immediate to extend the analysis to permit negotiations over multiple dimensions at once. For example, we might suppose that each country can make concessions on one dimension, but that these dimensions need not be the same across agents. To capture this possibility we simply expand the set of Jacobians we are choosing among. A typical new Jacobian would be constructed by taking some of the original Jacobians—say, $\mathbf{J}^1$ and $\mathbf{J}^2$—and building a new Jacobian by taking, say column 1 of $\mathbf{J}^1$ and column 2 of $\mathbf{J}^2$. To make this more concrete, suppose there are two countries 1 and 2, and two dimensions, say pollution (corresponding to $\mathbf{J}^1$)and trade (corresponding to $\mathbf{J}^2$). Let the first column of the Jacobians correspond to concessions by 1 and the second to concessions by 2. Our new Jacobian described above then describes the benefits of a negotiation in which country 1 makes concessions on pollution and country 2 on trade.

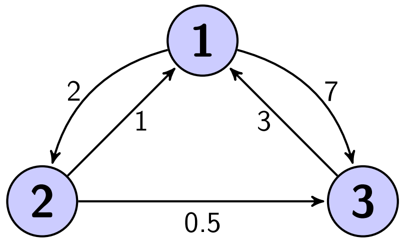
More generally, we can also allow each country to make concessions in multiple dimensions by constructing a new Jacobian where the ith column is a convex combination of the ith columns of the original Jacobians. This corresponds to a given agent taking positive actions in several dimensions simultaneously, with the weights in the convex combination determining the ratios among those actions.

9.2. **Choosing the negotiators.** A problem closely related to that of selecting an agenda is a planner's problem of deciding whom to include. Suppose, for example, that N countries can be selected to negotiate from a set of K countries. In practice, large negotiations may be infeasible or involve large unmodeled costs. We capture this by restricting the number of seats at the table to N. We take as a starting principle that a negotiation should be evaluated based on the marginal utilities of those involved—that is, on the Jacobian among the N countries invited to negotiate. This is consistent with our emphasis on measuring the scope for favor-trading or mutual benefit, rather than benefits to outside parties.

[23] In the transferable utility case, the usual Coasian argument applies: If one outcome is worth more units of the numeraire good than another, we can use transfers of the numeraire to make everyone better off. When transfers cannot actually be made, the case for making interpersonal comparisons based on agents' valuation of a commonly held numeraire good is weaker.

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ 7 & 0.5 & 0 \end{bmatrix}$$

(a)



(b)

Figure 2. A benefits matrix (a) and the corresponding weighted directed graph (b).

With this principle in mind, we can discuss in our framework the decision about whom to include. To do this, we construct N-by-N Jacobians from the original K-by-K Jacobians. For each dimension we form the set of Jacobians in which any N columns and the same same N rows of the original Jacobian are selected. Now we can simultaneously choose which countries to grant a seat at the table and which issue these countries should discuss by evaluating the N-by-N Jacobians using the index $r(\mathbf{B}(\cdot))$ axiomatized by Theorem 2.

9.3. **Changing the parameters of a negotiation.** So far we have discussed how to select the participants and the agenda from a fixed set of alternatives. A closely related question concerns the gains from relaxing the constraints imposed on the negotiations. What are the potential benefits from including an additional country or discussing an additional issue? Our marginal efficiency index provides a way of addressing such questions.

The parameters of the negotiation design characterize a set of possible negotiations that we can choose among. We can relax these constraints, choose among more potential negotiations, and see how much the highest available marginal efficiency index increases (it will always weakly increase as we are maximizing over a larger set). Such an exercise identifies when there are no additional gains from expanding the set in a certain way, reveals the directions in which it is best to expand the negotiation, and can help compare the gains from expanding the negotiations in different situations. More broadly, we can ask what it is about the data of the problem, i.e. the Jacobians, that determines the gains from adding people to the negotiation or else putting more issues on the table.

10. Interpreting the Spectral Radius

As mentioned in the introduction, a key conceptual implication of our analysis is a new connection between axiomatic analysis of negotiations and the theory of networks. In this section, we interpret the spectral radius of the benefits matrix—the main index that we axiomatize—as a network statistic. This will shed light on what aspects of the structure of externalities matter for the marginal efficiency of a negotiation.

The benefits matrix can be viewed as a network, in which there is a directed link from j to i if j's effort matters to i—i.e., if $B_{ij} > 0$—and the strength of this link is B_{ij}, which measures how much j's effort matters to i. This is illustrated in Figure 2. It will turn out that the spectral radius of $\mathbf{B}$ measures the aggregate strength of cycles in this network. Substantively, this corresponds to the following idea: The potential for the success of a given NTU negotiation depends on the ability of agents to compensate one another for the benefits they provide. These benefits must be balanced in a certain sense. If one agent can provide large marginal benefits to many others, it will be important that there are others who can collectively provide sufficient marginal benefits to this agent. This is the intuitive idea that is formalized by looking at cycles of cooperation.

A cycle in $\mathbf{B}$ is a sequence $(c(1), c(2), \ldots, c(\ell), c(\ell+1))$ of players that starts and ends at the same player $(c(\ell+1) = c(1))$, which $B_{c(k)c(k+1)} > 0$ for each k. Players may be visited multiple times in the sequence and so longer cycles can be made up by combining and repeating many shorter cycles. Letting $\mathcal{C}(\ell; \mathbf{B})$ be the set of all cycles of length ℓ in the benefit matrix $\mathbf{B}$, we define the value of a cycle $c \in \mathcal{C}(\ell; \mathbf{B})$ as

$$v(c; \mathbf{B}) = \prod_{t=1}^{\ell} B_{c(t)c(t+1)}. \tag{5}$$

So the value of a cycle c is simply the product of the edge weights. This definition entails that a cycle has a value, according to v, that depends on its "weakest link"—if some factors in this product are very small, then the value of the whole cycle is low. This is the sense in which balanced cycles of cooperation are more valuable.

It turns out that the spectral radius aggregates, in a particular way, the values of the cycles of cooperation that are present in a benefits matrix $\mathbf{B}$. Indeed, we have the following identity:[24]

$$r(\mathbf{B}) = \limsup_{\ell \to \infty} \left(\sum_{c \in \mathcal{C}(\ell; \mathbf{B})} v(c; \mathbf{B}) \right)^{1/\ell}.$$

Although it is intuitive that cycles of cooperation matter, Theorem 2 says that to be consistent with our axioms, a dimension should be evaluated according to the particular aggregation of cycles given above.[25]

11. Conclusions

In many settings multiple parties are involved in negotiations and a dimension for negotiations must be selected ahead of time. In each dimension, parties negotiate over how much costly effort to exert, and then the effort provides benefits to the other parties. We have proposed a marginal efficiency index, based only on marginal data (i.e., derivatives of payoffs), that can help parties involved in such negotiations decide which dimensions to select for negotiations.

When there are two agents, or when utilities are transferable, all parties should agree on the best dimension for negotiations. For any actions that can be agreed on in any other dimension, we can find a unanimously preferred alternative agreement in the best dimension—an agreement that generates higher marginal benefits without increasing anyone's costs. For the non-transferable utility two-agent case we identify the unique index that represents this ranking and show that it generates the same ranking as all marginal efficiency indexes that are: (i) invariant to rescaling of the units actions are measured (P1); (ii) invariant to alternative utility representations of the same preferences (P2); and (iii) rank entry wise better Jacobians (higher marginal benefits, lower marginal costs) higher (P3).

[24] See, e.g., Milnor (2001).

[25] See Elliott and Golub (2014) for a more extensive discussion that elaborates on the points made here. This discussion includes several examples that illustrate the import of cycles of cooperation, albeit in a different context.

When there are more than two agents, there may be no dimension that always dominates all other dimensions in the sense just discussed. However, there is a unique efficiency index that agrees with the transferable utility ranking when utilties are effectively transferable (P4), is continuous (P5) and that also satisfies P1 and P2. This index generalizes the two-agent index and is the spectral radius of the benefits matrix.

We summarize our results below:

TU index	$h(\mathbf{J})$	$\iff$	$\succeq^T$ TU Pareto ranking	(Proposition 1)
2 person NTU index	$\rho(\mathbf{J})$	$\iff$	$\succeq^P$ NTU Pareto ranking	(Proposition 2)
n person NTU index	$r(\mathbf{B}(\mathbf{J}))$	$\implies$	$\succeq^P$ NTU Pareto ranking	(Proposition 3)
	$\rho(\mathbf{J})$	$\iff$	P1, P2 and P3	(Theorem 1)
	$r(\mathbf{B}(\mathbf{J}))$	$\iff$	P1, P2, P4 and P5	(Theorem 2)

Although we have proposed three different indices with different foundations to apply on different domains, these indices agree where their domains overlap. We summarize these relationships in Table 1.

	$N = 2$ Agents	$N > 2$ Agents
TU	$h(\mathbf{J})$	$h(\mathbf{J})$
TU-Equivalent	$h(\mathbf{J}) = \rho(\mathbf{J}) = r(\mathbf{J})$	$h(\mathbf{J}) = r(\mathbf{J})$
NTU	$\rho(\mathbf{J}) = r(\mathbf{J})$	$r(\mathbf{J})$

TABLE 1. Our proposed indices and their equivalences across domains.

Overall, we view our results as providing a theory for ranking agendas based on marginal data. Our analysis easily extends to include multiple dimensions and different possible sets of negotiating agents. Indeed, as discussed in Section 8, it is ensuring that our framework is general enough to tackle these additional questions that generates its sharpness.

Appendix A. Omitted Proofs

Proof of Proposition 2.

We first show that $\mathbf{J} \succeq^P \mathbf{J}'$ implies $\rho(\mathbf{J}) \geq \rho(\mathbf{J}')$.

As $\mathbf{J} \succeq^P \mathbf{J}'$ for any $\mathbf{d}' \in \Delta_2$, there must exist a direction $\mathbf{d} \in \Delta_2$ such that $\mathbf{p}(\mathbf{d}; \mathbf{J}) \geq \mathbf{p}(\mathbf{d}'; \mathbf{J}')$. Taking in particular $\mathbf{d}' = (\frac{1}{2}, \frac{1}{2})$, there must exist a $\mathbf{d} \in \Delta_2$ such that:

$$\frac{J_{12}d_2}{-J_{11}d_1} \geq \frac{J'_{12}}{-J'_{11}} \qquad \text{and} \qquad \frac{J_{21}d_1}{-J_{22}d_2} \geq \frac{J'_{21}}{-J'_{22}}.$$

As both sides of both the above inequalities are positive[26], we can multiply the inequalities to find

$$\frac{J_{12}J_{21}}{J_{11}J_{22}} \geq \frac{J'_{12}J'_{21}}{J'_{11}J'_{22}}, \tag{6}$$

which is precisely the statement that $\rho(\mathbf{J}^k) \geq \rho(\mathbf{J}^{k'})$.

Conversely, suppose $\rho(\mathbf{J}) \geq \rho(\mathbf{J}')$, i.e., that (6) holds; we will show that $\mathbf{J} \succeq^P \mathbf{J}'$. Take any $\mathbf{d}' \in \Delta_2$. Choose $\mathbf{d} \in \Delta_2$ so that

$$\frac{J_{12}d_2}{-J_{11}d_1} = \frac{J'_{12}d'_2}{-J'_{11}d'_1} \tag{7}$$

It is always possible to find such a $\mathbf{d} \in \Delta_2$ because d_1 and d_2 can be selected to make d_1/d_2 equal to any positive number. Flipping the fractions on both sides of this equation and multiplying the resulting equation by (6) above yields:

$$\frac{J_{21}d_1}{-J_{22}d_2} \geq \frac{J'_{21}d'_1}{-J'_{22}d'_2}. \tag{8}$$

Statements (7) and (8) entail that $\mathbf{p}(\mathbf{d}; \mathbf{J}) \geq \mathbf{p}(\mathbf{d}'; \mathbf{J}')$. ∎

Proof of Lemma 2. First we prove that $\mathbf{J} \in \mathcal{H} \cap \mathcal{F}$ implies $\mathbf{J} \in \mathcal{E}$, i.e. that for any $b > 0$, $c \geq 0$, there exists $\underline{c} \geq 0$ such that $\mathrm{FLP}(\mathbf{J}; b, c) = \mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c')$ for all $c' \geq \underline{c}$.

We begin by establishing that $\mathrm{FLP}(\mathbf{J}; b, c) \subseteq \mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c')$ for all large enough c'.

Suppose $\mathbf{v} \in \mathrm{FLP}(\mathbf{J}; b, c)$; i.e. $\mathbf{v} = \mathbf{J}\mathbf{a} + \mathbf{m}$ for some some $\mathbf{a}$ satisfying $\sum_i (-J_{ii}) a_i = b$ and, for all i, we have $(-J_{ii}) a_i \geq -c$. We seek to show that $\mathbf{v} \in \mathrm{FLP}^{\mathrm{NT}}(\mathbf{J}; b, c')$ for some c'—i.e., that $\mathbf{v} = \mathbf{J}\mathbf{a}'$ for some $\mathbf{a}'$ satisfying $\sum_i (-J_{ii}) a'_i = b$ and, for all i, $(-J_{ii}) a_i \geq -c'$. As $\mathbf{J}$ has full rank, it is invertible, and we can choose $\mathbf{a}' = \mathbf{J}^{-1}\mathbf{v}$.

Next we will verify that the budget constraint $\sum_i (-J_{ii}) a'_i = b$ holds. To this end, note that:

$$\begin{aligned}
\mathbf{1}^{\mathsf{T}}\mathbf{J}\mathbf{a}' &= \mathbf{1}^{\mathsf{T}}\mathbf{J}\mathbf{J}^{-1}\mathbf{v} && \text{definition of } \mathbf{a}' \\
&= \mathbf{1}^{\mathsf{T}}\mathbf{v} \\
&= \mathbf{1}^{\mathsf{T}}\mathbf{J}\mathbf{a} + \mathbf{1}^{\mathsf{T}}\mathbf{m} && \text{definition of } \mathbf{v} \\
&= \mathbf{1}^{\mathsf{T}}\mathbf{J}\mathbf{a} && \text{transfers sum to 0.}
\end{aligned}$$

[26]Recall that all matrices in the domain $\mathcal{J}$ are irreducible. The irreducibility assumption in the $N = 2$ case implies $J_{12} > 0$ and $J_{21} > 0$.

Let $\gamma = \sum_{i:i\neq j} \frac{J_{ij}}{-J_{jj}}$ for any j, which does not depend on j by definition of $\mathcal{H}$. We can write

$$\mathbf{1}^\mathsf{T}\mathbf{J}\mathbf{a} = \sum_{i,j} J_{ij}a_j = \sum_{i,j} \frac{J_{ij}}{-J_{jj}}[a_j(-J_{jj})] = \sum_j a_j(-J_{jj}) \sum_i \frac{J_{ij}}{-J_{jj}} = (\gamma - 1)\sum_j a_j(-J_{jj}).$$

The analogous equations hold when $\mathbf{a}$ is replaced by $\mathbf{a}'$. And, as we have seen, $\mathbf{1}^\mathsf{T}\mathbf{J}\mathbf{a} = \mathbf{1}^\mathsf{T}\mathbf{J}\mathbf{a}'$. So, as long as $\gamma \neq 1$ (something we will show in a moment), we have $\sum_j a_j(-J_{jj}) = \sum_j a'_j(-J_{jj})$, and we conclude that $\mathbf{a}'$ satisfies the budget constraint, because $\mathbf{a}$ does. Finally, by making c' large enough, $(-J_{ii})a_i \geq -c'$ can be ensured for all i.

Now $\gamma \neq 1$ does hold for our $\mathbf{J}$, because $\gamma = 0$ implies $\mathbf{1}^\mathsf{T}\mathbf{J} = \mathbf{0}$, contradicting the assumption that $\mathbf{J}$ has full rank.

We now show that $\text{FLP}(\mathbf{J}; b, c) \supseteq \text{FLP}^{\text{NT}}(\mathbf{J}; b, c')$ for any $c, c' \geq 0$. Consider any $\mathbf{v} \in \text{FLP}^{\text{NT}}(\mathbf{J}; b, c')$, and let $c \geq 0$ be arbitrary. We will show that $\mathbf{v} = \mathbf{J}\mathbf{a}' + \mathbf{m}$ for some $\mathbf{a}', \mathbf{m}$ satisfying the conditions in the definition of $\text{FLP}(\mathbf{J}; b, c)$. Set $a'_i = \frac{b}{n(-J_{ii})}$ and $m'_i = v_i - \sum_j J_{ij}a'_j$ for all i. As $\mathbf{J} \in \mathcal{H}$, for some $\gamma > 0$ we know that $\sum_i \frac{J_{ij}}{J_{jj}} = \gamma - 1$ for all j. Thus:

$$\sum_i v_i = \sum_i \sum_j J_{ij}a_j = \sum_j (-J_{jj})a_j \sum_i \frac{J_{ij}}{(-J_{jj})} = (\gamma - 1)\sum_j (-J_{jj})a_j = (\gamma - 1)b.$$

Similarly,

$$\sum_i \sum_j J_{ij}a'_j = \sum_j (-J_{jj})a'_j \sum_i \frac{J_{ij}}{(-J_{jj})} = (\gamma - 1)\sum_j (-J_{jj})\frac{b}{n(-J_{ii})} = (\gamma - 1)b.$$

We therefore have that $\sum_i m_i = \sum_i v_i - \sum_i \sum_j J_{ij}a'_j = 0$ and the proposed transfers are feasible. By construction $(-J_{ii})a'_i = b/n > 0$ so the backsliding constraint is satisfied for any $c \geq 0$. Also, $\sum_i (-J_{ii})a_i = n(b/n) = b$, so the budget constraint is satisfied. Finally,

$$\sum_i J_{ij}a'_j + m_i = \sum_i J_{ij}a'_j + v_i(\mathbf{a}) - \sum_j J_{ij}a'_j = v_i.$$

Thus $\text{FLP}(\mathbf{J}; b, c) \supseteq \text{FLP}^{\text{NT}}(\mathbf{J}; b, c')$ for any $c, c' \geq 0$. Combining the two inclusions we conclude that there exists an $\underline{c} \geq 0$ such that $\text{FLP}(\mathbf{J}; b, c) = \text{FLP}^{\text{NT}}(\mathbf{J}; b, c')$ for all $c' \geq \underline{c}$ and so $\mathbf{J} \in \mathcal{E}$.

For the other direction, we suppose $\mathcal{J} \in \mathcal{E} \cap \mathcal{F}$ and show $\mathcal{J} \in \mathcal{H}$. The proof is by contradiction. We show that if $\mathcal{J} \notin \mathcal{H}$, then relaxing the backsliding constraint increases the total surplus that can be generated, implying that $\mathcal{J} \notin \mathcal{E}$. More formally, if $\mathcal{J} \notin \mathcal{H}$ then there is a k such that

$$k \in \operatorname*{argmin}_{j \in N} \sum_{i:i\neq j} \frac{J_{ij}}{-J_{jj}}$$

and

$$\sum_{i:i\neq j} \frac{J_{ik}}{-J_{kk}} < h(\mathbf{J}).$$

Take any $k' \in \operatorname{argmax}_j \sum_{i\neq j} \frac{J_{ij}}{-J_{jj}}$. Applying Proposition 1, the payoff vectors $\mathbf{v}$ maximizing $\sum_i v_i$ subject to the budget and backsliding constraints are those such that

$$\sum_i v_i = \sum_i J_{ik'}a'_k = \sum_i J_{ik'}\frac{(b + (n-1)c)}{J_{k'k'}} = (h(\mathbf{J}) - 1)(b + (n-1)c) \tag{9}$$

and $v_i \geq 0$ for all i. Call this set of payoff vectors V. As $\mathcal{J} \in \mathcal{E}$ there is a $\underline{c} \geq 0$ such that for all $c' \geq \underline{c}$ all these payoffs are obtainable without transfers. In particular, we can find a $\mathbf{v} \in \text{FLP}^{\text{NT}}(\mathbf{J}; b, \underline{c})$ with all positive entries satisfying (9). Suppose we now relax the backsliding constraint further by selecting some $c' > \underline{c}$. We can now strictly reduce the action k takes and strictly increase the action k' takes, continuing to exactly satisfy the budget constraint. By construction then we have new payoffs $\mathbf{v}' \in \text{FLP}^{\text{NT}}(\mathbf{J}; b, c')$. However, by the choice of k and k' we have $\sum_i v_i' > (h(\mathbf{J}) - 1)(b + (n-1)c)$ and so $\mathbf{v}' \notin \text{FLP}(\mathbf{J}; b, c)$ contradicting the premise that $\mathbf{J} \in \mathcal{E}$. ∎

Proof of Proposition 3.

Let $\mathbf{B} = \mathbf{B}(\mathbf{J})$ and $\mathbf{B}' = \mathbf{B}(\mathbf{J}')$. Also define $\widetilde{\mathbf{J}} = \mathbf{B} - \mathbf{I}$ and $\widetilde{\mathbf{J}'} = \mathbf{B}' - \mathbf{I}$. It can easily be verified that $\mathbf{J} \succeq^P \mathbf{J}'$ holds if and only if $\widetilde{\mathbf{J}} \succeq^P \widetilde{\mathbf{J}'}$. Let $\mathbf{d}'$ be a right-hand Perron vector of $\mathbf{B}'$, i.e., a nonzero vector such that $\mathbf{B}'\mathbf{d}' = r(\mathbf{B}')\mathbf{d}'$. By definition of $\succeq^P$, the ranking $\widetilde{\mathbf{J}} \succeq^P \widetilde{\mathbf{J}'}$ implies there is a vector $\mathbf{d}$ such that $\mathbf{d} \leq \mathbf{d}'$ and $\mathbf{B}\mathbf{d} \geq \mathbf{B}'\mathbf{d}'$, with the latter inequality being strict in some entry. From this it follows that

$$\mathbf{B}\mathbf{d}' \geq r(\mathbf{B}')\mathbf{d}' \tag{10}$$

with strict inequality in some entry. Now recall that the Collatz–Wielandt formula (Meyer, 2000, equation 8.3.3) says that

$$r(\mathbf{B}) = \sup_{\mathbf{v} \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}} \min_i \frac{[\mathbf{B}\mathbf{v}]_i}{v_i},$$

so (10) implies that $r(\mathbf{B}) > r(\mathbf{B}')$. ∎

Appendix B. Multiple Rankings are Consistent with P1-P3

In this appendix we show that there are two different functions on the domain $\mathcal{J}$, both consistent with P1, P2, and P3, that produce different rankings of Jacobians when $N > 2$.

Let $\widehat{f}(\mathbf{J}) = r(\mathbf{B}(\mathbf{J}))$ and $f(\mathbf{J}) = \text{trace}(\mathbf{B}(\mathbf{J})^2)$. By Theorem 2, $\widehat{f}$ satisfies P1 and P2. That it satisfies P3 is a standard fact.[27] To establish the same for f, we argue as follows. It clearly satisfies P1 because it depends only on $\mathbf{B}(\mathbf{J})$ (recall Remark 1). It satisfies P2 because for any diagonal matrix $\mathbf{S}$ with positive diagonal entries, we have

$$f(\mathbf{J}\mathbf{S}) = \text{trace}((\mathbf{S}^{-1}\mathbf{B}(\mathbf{J})\mathbf{S})^2) = \text{trace}(\mathbf{S}^{-1}\mathbf{B}(\mathbf{J})^2\mathbf{S}) = \text{trace}(\mathbf{B}(\mathbf{J})^2).$$

The last equality follows from the fact that that conjugating by a diagonal $\mathbf{S}$ does not change any of the diagonal entries of a matrix. So P2 holds. Finally, P3 holds because (recalling that any $\mathbf{J} \in \mathcal{J}$ has positive off-diagonal entries), the diagonal entry $[\mathbf{B}(\mathbf{J})^2]_{ii}$ is equal to $\sum_k [\mathbf{B}(\mathbf{J})]_{ik}[\mathbf{B}(\mathbf{J})]_{ki}$. This strictly increases when all off-diagonal entries of $\mathbf{J}$ are increased.

It remains to show that f and $\widehat{f}$ represent different orders on $\mathcal{J}$. Define

$$\mathbf{J} = \begin{pmatrix} -1 & 1/2 & 0 \\ 1/2 & -1 & 1/2 \\ 0 & 1/2 & -1 \end{pmatrix} \qquad \mathbf{J}' = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

We can compute $f(\mathbf{J}) = 1 > 0 = f(\mathbf{J}')$. On the other hand, $\widehat{f}(\mathbf{J}) = 2^{-1/2} < 1 = \widehat{f}(\mathbf{J}')$.

[27] Note that if we have $J'_{ij} > J_{ij}$ whenever $i \neq j$, and $J'_{ii} \geq J_{ii}$ for all i, then the same inequalities hold for the corresponding entries of the corresponding benefits matrices: $B'_{ij} > B_{ij}$ whenever $i \neq j$, and $B'_{ii} \geq B_{ii}$. Then monotonicity follows by the Collatz–Wielandt formula (Meyer, 2000, equation 8.3.3) and irreducibility of $\mathbf{B}$.

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