SUPPLEMENTARY APPENDIX

INEFFICIENCIES IN NETWORKED MARKETS

This document contains supporting material for 'Inefficiencies in Networked Markets.' Many additional results are stated that are not included in the main paper. Proofs are collated at the back. The following sections are included:

(1) Shapley and Shubik (1972)
(2) Network Decomposition Extended Example
(3) Payoff Equivalence with an Alternating Offer Bargaining Game
(4) Network Decomposition with Deleted Links
(5) Existence of a stable network
(6) Stability of the Efficient Network
(7) Absolute Inefficiency Bound
(8) Vertical Differentiation
(9) Comparative Statics

SA-1. Shapley and Shubik (1972)

In this section we discuss the results from Shapley and Shubik (1972) that we use in Section 4 of the paper.

Consider a stable bargaining outcome $(\mu^S, \mathbf{u}^S, \mathbf{v}^S)$. It is easy to see that i and $\mu^S(i)$ must split the gains from trade they generate between them in a pairwise stable outcome. If $u_i^S + v_{\mu^S(i)}^S < \alpha_{i\mu^S(i)}$, buyer i and seller j would have a profitable pairwise deviation, so $u_i^S + v_{\mu^S(i)}^S \geq \alpha_{i\mu^S(i)}$. Summing over all buyers and sellers: $\sum_{i\in\mathbf{P}} u_i^S + \sum_{j\in\mathbf{Q}} v_{\mu^S(i)}^S \geq \sum_{i\in\mathbf{P}} \alpha_{i\mu^S(i)}$. As the total surplus available to distribute is $\sum_{i\in\mathbf{P}} \alpha_{i\mu^S(i)}$, it must be that $u_i^S + v_{\mu^S(i)}^S = \alpha_{i\mu^S(i)}$ for all buyers $i \in \mathbf{P}$.

We now show that if there are no profitable pairwise deviations, there are also no profitable coalitional deviations (the set of stable outcomes coincides with the set of core outcomes). Suppose, towards a contradiction, that there exists a profitable coalitional deviation among the agents $\widehat{\mathbf{P}} \cup \widehat{\mathbf{Q}}$ but no profitable pairwise deviation. Let S be the maximum surplus such a coalition could generate by matching among themselves. Denote this match μ', so that for all $i \in \widehat{\mathbf{P}}$, $\mu'(i) \in \widehat{\mathbf{Q}}$ and $S = \sum_{i\in\widehat{\mathbf{P}}} \alpha_{i\mu'(i)}$. For the deviation to be profitable, initial payoffs must sum to less than this surplus: $\sum_{i\in\widehat{\mathbf{P}}} u_i^S + \sum_{j\in\widehat{\mathbf{Q}}} v_j^S < S$. However, $u_i^S + v_{\mu'(i)}^S \geq \alpha_{i\mu'(i)}$ for all $i \in \widehat{\mathbf{P}}$, as there is no profitable pairwise deviation. Summing over all $i \in \widehat{\mathbf{P}}$, we get that $\sum_{i\in\widehat{\mathbf{P}}} u_i^S + \sum_{j\in\widehat{\mathbf{Q}}} v_j^S \geq S$. This is a contradiction, hence the pairwise stable outcomes coincide with the core outcomes.

An immediate observation is that the grand coalition would have a profitable deviation for any $\boldsymbol{\mu}^S \neq \boldsymbol{\mu}^*$, and so $\boldsymbol{\mu}^*$ must be the unique pairwise stable match. For any match other than $\boldsymbol{\mu}^*$, the match $\boldsymbol{\mu}^*$ could be implemented, and the additional surplus used to give everyone a payoff strictly above their current payoff.

Finally, we show that the set of stable payoffs forms a complete lattice for the partial ordering of buyers' payoffs, where $\mathbf{u} > \mathbf{u}'$ if and only if $u_i > u_i'$ for all $i \in \mathbf{P}$. Consider two payoff vectors, $\mathbf{u}$ and $\mathbf{u}'$. We need to show that the payoff vector $\widehat{\mathbf{u}} = \mathbf{u} \vee \mathbf{u}'$ such that for all i, $\hat{u}_i = \max(u_i, u_i')$ is also stable, and the payoff vector $\widetilde{\mathbf{u}} = \mathbf{u} \wedge \mathbf{u}'$ such that, $\tilde{u}_i = \min(u_i, u_i')$ is stable. We focus here on showing that $\widehat{\mathbf{u}}$ is stable; the argument for $\widetilde{\mathbf{u}}$ is symmetric. Consider any buyer–seller pair i, j who are not matched. As the match is pairwise stable, for both payoffs $\mathbf{u}$ and $\mathbf{u}'$, we have

$$u_i + v_j \geq \alpha_{ij}$$
$$u_i' + v_j' \geq \alpha_{ij}$$

In addition, we know that $u_{\mu(j)^*} + v_j = u'_{\mu(j)^*} + v_j' = \alpha_{\mu(j)^*j}$. Consider now the buyer payoffs $\mathbf{u} \vee \mathbf{u}'$. At these payoffs, we have

$$\max\{u_i, u_i'\} + \min\{v_j, v_j'\} \geq \max u_i, u_i' + \min\{\alpha_{ij} - u_i, \alpha_{ij} - u_i'\} = \alpha_{ij},$$

so i and j still do not have a profitable pairwise deviation.

SA-2. Network Decomposition Extended Example

This section considers a more involved example to help illustrate how the tools developed in Section 4 of the main document can be applied.

From Figure SA-1 it is not immediately clear what s_4's outside option will be, or what role the different links in the network play in establishing s_4's outside option. However, the multi-unit auction algorithm from Démange, Gale, and Sotomayor (1986) can be used to find s_4's outside option (their minimum core payoff). The multi-unit auction

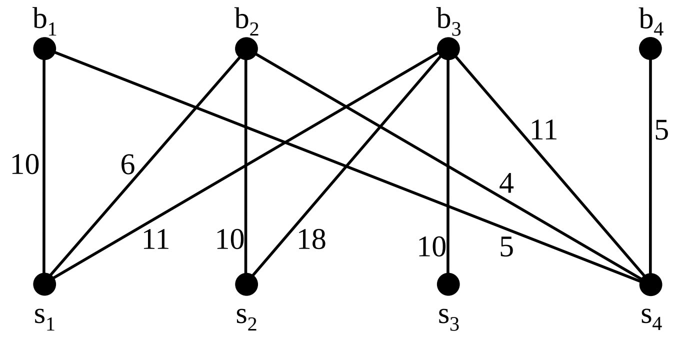


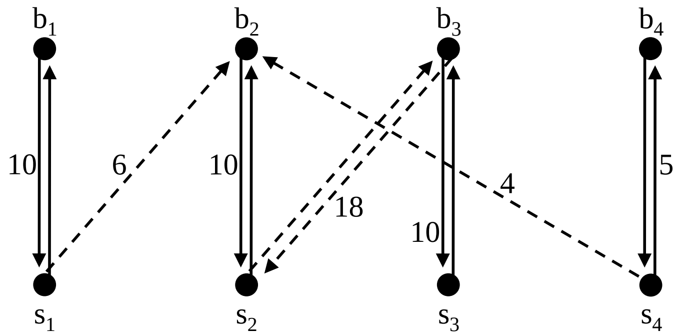
Figure SA-1. A more complicated network

algorithm works approximately as follows: Set the prices charged by all sellers to 0 and consider which buyers want to trade with which sellers. Increase the price charged by each overdemanded seller by one unit. Repeat this process until no seller is overdemanded. In the above example, the prices charged by the sellers and the set of overdemanded sellers would evolve as shown in Table SA-2. The prices are shown for each round and the superscripts identify which buyers demand trade with which sellers at these prices. In the final price vector, no seller is overdemanded as all buyers can be matched to one of their most preferred sellers. The buyer matched to each seller in this final round is highlighted, as are the prices traded at, which correspond to sellers' outside options.

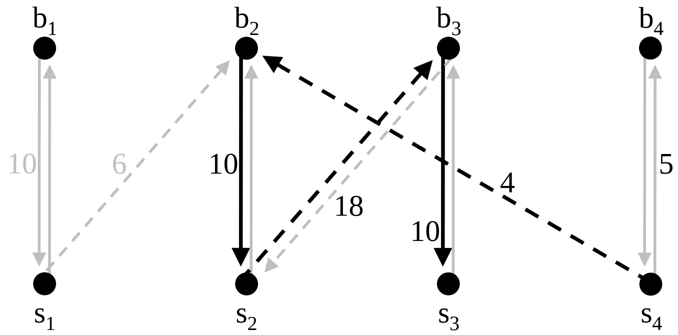
	Round								
	1	2	3	4	5	6	7	8	9
s_1	0^{b_1}	0^{b_1}	0^{b_1}	0^{b_1}	$0^{b_1,b_2}$	$1^{b_1,b_2}$	$2^{b_1,b_2}$	$3^{b_1,b_2}$	$\mathbf{4}^{\mathbf{b_1},b_2}$
s_2	$0^{b_2,b_3}$	$1^{b_2,b_3}$	$2^{b_2,b_3}$	$3^{b_2,b_3}$	$4^{b_2,b_3}$	$5^{b_2,b_3}$	$6^{b_2,b_3}$	$7^{b_2,b_3}$	$\mathbf{8}^{\mathbf{b_2},b_3}$
s_3	$0^{\emptyset}$	$0^{\emptyset}$	$0^{\emptyset}$	$0^{\emptyset}$	$0^{\emptyset}$	$0^{\emptyset}$	$0^{\emptyset}$	$0^{\emptyset}$	$\mathbf{0}^{\mathbf{b_3}}$
s_4	0^{b_4}	0^{b_4}	0^{b_4}	0^{b_4}	0^{b_4}	0^{b_4}	$0^{b_4,b_2}$	$1^{b_4,b_2}$	$\mathbf{2}^{\mathbf{b_4},b_2}$

The assignment auction algorithm identifies sellers' outside options. However, it is not clear which links affect which sellers' outside options and how. The network decomposition algorithm provides this information. First, links used for trade and those used to establish outside options are identified. This is shown in Figure SA-2a. Solid directed links indicate trade relationships, and dashed directed links indicate which agents use which links to establish outside options.

From the directed network decomposition, it is straightforward to find agents' outside option chains. Figure SA-2b highlights s_4's outside option chain. From Theorem 1, outside options can be found by alternatively adding and then subtracting the values of the links in an agent's outside option chain: Seller s_4's outside option is $a_{24} - a_{22} + a_{32} - a_{33} = 4 - 10 + 18 - 10 = 2$. This is the same outside option found by the



(A) Directed network decomposition



(B) s_4's outside option chain

FIGURE SA-2. Applying the directed network decomposition algorithm

assignment auction algorithm. The value of Theorem 1 is in showing how the links identified by the network decomposition determine each agent's payoff. Furthermore, the assignment auction algorithm provides an alternative methodology through which the network decomposition is possible. Each seller's outside trade partner is the buyer who is willing to trade with him, but does not, when the algorithm ends.

SA-3. Payoff Equivalence with an Alternating-Offer Bargaining Game

This section demonstrates an equivalence between the payoffs and matches generated by the non-cooperative alternating-offers approach of Corominas-Bosch (2004) and the payoffs and matches identified in Section 4 of the main document. Corominas-Bosch considers (bipartite) networks of buyers and sellers where, as here, a link between a buyer and seller permits trade but, unlike here, the value of trade is constant across buyer–seller pairs. To show this equivalence, this section will consider only this more restricted environment:[1]

[1] It is not straightforward to extend the Corominas-Bosch environment to include heterogeneous gains from trade.

Assumption: The value of trade between every buyer–seller pair is either 1 or 0.

The alternating-offer bargaining game modeled by Corominas-Bosch proceeds as follows: First, each agent on one side of the market simultaneously makes a uniform offer to all agents on the other side of the market he is connected to. Agents receiving offers then decide which, if any, offer to accept. If an agent accepts an offer, trade occurs at the proposed terms of trade and that buyer–seller pair exits the market. Each remaining agent who was initially on the receiving side of the market then simultaneously makes a uniform offer to all agents on the other side of the market he is connected to, and so on.

Corominas-Bosch defines three types of networks, as follows:[2]

- $\mathbf{L}^S$ if n (the number of sellers) $> m$ (the number of buyers) and any set of sellers of size smaller than or equal to m can be simultaneously matched to different buyers.
- $\mathbf{L}^B$ if $m > n$ and any set of buyers of size smaller than or equal to n can be simultaneously matched to different sellers.
- $\mathbf{L}^E$ if $n = m$ and all sellers (equivalently all buyers) in $\mathbf{L}$ can be simultaneously matched.

Then the following is shown:

Lemma. *(Corominas-Bosch, Proposition 4) There exists a perfect equilibrium payoff in the Corominas-Bosch alternating-offer bargaining game in which the following hold:*

(*i*) *Sellers in an* $\mathbf{L}^B$ *network get 1, and buyers get 0.*
(*ii*) *Sellers in an* $\mathbf{L}^S$ *network get 0, and buyers get 1.*
(*iii*) *Sellers in an* $\mathbf{L}^E$ *network get* $(1 - \beta_{\text{C-B}})$*, and buyers get* $\beta_{\text{C-B}}$*.*

$\beta_{\text{C-B}}$ can be thought of as the relative bargaining power of buyers and sellers, and is function of the discount factor δ reflecting first mover advantage as in the standard Rubinstein–Stahl alternating-offers bargaining game (i.e., $\beta_{\text{C-B}} = 1/(1 + \delta)$ if buyers make the first offer).

Furthermore, Corominas-Bosch shows that any network can be decomposed into subnetworks of the types $\mathbf{L}^S$, $\mathbf{L}^B$, and $\mathbf{L}^E$, with the bargaining solution in these subnetworks the

[2] The marriage theorem is used to simplify these conditions. Consider, for example, a $\mathbf{L}^S$ network. To check that a network is a $\mathbf{L}^S$-type network directly from the definition, it would be necessary to check that within every subset of sellers with up to m members, each individual seller can be matched with a different buyer. The marriage theorem shows that, instead, it can simply be checked that every subset with up to m members can be jointly matched to at least as many buyers.

same as the bargaining solution for the entire network. This is achieved by the deletion of links that do not affect the bargaining outcomes.

Equivalence between the Corominas-Bosch outcomes and the outcomes from Theorem 1 can now be stated:

Proposition SA-1. *For any network* $\mathbf{L}$*, where the gains from trade between any connected buyer and seller over the network is 1, there is a perfect equilibrium payoff of the Corominas-Bosch alternating-offer bargaining game that implements the same matches and coincides with the payoffs from Theorem 1 for* $\beta_{C\text{-}B} = \beta$.

Proposition SA-1 is proved in Appendix A. There are two steps to the proof. First, it is shown that in an $\mathbf{L}^S$, $\mathbf{L}^B$, or $\mathbf{L}^E$ network the payoffs of players from the Corominas-Bosch alternating-offer bargaining game coincide with payoffs identified in Section 4 of the main document for $\beta_{\text{C-B}} = \beta$. The second step of the proof shows that the same decomposition of networks as imposed by Corominas-Bosch is possible without affecting payoffs. Indeed the Corominas-Bosch decomposition deletes only the links that are also deleted by the network decomposition algorithm.

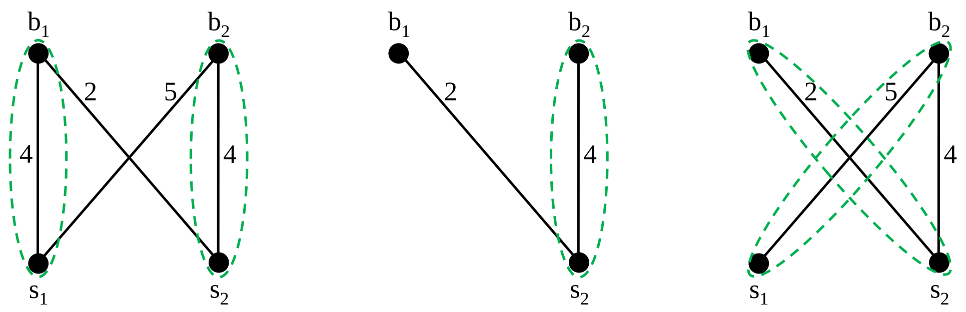
Proposition SA-1 provides a non-cooperative justification for the bargaining solution proposed in Section 4 of the main document. It also provides some justification for Corominas-Bosch focusing on the perfect equilibrium payoffs she focuses on.

Unfortunately, the Corominas-Bosch approach cannot easily be extended to environments with heterogeneous gains from trade. In some cases a heterogeneous gains from trade network can be split into a number of homogeneous gains from trade networks, where the payoffs in these networks are generated and then summed to generate the payoffs in the heterogeneous network. However, this additivity does not often hold, and analysis of the whole network at once must be undertaken. This prevents the repeated application of the tools developed by Corominas-Bosch and provides a role for the alternative methodology employed in this paper. This emphasizes that the heterogeneous gains from trade environment is richer than, and fundamentally different from, the homogeneous gains from trade environment.[3]

[3] With homogeneous gains from trade, outside option links can never benefit both agents they connect. This will be particularly important when considering network formation, because if both agents have to contribute towards the costs of forming a link, then no outside option links will ever be formed in the homogeneous gains from trade case.

SA-4. Network Decomposition with Deleted Links

This section shows that outside trade partners cannot always be identified in the network decomposition algorithm by removing the link an agent trades over and considering whom he rematches to, instead of removing his trade partner.



(A) Network $\mathbf{L}$ (B) Network $\mathbf{L} \setminus \{s_1\}$ (C) Network $\mathbf{L} \setminus \{l_{1,1}\}$

Figure SA-3. Matches are identified in the above networks by the dashed ovals.

Consider b_1's outside option. First, apply the standard network decomposition algorithm to network $\mathbf{L}$ in Figure SA-3a. When s_1 is removed from the network, b_1 does not rematch and has no outside trade partner. This suggests that b_1's outside option is 0. Suppose now that the network decomposition algorithm were run by removing the link b_1 trades over, $l_{1,1}$, instead of s_1. In this reduced network, which is shown in Figure SA-3c, seller s_2 would appear to be b_1's outside trade partner. Consider again the network $\mathbf{L}$, and suppose that b_1 received a payoff of 0, s_1 a payoff of 4, b_2 a payoff of 1, and s_2 a payoff of 3. Despite b_1 receiving a payoff of 0, this outcome is pairwise stable and b_1 has no profitable pairwise deviation. Also, b_1 has no outside option. Thus the rematching of b_1 when the link to his trade partner was removed was misleading.

SA-5. Existence of a stable network

Suppose first that link-formation costs (investments) are split in exogenously determined proportions.

Proposition SA-2. *For all potential gains from trade* $\mathbf{a}$, *all levels of bargaining power* β *and all cost shares* $\gamma \in (0, 1)$, *there exists a stable network.*

The proof of Proposition SA-2 is in Section A.2. It is constructive and generates the following implication:

Corollary SA-1. *For all potential gains from trade* ($\mathbf{a}$)*, all levels of bargaining power* (β) *and all cost shares* ($\gamma \in (0,1)$)*, there exists a stable network with no coordination inefficiency.*

Corollary SA-1 provides some additional motivation for focusing on underinvestment and overinvestment inefficiency.

Suppose now that investment shares are negotiated endogenously.

Proposition SA-3. *When cost shares are negotiated there exist potential gains from trade* ($\mathbf{a}$) *and levels of bargaining power* (β) *such that there is no network that is pairwise Nash stable with transfers.*

SA-6. Stability of the Efficient Network

When cost shares are exogenous, underinvestment in trade links is never a problem if $\beta = \gamma$. In this case, any unmatched buyer and seller who could increase the net gains from trade by forming a link will both find it profitable to do so. Furthermore, when $\beta = \gamma \in \{0,1\}$, so that the same side of the market has all the bargaining power and pays all the costs of forming links, the efficient network will be stable. This is because links are unilaterally formed by the side of the market that already extracts the maximum possible rents from their trade partner, hence the agent on that side of the market cannot benefit further from an outside option link. There is thus no underinvestment or overinvestment in any stable network when $\beta = \gamma \in \{0,1\}$ and the efficient network is stable. However, if ($\beta = 1$ and $\gamma < 1$) or ($\beta = 0$ and $\gamma > 0$), the unique stable network will be the empty network and all potential gains from trade will be lost to underinvestment inefficiency.

When investment cost shares are negotiated, the efficient network will be stable when no two buyers' most preferred seller is the same and no two sellers' most preferred buyer is the same. There is then no value to any outside option. No agent can threaten to trade with someone other than his trade partner, because no one other than his trade partner will be willing to trade with him. As negotiation prevents any underinvestment the efficient network is then stable. Furthermore, as $c \to 0$ the efficient network will be stable if and only if preferences meet this anti-assortativity condition. However, this condition is unlikely to be met in many applications. Indeed, it is never met when there are different numbers of buyers and sellers with positive potential gains from trade. These results are summarized in Proposition SA-4.

Proposition SA-4. *When cost shares are exogenous, the following hold:*

(*i*) *If* $\gamma \in [0,1] = \beta$, *there will be no underinvestment in any stable network.*

(*ii*) *If* $\gamma = \beta = 1$ *or* $\gamma = \beta = 0$, *the efficient network will be stable and there will be no underinvestment or overinvestment in any stable network.*

(*iii*) *If* ($\beta = 1$ *and* $\gamma < 1$) *or* ($\beta = 0$ *and* $\gamma > 0$), *all potential gains from trade will be lost to underinvestment and the unique stable network will be the empty network.*

When cost shares are endogenous, the following hold:

(*i*) *The efficient network will be stable if the following anti-assortativity conditions hold:*

(*a*) $\text{argmax}_j(a_{ij}) \neq \text{argmax}_j(a_{i',j})$ *for all* $i \neq i'$; and

(*b*) $\text{argmax}_i(a_{ij}) \neq \text{argmax}_i(a_{ij'})$ *for all* $j \neq j'$.

(*ii*) *If* $\beta \in (0,1)$, *then as* $c \to 0$ *the efficient network will be stable if and only if the above anti-assortativity conditions hold.*

(*iii*) *If the same network* $\mathbf{L}^e$ *is efficient for all* $c \in [\underline{c}, \overline{c}]$, *then there exists a threshold* $c^* \in \Re$ *such that for all* $c \in [\underline{c}, \overline{c}]$ *the efficient network is stable if and only if* $c > c^*$.

(*iv*) *There exist potential gains from trade* $\mathbf{a}$ *and* $c' < c''$ *such that* $\mathbf{L}^e(c') \neq \mathbf{L}^e(c'')$, $\mathbf{L}^e(c')$ *is stable, and* $\mathbf{L}^e(c'')$ *is not stable, where* $\mathbf{L}^e(c)$ *is the efficient network for cost of link formation* c.

SA-7. Absolute Inefficiency Bound

So far, efficiency losses have been considered relative to the potential net gains from trade. It is also possible to bound the absolute size of efficiency losses when cost sharing is endogenous:

Proposition SA-5. *In any stable network* $\mathbf{L}$ *with* K *links formed and endogenous cost sharing, the absolute magnitude of inefficiency is bounded such that* $NGT(\mathbf{L}^e) - NGT(\mathbf{L}) \leq Kc$.

Proposition SA-5 has a couple of interesting implications. First, as $c \to 0$ all stable networks generate net gains from trade approaching that generated by the efficient network and there cannot be any coordination problems. Second, for $c > 0$, stable networks with fewer links have a tighter inefficiency bound. It is the possibility of overinvestment rather than coordination inefficiency that drives the inefficiency bound.

SA-8. Vertical Differentiation

The results in this section are closely related to those in Felli and Roberts (2002). The main difference is that here pairwise stability underlies the bargained outcomes, while Felli and Roberts model Betrand competition.

It will be assumed that there are increasing differences in the gains from trade: Buyers $i \in \{1, \ldots, m+1\}$ and sellers $j \in \{1, \ldots, n+1\}$ can be indexed so that if $i' < i''$ and $j' < j''$, then $a_{i'j'} - a_{i'j''} > a_{i''j'} - a_{i''j''}$, where $b_{m+1} = s_{n+1} = \emptyset$ and $a_{ij} = 0$ if either $i = m+1$ or $j = n+1$.

There are a number of immediate implications of the increasing differences condition. First, sellers' products are vertically differentiated, hence the quality of their products is unambiguously ranked by all buyers in the same way, with seller s_1 offering the highest quality product: for all $j'' > j'$ and all i', $a_{i'j'} > a_{i'j''}$. This follows from setting $i'' = m+1$. Second, buyers can be ranked in their preferences for quality, with buyer b_1 having the strongest preference for high quality. If a buyer prefers a higher-quality to a lower-quality good *by more* than another buyer, then he prefers any higher-quality good to any lower-quality good *by more* than the other buyer. Equivalently, because of the symmetry of buyers and sellers, the quality of buyers can be unambiguously ranked by sellers, and sellers' preferences for quality can be ranked.

Proposition SA-6. *When the increasing differences condition holds and the complete network has been formed then for all* $k \leq \min(m, n)$*:*

(*i*) *Buyer* b_k *will be matched to seller* s_k.
(*ii*) *Buyer* b_k*'s trade partner will be* s_{k+1}.
(*iii*) *Seller* s_k*'s outside trade partner will be buyer* b_{k+1}.
(*iv*) $\underline{u}_{b_k} > \underline{u}_{b_{k+1}}$.
(*v*) $\underline{v}_{s_k} > \underline{v}_{s_{k+1}}$.

The results of Proposition SA-6 can be summarized in the directed network representation of the network, shown in Figure SA-4, where solid links identify trade partners and dotted links identify outside trade partners.

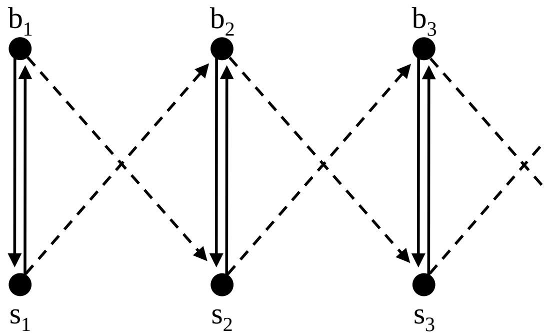


FIGURE SA-4. Outside option chains for vertically differentiated agents

With increasing differences in **a**, the efficient network links a buyer to a seller if and only if that seller has the same quality ranking as the buyer and $a_{b_k s_k} \geq c$. Figure SA-4 shows that no buyer and seller use the same link to establish an outside option when the complete network has been formed. Furthermore, it will be shown that no buyer and seller can ever benefit from the same outside option link. This prevents the formation of any outside option links when investment cost shares are separate and both agents must make some positive investment.

Proposition SA-7. *Let K' be defined by $a_{b_{K'} s_{K'}} \geq c > a_{b_{K'+1} s_{K'+1}}$. If there are increasing differences in* **a***, then for all $\beta \in (0,1)$ the following hold:*

(*i*) *When cost shares are exogenous and $\gamma \in (0,1)$, there will be no overinvestment inefficiency in any stable network and the efficient network will be stable if and only if* $\min\{\beta/\gamma, (1-\beta)/(1-\gamma)\} a_{b_{K'} s_{K'}} \geq c$.

(*ii*) *When cost shares are endogenous the efficient network will be stable if and only if* $\max\{(1-\beta)(a_{12}-a_{22}), \beta(a_{21}-a_{22})\} \leq c$.

Part (*i*) of Proposition SA-7 shows that when cost shares are exogenous and investment by both the buyer and seller is required for a link to be formed, the efficient network will be stable if and only if the lowest quality buyer–seller pair who form a link in the efficient network are both willing to form the link. Part (*ii*) shows that when cost shares are endogenous, the efficient network will be stable if and only if the highest quality buyer does not want to form an outside option link to the second highest quality seller, and the highest quality seller does not want to form an outside option link to the second highest quality buyer. Proposition SA-7 reaffirms that overinvestment inefficiency is most problematic when investments are negotiated, while underinvestment inefficiency is more problematic when costs are exogenous.

SA-9. Comparative Statics

This section first presents comparative statics results for bargaining over a formed network and then develops comparative statics results for network formation.

Proposition SA-8. *Adding a seller (and links connecting this seller to a formed network) weakly increases the payoffs of all buyers and weakly decreases the payoffs of all other sellers. Adding a buyer (and links connecting this buyer to a formed network) weakly increases the payoffs of all sellers and weakly decreases the payoffs of all other buyers.*

While bargaining over a formed network generates positive results, it is much harder to make any general comparative static claims about network formation. With endogenous network formation, comparative static results are negative—the affect of parameter changes are non-monotonic. Even when attention is restricted to cases where there is a unique stable network, so that there is no question as to which stable network will be formed, the effects of parameter changes are not predictable.

Increasing the cost of link formation (c) can either increases or reduce the resources wasted on forming outside option links: An increase in c can result in fewer outside options being formed or just increase the resources spent on forming the same links. Also, an increase in c can either reduce or increase an agent's payoff. When costs are shared endogenously, or split such that both buyers and sellers have to make some contribution $\gamma \in (0,1)$, this is not especially surprising: An increase in c can result in a trade partner no longer forming an outside option link, thereby increasing an agent's payoff. However, even when only buyers contribute towards c ($\gamma = 1$), they can still benefit from an increase in c. This is shown in the example below:

Suppose that the potential gains from trade $\mathbf{a}$ are as shown in Figure SA-5, and that buyers pay for links to be formed ($\gamma = 1$) and $\beta = \frac{3}{4}$.

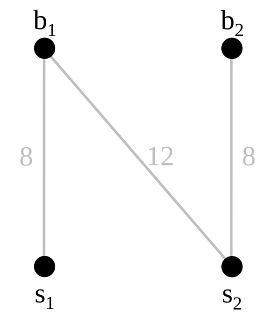
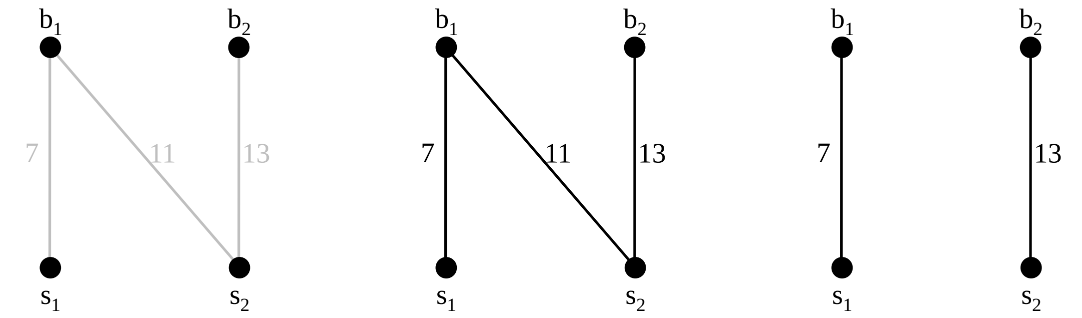


FIGURE SA-5. A reduction in the cost of link formation can reduce u_{b_2}

The complete network is stable only for $c < 1$. In the complete network, b_1 forms a link to s_2 to establish an outside option and improve their terms of trade with s_1. However, this link also provides an outside option to s_2. In the complete network, b_2 receives a payoff of $3 - c$. If the cost of link formation is now increased to a value above 1, so that $c \in (1, 3)$, then it is no longer worthwhile for b_1 to form an outside option link to s_2 and the unique stable network is the efficient network. In this new network, b_2 receives a payoff of $6 - c$, which is greater than the payoff they received before the increase in c.

Increasing an agent's bargaining power β can also reduce their payoff. This is shown in the example below. Suppose that only sellers can form links ($\gamma = 0$), and that $c = 2$ and the gains from trade matrix is as shown in Figure SA-6a.



(A) Underlying network (B) Stable $\beta \in (\frac{1}{2}, \frac{2}{3})$ (C) Stable $\beta < \frac{1}{2}$

FIGURE SA-6. Stable networks for different β

When $\beta \in (1/2, 2/3)$, the complete network shown in Figure SA-6b is stable, because s_1 and s_2 are willing to pay for the traded-over links and because seller s_2's payoff is increased by $4\beta - c > 0$ when they form the outside option link $l_{b_1 s_2}$. Buyer b_2's payoff is therefore 9β. Suppose now that β is decreased to a value below 1/2. With this lower β, the complete network is no longer stable—seller s_2 would delete their outside option link to buyer b_1, and the new stable network is shown in Figure SA-6c. Buyer b_2's payoff in the new stable network is 13β. As a function of β, b_2's payoff is therefore discontinuous at $\beta = \frac{1}{2}$. Their payoff jumps from $u_{b_2} = 6.5$ when link $l_{b_1 s_2}$ is not formed to $u_{b_2} = 4.5$ as link $l_{b_1 s_2}$ is formed. An increase in β about this point can therefore reduce buyer b_2's payoff.

The above example shows that reduced bargaining power can increase an agent's payoff by reducing the incentives of their trade partner to form an outside option link. However, even if a buyer's trade partner cannot form links ($\gamma = 1$), this buyer can still benefit from a reduction in their bargaining power, which might induce another buyer in their outside option chain to form an outside option link, thereby increasing their payoff.

The difficulty in generating positive comparative static results for investments emphasizes the importance of case-by-case analysis for any intervention in the market. For example, a government policy to reduce the costs of link formation could result in more resources being wasted on link formation without affecting which buyers trade with which sellers.

Appendix A. Proofs

A.1. **Proof of Proposition SA-1.**

Proof. It is useful to first prove the following lemma:

Lemma SA-1. *Section 4 of the main document shows how agents' payoffs can be identified from the network decomposition algorithm. These payoffs are such that for a formed network the following hold:*

- (i) *Sellers in an* $\mathbf{L}^B$ *network get 1, and buyers get 0.*
- (ii) *Sellers in an* $\mathbf{L}^S$ *network get 0, and buyers get 1.*
- (iii) *Sellers in an* $\mathbf{L}^E$ *network get* $(1-\beta)$*, and buyers get* β*, where* β *is buyers' bargaining power as defined in Section 4 of the main document.*

Proof. **Part** (i): The payoffs identified in Section 4 of the main document are a convex combination of the buyer-optimal and seller-optimal core payoffs. It is therefore sufficient to show that both the buyer-optimal and seller-optimal points coincide with buyers always getting 0 and sellers always getting 1. To find the seller-optimal point the multi-unit auction algorithm considered by Démange, Gale, and Sotomayor (1986) can be run with buyers and sellers switched. This multi-unit auction algorithm is initialized with buyers' shares of surplus equal to 0, and then these shares are increased where there is excess demand for trade with a buyer (as in Section SA-2). In an $\mathbf{L}^B$ network, there are more buyers than sellers, and any subset of the m buyers of size n is able to simultaneously match to all n sellers. As the surplus from trade is always 1, sellers are indifferent about which buyer they trade with, and all n sellers can be matched with different buyers, so that there will be no excess demand for trade with any one buyer. Thus the algorithm terminates immediately, and sellers all receive a payoff of 1, while buyers all receive a payoff of 0.

Consider now the buyer-optimal point in an $\mathbf{L}^B$ network. This can be found by running the same multi-unit auction algorithm, but with buyers bidding up sellers' prices. Let the subset of sellers $\underline{\mathbf{\Psi}}_S \subseteq \mathbf{S}$ be defined as

$$\underline{\Psi}_S \equiv \{j \in \mathbf{S} : j \in \operatorname*{argmin}_{j \in \mathbf{S}} v_j\}$$

That is, $\underline{\mathbf{\Psi}}_S$ is the set of sellers charging the lowest price.

By definition of $\mathbf{L}^B$, any subset of buyers less than or equal to the number of sellers can be matched to different sellers over the network. It will be shown that a subset of sellers $(\mathbf{\Psi}_S)$ of size $|\mathbf{\Psi}_S|$ must be jointly connected to at least $m - n + |\mathbf{\Psi}_S|$ buyers. Suppose that, for some $k \geq 0$, $m - n + |\mathbf{\Psi}_S| - k$ buyers were connected to a subset of sellers $\mathbf{\Psi}_S$. The complement of the subset of buyers connected to this subset of sellers must then be of size $m-(m-n+|\mathbf{\Psi}_S|-k) = n-|\mathbf{\Psi}_S|+k$. This is the number of buyers not connected to any seller $s \in \mathbf{\Psi}_S$. By definition of $\mathbf{L}^B$, these buyers must be able to connect to at least $n - |\mathbf{\Psi}_S| + k$ different sellers. However, they can connect only to sellers not in $\mathbf{\Psi}_S$, and so there are only $n - |\mathbf{\Psi}_S|$ different sellers for them to connect to. Thus $n - |\mathbf{\Psi}_S| + k \leq n - |\mathbf{\Psi}_S|$, which implies that $k = 0$.

As there are more buyers than sellers ($m > n$) in an $\mathbf{L}^B$ network, and the subset of sellers $\underline{\mathbf{\Psi}}_S$ of size $|\underline{\mathbf{\Psi}}_S|$ must be connected to at least $m - n + |\underline{\mathbf{\Psi}}_S|$ buyers, the subset of sellers $\underline{\mathbf{\Psi}}_S$ will be connected to strictly more than $|\underline{\mathbf{\Psi}}_S|$ buyers. Thus there will be excess demand by buyers for trade with at least one $s \in \underline{\mathbf{\Psi}}_S$ if the price charged by sellers in $\underline{\mathbf{\Psi}}_S$ is less than 1. When the multi-unit auction algorithm terminates, the lowest price charged by any seller must therefore be greater than or equal to 1. By individual rationality, prices cannot increase to a value above 1, and so sellers all receive a payoff of 1, while buyers all receive a payoff of 0.

Part (ii): This proof is the same as Part (i) with the roles of buyers and sellers reversed.

Part (iii): In an $\mathbf{L}^E$ network, a matching of all sellers to different buyers and all buyers to different sellers is always possible. Thus both the buyer-optimal and seller-optimal versions of the multi-unit auction algorithm terminate immediately. Under the buyer-optimal auction algorithm, the buyers receive a payoff of 1, while sellers receive a payoff of 0; and under the seller-optimal auction algorithm, sellers receive a payoff of 1, while buyers receive a payoff of 0. The payoffs identified in Section 4 of the main document are then the convex combinations of these points, generating payoffs of $1 - \beta$ for sellers and β for buyers.

□

Now that Lemma SA-1 has been proved, it remains to be shown that the Corominas-Bosch decomposition does not affect the payoffs identified in Section 4 of the main document.

Corominas-Bosch decomposes her network through an algorithm that deletes links to generate subnetworks. The algorithm deletes links connecting buyers and sellers such that for each link deleted, either the buyer is left in an $\mathbf{L}^S$ network (hence the buyer extracts all the surplus anyway) or the seller is left in an $\mathbf{L}^B$ network (hence the seller extracts all the surplus anyway).

Consider deleting a link l_{ij} that leaves buyer i in an $\mathbf{L}^S$ network. In the $\mathbf{L}^S$ network, i extracts all the surplus, and so the deleted link did not affect the price received by this buyer. Further, j would not have wanted to trade over this link either, as i would have extracted the entire surplus and trade over l_{ij} cannot have left j with a positive payoff. Thus deleting the link does not affect the trade partner or payoff of the buyer or seller. An equivalent argument can be made about deletion of links to a seller that would leave the seller in an $\mathbf{L}^B$ network. □

A.2. **Proof of Proposition SA-2.**

Proof. It is useful to first introduce the concept of an improving path. An improving path is a sequence of networks where each element of the sequence represents a profitable deviation made from the previous network by an agent:

Definition: An improving path from a network $\mathbf{L}_1$ to a network $\mathbf{L}_K$ is a finite sequence of networks $(\mathbf{L}_1, \ldots, \mathbf{L}_K)$ such that for all $k \in \{1, \ldots, K-1\}$ at least one of the following conditions holds, and for $\mathbf{L}_K$ none of them hold:

(i) There exists $l_{i'j'}$ such that $\mathbf{L}_{k+1} = \mathbf{L}_k \cup \{l_{i'j'}\}$ and $u_{i'}(\mathbf{L}_{k+1}) - \gamma c \geq u_{i'}(\mathbf{L}_k)$ and $v_{j'}(\mathbf{L}_{k+1}) - (1-\gamma)c \geq v_{j'}(\mathbf{L}_k)$.

(ii) There exists an agent i' and a set of links $\hat{\mathbf{L}}^{i'}$ connected to i' such that $\mathbf{L}_{k+1} = \mathbf{L}_k \setminus \hat{\mathbf{L}}^{i'}$ and $u_{i'}(\mathbf{L}_{k+1}) + |\hat{\mathbf{L}}^{i'}|\gamma c \geq u_{i'}(\mathbf{L}_k)$, where $|\hat{\mathbf{L}}^{i'}|$ is the cardinality of $\hat{\mathbf{L}}^{i'}$.

(*iii*) There exists an agent j' and a set of links $\hat{\mathbf{L}}^{j'}$ connected to agent j' such that $\mathbf{L}_{k+1} = \mathbf{L}_k \setminus \hat{\mathbf{L}}^{j'}$ and $v_{j'}(\mathbf{L}_{k+1}) + |\hat{\mathbf{L}}^{j'}|(1-\gamma)c \geq v_{j'}(\mathbf{L}_k)$, where $|\hat{\mathbf{L}}^{j'}|$ is the cardinality of $\hat{\mathbf{L}}^{i'}$.

The existence of a pairwise Nash stable network can be shown by finding an improving path (which will terminate by definition) from some initialization of the network. This is because when an improving path terminates, there are no profitable deviations to prevent this network from being pairwise Nash stable.

As $\gamma \in (0,1)$, each formed link must benefit both the buyer and the seller. Initialize the network as follows:

(1) Find the efficient network.
(2) Remove any links that are not pairwise stable.

This network exists, because there is always an efficient network (and this network is generically unique). From this initialization, it will be shown that an improving path exists. Thus a stable network will be found.

First, it will be shown that none of the links present in this initialized network can ever be deleted and that no new links are formed for the purpose of trade. Second, it will be shown that there exists an improving path where no trade links are ever deleted.

All links in the initialized network are used for trade. Consider round k of an improvement path. Suppose that the same links are traded over as in the initialization. There cannot exist a next step in an improvement path where a link is formed with an unconnected agent. Such a link would not benefit the unconnected agent (and would therefore not be formed) unless it were used for trade. If it could be used for trade, it would have been formed in the initialized network. Consider now two agents who already trade, but not with each other. They cannot form a link between them that will be used for trade, because such a link would have been formed in the initialized network. The only possible links that can be formed are outside option links that connect two agents that currently trade.

For a trade link to be deleted, it must be the case that one of the agents it connects does not benefit from it. This can occur only if the outside option of that agent's trade partner is sufficiently high. Suppose that this did occur, and that the agent deleting the link is i'. It must then be the case that $\gamma c > a_{i'\mu^*(i')} - \underline{v}_{\mu^*(i')}$. Thus, $\gamma c > a_{i'\mu^*(i')} - \sum_{l \in \mathbf{L}_{s\to b}^{\mu^*(i')}} a_l + \sum_{l \in \mathbf{L}_{b\to s}^{\mu^*(i')}} a_l$. However, the current trade links are all in the efficient network. Thus $a_{i'\mu^*(i')} + \sum_{l \in \mathbf{L}_{b\to s}^{\mu^*(i')}} a_l - \sum_{l \in \mathbf{L}_{s\to b}^{\mu^*(i')}} a_l > c$; otherwise, the network with trade over the outside option links would have been more efficient (had higher net gains from trade). This is a contradiction, and so none of the trade links formed in the initialization can ever be deleted.

As no new trade links are formed in an improving path, no outside option links can become trade links without deleting an existing trade link. Thus as existing trade links are not deleted, the set of links traded over remains constant. It therefore follows that along any improving path, only outside option links are formed or deleted.

Each outside option link that is formed must increase the outside options of the two agents it links, and it must also weakly increase the outside options of all other agents in the network. Furthermore the positive uni-directional complementarity between outside option links means that a formed outside option link will be deleted only when a more profitable outside option link becomes available and replaces it. This can happen only if other outside option links formed in the interim period make the new outside option link better than the old one. The links formed in this interim period could also have been formed prior to the first outside option link. This is because outside option chains never cycle, and so the first outside option link cannot be incentivizing the formation of these links. Thus there must always exist an improvement path where no outside option links are deleted and in each step of the improving path an outside option link is added.[4]

□

A.3. **Proof of Corollary SA-1.**

Proof. By construction of the improving path in the proof of Proposition SA-2, there exists a stable network where all traded-over links are also traded over in the efficient network. To move from this stable network to the efficient network, the following two steps can be taken: First, links that are used for the purpose of trade in the efficient network but that are not in the stable network are added. This removes all underinvestment inefficiency. Second, links not traded over can be removed eliminating all overinvestment inefficiency and yielding the efficient network. Thus there is no coordination inefficiency. □

A.4. **Proof of Proposition SA-3.**

Proof. The proof is by counterexample. Consider the potential gains from trade shown in Figure SA-7 for $\beta = \frac{7}{8}$ and $c = 6$.

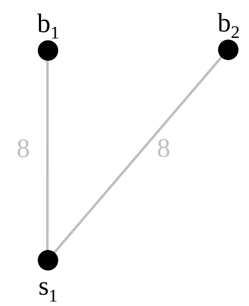


FIGURE SA-7. Potential gains from trade

There are four networks that could be formed: $\emptyset, \{l_{1,1}\}, \{l_{2,1}\},$ and $\{l_{1,1}, l_{2,1}\}$. It will be shown that there is a profitable deviation leading away from each of these possible networks, hence none of them is pairwise Nash stable with transfers.

The payoffs of each agent in these possible networks are given in Table 1. As cost sharing is determined endogenously, there is often a range of possible cost shares. To capture this, the

[4] The number of outside option links in the network can be thought of as the output of a function that is increasing along every step of the improvement path. Jackson and Watts (2001) shows that the existence of such a function can be used to establish existence of a stable network.

cost shares are stated explicitly, although these are not predetermined. Buyer b_1's cost share of link $l_{1,1}$ is given by γ_1, and b_2's cost share of link $l_{2,1}$ is given by γ_2.

Possible Networks

	$\emptyset$	$\{l_{1,1}\}$	$\{l_{2,1}\}$	$\{l_{1,1}, l_{2,1}\}$
u_1	0	$7-\gamma_1 6$	0	$-\gamma_1 6$
u_2	0	0	$7-\gamma_2 6$	$-\gamma_2 6$
v_1	0	$1-(1-\gamma_1)6$	$1-(1-\gamma_2)6$	$8-(2-\gamma_1-\gamma_2)6$
u_1+v_1	0	2	$1-(1-\gamma_2)6$	$2-(1-\gamma_2)6$
u_2+v_1	0	$1-(1-\gamma_1)6$	2	$2-(1-\gamma_1)6$
$u_1+u_2+v_1$	0	2	2	-4

TABLE 1. Payoffs from possible networks

Consider first the empty network. From Table 1 forming link $l_{1,1}$ is jointly profitable for b_1 and s_1, and so moving to this network is a profitable deviation from the empty network. Consider the network $\{l_{1,1}\}$. Regardless of how the cost of forming link $l_{1,1}$ is split (i.e., regardless of γ_1), forming link $l_{2,1}$ is jointly profitable for b_2 and s_1 (there exists a γ_2 such that both b_2 and s_1's profits weakly increase and one of their profits strictly increases).[5] Thus there is a profitable deviation away from $\{l_{1,1}\}$. By an equivalent argument, the network $\{l_{2,1}\}$ is not stable. Finally, the network $\{l_{1,1}, l_{2,1}\}$ results in a negative joint payoff, and so at least one agent must receive a negative payoff regardless of γ_1 and γ_2. This agent will find it profitable to delete all their links, and so there is also a profitable deviation away from this network.

As there exists a profitable deviation leading away from each of the possible networks, regardless of the cost shares reached in these networks, there is no network that is pairwise Nash stable with transfers. □

A.5. **Proof of Proposition SA-4.**

Proof. **Exogenous costs**

It is convenient to first prove the following lemma:

Lemma SA-2.

(*i*) *When cost sharing is exogenous and* $\gamma \in (0,1)$ *the efficient network will be pairwise Nash stable if and only if for all links* l_{ij} *such that* $j \neq \mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))$, $(1-\beta)(a_{ij} - a_{\mu^*(j)j}) \leq \gamma c$ *or* $\beta(a_{ij} - a_{i\mu^*(i)}) \leq (1-\gamma)c$, *and for all links* $l_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}$, $\beta a_{i\mu^*(i)} \geq \gamma c$ *and* $(1-\beta)a_{i\mu^*(i)} \geq (1-\gamma)c$.

(*i*) *If only buyers can pay towards the cost of link formation* ($\gamma = 1$)*, the efficient network will be pairwise stable if and only if for all efficiently matched buyers (that is, for all i such that* $\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e)) \neq i$), $\beta a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} \geq c$ *and* $\max_j\{a_{ij} - a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j}\} \leq \frac{c}{1-\beta}$.

[5] Seller s_1 will necessarily pay the entire cost of forming the link ($\gamma_2 = 0$) because b_2 does not benefit from it.

Proof. To simplify notation, for this proof we use $\mu^*(.)$ for $\mu^*(.;\boldsymbol{\alpha}(\mathbf{L}^e))$.

Part (i): It has already been shown that in the efficient network there are never sufficient incentives for additional links to be formed for the purpose of trade. See the proof of Proposition SA-2.

The efficient network is then stable as long as there are no incentives to form outside option links or to delete trade links. For a trade link to avoid being deleted, *both* the buyer and seller must benefit from it. As there are no outside options in the efficient network, a buyer i''s benefit from a trade link is $\beta a_{i',\mu^*(i')}$, while a seller j''s benefit is $(1-\beta)a_{\mu^*(j')j'}$.

To avoid the formation of outside option links, the following must hold: For every potential outside option link, *either* the buyer or seller must receive benefits lower than their costs of forming it. As the formed outside option chain would involve only one outside option link, a buyer i' will benefit from an outside option link to j' by an amount $(1-\beta)(a_{i',j'} - a_{\mu^*(j')j'})$ (see Theorem 1 in Section 4 of the main document). Similarly, a seller j' will benefit from this outside option link by an amount $\beta(a_{i',j'} - a_{i',\mu^*(i')})$.

Part (ii): By the proof of Proposition SA-2, no buyer will ever want to form a trade link.

The benefit to a buyer i' of trading over a link in the efficient network is $\beta a_{i',\mu^*(i')}$. Thus for a buyer to maintain this traded-over link, it must be the case that $\beta a_{i',\mu^*(i')} \geq c$.

Unmatched buyers can never benefit from an outside option link. Matched buyers will not form an outside option link $l_{i',j'}$ (where $j' \neq \mu^*(i')$) if and only if the costs of doing so are greater than the benefits: $c \geq (1-\beta)(a_{i',j'} - a_{\mu^*(j')j'})$. These conditions can be consolidated by considering whether a buyer i' would form a link to *any* seller. Buyer i' will not want to form a link to any seller if and only if $c \geq (1-\beta)(\max_{j'\neq\mu^*(i')}\{a_{i',j'} - a_{\mu^*(j')j'}\})$. □

Consider a buyer i and seller $\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))$ such that $a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} > c$. From Lemma SA-2, when $\gamma \in (0,1)$ there will be underinvestment if and only if either $\beta a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} < \gamma c$ or $(1-\beta)a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} < (1-\gamma)c$. Equivalently there will be underinvestment if and only if $\min\{\frac{\beta}{\gamma}, \frac{(1-\beta)}{(1-\gamma)}\} < \frac{c}{a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}}$. As $l_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}$ was formed in the efficient network, $a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} \geq c$.

Part (i): When $\beta = \gamma$, the above condition becomes $\min\{\frac{\beta}{\gamma}, \frac{(1-\beta)}{(1-\gamma)}\} = 1 \geq \frac{c}{a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}}$ and there is no under-investment.

Part (ii): When $\gamma = \beta = 1$ or $\gamma = \beta = 0$, the condition in part $(ii)b$ of Lemma SA-2 is met and there is no overinvestment (as well as no underinvestment by Part (i)) in any stable network. There is thus no profitable deviation away from the efficient network.

Part (iii): When $\gamma \in (0,1)$, each component of each stable network is arranged into a chain. Outside option links must benefit both the connected buyer and seller for them to both invest in it, each agent can benefit from at most one outside option, and, from Lemma 1 outside option chains never cycle. In any network other than the empty network, there must then exist a buyer and a seller each of whom has a trade link but no outside option link. However, if $\beta = 0$, then this buyer will not be willing to invest in his trade link; and if $\beta = 1$, this seller will not invest in his trade link. Thus the unique stable network will be the empty network.

Endogenous costs

It is helpful to first prove the following lemma:

Lemma SA-3. *When cost sharing is endogenous, the efficient network will be pairwise Nash stable with transfers if and only if for all i, j such that $j \neq \mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))$, the following holds:*

$$\mathbf{1}\left(a_{ij} > a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j}\right)(1-\beta)(a_{ij} - a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j}) \\ +\mathbf{1}\left(a_{ij} > a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}\right)\beta(a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}) \leq c,$$

where $\mathbf{1}$ *is an indicator function taking value* 1 *if the condition to the right of it holds and* 0 *otherwise.*

Proof. By the proof of Proposition SA-2, there are no incentives to form links for the purpose of trade. None of the links already formed will be deleted either. This is because the benefit the connected buyer and seller jointly receive from the link is given by a_{ij}, and as this link was formed in the efficient network, $a_{ij} \geq c$. The only remaining deviation to consider is the addition of links to the efficient network for the purpose of forming an outside option.

Consider first the incentives of seller j to form an outside option link. For link l_{ij} to provide j with an outside option in $\mathbf{L}^e \cup \{l_{ij}\}$, i must want to trade with j rather than $\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))$ when all sellers receive a price of zero: $a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} > 0$. Furthermore, if l_{ij} is added to $\mathbf{L}^e$, i will bid up j's price, when buyers have all the bargaining power, by $(a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))})$. Seller j will therefore be willing to pay up to $\beta(a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))})$ to form the link. An equivalent argument for buyer i shows that they will value the link only when $a_{ij} > a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e)),j}$ and will be willing to pay up to $(1-\beta)(a_{ij} > a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e)),j})$ to form it. The link l_{ij} can never reduce buyer i or seller j's payoff and will be formed if and only if

$$\mathbf{1}(a_{ij} > a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j})\left[(1-\beta)(a_{ij} - a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j})\right] \\ +\mathbf{1}(a_{ij} > a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))})\left[\beta(a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))})\right] > c.$$

□

Part (i): If all agents' most preferred trade partners, were they to extract all the gains from any trade, are different, then all agents can be matched, so that $a_{i\mu(i)} = \max_j a_{ij}$ and $a_{\mu(j)i} = \max_i a_{ij}$. Thus $a_{i\mu(i)} \geq a_{ij}$ for all $j \neq \mu(i)$ and $a_{\mu(j)j} \geq a_{ij}$ for all $i \neq \mu(j)$. For this match, all the indicator functions in Lemma SA-3 will be 0 and so the efficient network will be stable.

Part (ii): From Lemma SA-3, as $c \to 0$ the efficient network will be stable only if $a_{i\mu^*(i\boldsymbol{\alpha}(\mathbf{L}^e))} > a_{ij}$ for all $j \neq \mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))$ and $a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j} > a_{ij}$, for all $i \neq \mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))$. If any of these conditions did not hold, then at least one agent would form an outside option link. Necessary conditions for this are that $\text{argmax}_j(a_{ij}) \neq \text{argmax}_j(a_{i',j})$ for all $i \neq i'$ and $\text{argmax}_i(a_{ij}) \neq \text{argmax}_i(a_{ij'})$ for all $j \neq j'$.

Part (iii): Consider a network $\mathbf{L}^e$ that is efficient for all $c \in [\underline{c}, \overline{c}]$. Two cases are trivial: If $\mathbf{L}^e$ is not stable for any $c \in [\underline{c}, \overline{c}]$, set $c^* < \underline{c}$. If $\mathbf{L}^e$ is stable for all $c \in [\underline{c}, \overline{c}]$, set $c^* > \overline{c}$. In all other cases, there exists a $c' \in [\underline{c}, \overline{c}]$ for which $\mathbf{L}^e$ is unstable and a $c'' \in [\underline{c}, \overline{c}]$ for which $\mathbf{L}^e$ is stable.

At c' there must exist an outside option link l_{ij}, $j \neq \mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))$, that it is jointly profitable for i and j to form in the efficient network:

$$\mathbf{1}(a_{ij} > a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j}) \left[(1-\beta)(a_{ij} - a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j})\right] + \mathbf{1}(a_{ij} > a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}) \left[\beta(a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))})\right] \geq c'.$$

Reducing c' can only increase the incentives to form this link. Thus $\mathbf{L}^e$ will be unstable for all $c \in [\underline{c}, c']$.

At c'' no outside option links are profitably formed: For any potential link between a buyer i and seller $j \neq \mu^*(i; \boldsymbol{\alpha}(\mathbf{L}^e))$, the following condition must hold:

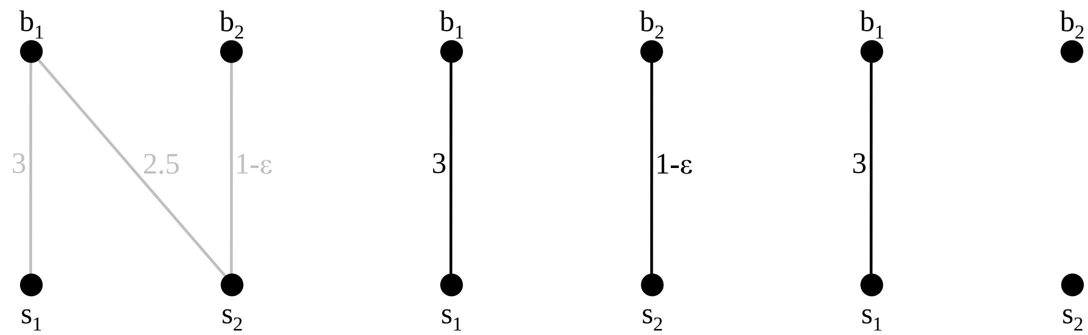
$$\mathbf{1}(a_{ij} > a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j}) \left[(1-\beta)(a_{ij} - a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j})\right] + \mathbf{1}(a_{ij} > a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}) \left[\beta(a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))})\right] \leq c''. \quad (1)$$

Increasing c'' can then only further reduce the incentives to form each such potential outside option link, and $\mathbf{L}^e$ will continue to be stable for all $c \in [c'', \overline{c}]$.

There thus exists a c^* such that $\underline{c} \leq c' \leq c^* \leq c'' \leq \overline{c}$ and the efficient network $\mathbf{L}^e$ is stable if and only if $c \geq c^*$.

Part (iv):

Consider the network shown in Figure SA-8a, where $\beta = \frac{1}{2}$. For $c = 1 - 2\varepsilon$, ε small and positive, the efficient network is shown in Figure SA-8b. In this network, there are no outside option links that it would be profitable to form, so the efficient network is stable. Suppose now that c increased to $c = 1 - \frac{\varepsilon}{2}$. The efficient network is now the network shown in Figure SA-8c. In this network, there are incentives for buyer b_1 and seller s_2 to form an outside option link, hence the efficient network is not stable.



(A) Potential links (B) $\mathbf{L}^e(c = 1 - 2\varepsilon)$ (C) $\mathbf{L}^e(c = 1 - \frac{\varepsilon}{2})$

FIGURE SA-8. Different efficient networks for different levels of c

□

A.6. **Proof of Proposition SA-5.**

Proof.

$$NGT(\mathbf{L}^e) - NGT(\mathbf{L}) = \left(\sum_{i=1}^{m} u_i(\mathbf{L}^e) + \sum_{j=1}^{n} v_j(\mathbf{L}^e) - K'c\right) - \left(\sum_{i=1}^{m} u_i(\mathbf{L}) + \sum_{j=1}^{n} v_j(\mathbf{L}) - Kc\right),$$

where $K' \leq K$ links are formed in the efficient network. Now consider a buyer i and seller j that would trade with each other in the efficient network. If i and j are not connected in the stable network, then there must not be sufficient incentives for them to form the link l_{ij}: $u_i(\mathbf{L}) + v_j(\mathbf{L}) \geq a_{ij} - c$. If i and j are connected in the stable network, then $u_i(\mathbf{L}) + v_j(\mathbf{L}) \geq a_{ij}$. As i and j would trade in the efficient network, $u_i(\mathbf{L}^e) + v_j(\mathbf{L}^e) = a_{ij}$. Also, regardless of whether i and j are connected, $u_i(\mathbf{L}) + v_j(\mathbf{L}) \geq u_i(\mathbf{L}^e) + v_j(\mathbf{L}^e) - c$. Furthermore, in the stable network, buyer i and seller j must receive payoffs $u_i(\mathbf{L}) \geq 0$ and $v_j(\mathbf{L}) \geq 0$, respectively. Summing these inequalities across all buyers and all sellers:

$$\begin{aligned}
\sum_{i=1}^{m} u_i(\mathbf{L}) + \sum_{j=1}^{n} v_j(\mathbf{L}) &\geq \sum_{i=1}^{m} u_i(\mathbf{L}^e) + \sum_{j=1}^{n} v_j(\mathbf{L}^e) - cK' \\
\left(\sum_{i=1}^{m} u_i(\mathbf{L}^e) + \sum_{j=1}^{n} v_j(\mathbf{L}^e) - K'c\right) &\leq \left(\sum_{i=1}^{m} u_i(\mathbf{L}) + \sum_{j=1}^{n} v_j(\mathbf{L}) - Kc\right) + Kc \\
NGT(\mathbf{L}^e) - NGT(\mathbf{L}) &\leq Kc
\end{aligned}$$

□

A.7. **Proof of proposition SA-6.**

Proof. **Part** (i): In the bargaining outcomes considered, buyers and sellers are matched to maximize the possible gains from trade. Suppose that $\mu(b_1) \neq s_1$. Matching b_1 to s_1 and $\mu(s_1)$ to $\mu(b_1)$ will increase the gains from trade: $a_{b_1 s_1} + a_{\mu(s_1)\mu(b_1)} > a_{b_1\mu(b_1)} + a_{\mu(s_1)s_1}$ by the increasing differences condition. Consider now matches such that $\mu(b_k) = s_k$ for all $k < K$, but $\mu(b_K) \neq s_K$. Matching b_K to s_K and $\mu(s_K)$ to $\mu(b_K)$ will increase the gains from trade: $a_{b_K s_K} + a_{\mu(s_K)\mu(b_K)} > a_{b_K\mu(b_K)} + a_{\mu(s_K)s_K}$ by the increasing difference condition. By induction, for any match where $\exists k \leq \min(m,n), \mu(b_k) \neq s_k$, the surplus can be increased by rematching. Thus b_k will be matched to s_k, for all $k \leq \min(m,n)$.

Part (ii): Agents' outside trade partners are found by removing their trade partner from the network and considering the optimal rematching. Suppose that $\mu(b_k)$ is removed from the network. Sellers ranked below $\mu(b_k)$ now move up in the rankings. Part (i) can then be applied to determine the new matches. Thus b_k will be matched to the seller ranked in the kth position in the new network, which will be the seller ranked in the $k+1$ position in the original network: $\eta(b_k) = s_{k+1}$.

Part (iii): Suppose that $\mu(s_k)$ is removed from the network. Buyers ranked below $\mu(s_k)$ now move up in the rankings. Part (i) can then be applied to determine the new matches. Thus s_k will be matched to the buyer ranked in the kth position in the new network, which will be the buyer ranked in the $k+1$ position in the original network: $\eta(s_k) = b_{k+1}$.

Part (*iv*): Not that all agents' outside trade partners have been identified, Theorem 1 from Section 4 of the main document can be applied to determine agents' outside trade values. It follows immediately that $\underline{u}_{b_k} = \underline{u}_{b_{k+1}} + (a_{b_k s_{k+1}} - a_{b_{k+1} s_{k+1}})$. Thus $\underline{u}_{b_k} > \underline{u}_{b_{k+1}}$.

Part (*v*): Applying Theorem 1 from Section 4 of the main document, it follows immediately that $\underline{v}_{s_k} = \underline{v}_{s_{k+1}} + (a_{b_{k+1} s_k} - a_{b_{k+1} s_{k+1}})$. Thus $\underline{v}_{s_k} > \underline{v}_{s_{k+1}}$.

□

A.8. **Proof of Proposition SA-7.**

Proof. **Part** (*i*): When $\beta \in (0,1)$ and cost shares are exogenous, the only outside option links that can be formed are those that benefit both connected agents. Under the increasing difference condition, the efficient network is such that for all potential outside option links $\{l_{ij} \notin \mathbf{L}^e \text{ where } a_{ij} > c\}$, either $i > j$ or $j > i$. For those with $i > j$, $a_{ij} - a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} < 0$, so j does not benefit from the outside option link l_{ij}. For those with $j > i$, $a_{ij} - a_{\mu^*(j;\boldsymbol{\alpha}(\mathbf{L}^e))j} < 0$, so i doesn't benefit from the outside option link l_{ij}. Thus there are no potential outside option links that benefit both connecting agents, and so no outside option links will ever be formed.

The efficient network will therefore be stable as long as there are sufficient incentives for all trade links present to be formed: $\min\{\beta a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} - \gamma c, (1-\beta) a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} - (1-\gamma)c\} > 0$ for all i such that $a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} > c$. Increasing differences implies that $a_{K\mu^*(K;\boldsymbol{\alpha}(\mathbf{L}^e))} \leq a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))}$ for all i such that $a_{i\mu^*(i;\boldsymbol{\alpha}(\mathbf{L}^e))} > c$, and so the efficient network will be stable if and only if $\min\{\beta/\gamma, (1-\beta)(1-\gamma)\} a_{b_{K'} s_{K'}} \geq c$.

Part (*ii*): Suppose the efficient network is formed and buyer k is matched to seller k. Buyer k's highest possible payoff from forming an outside option link is from forming a link to seller $k+1$. This would generate an outside option for buyer k of $a_{k,k+1} - a_{k+1,k+1}$ which, by the increasing differences condition, is weakly greater than $a_{kj} - a_{jj}$ for all j. Further, as $a_{12} - a_{22} \geq a_{kk+1} - a_{k+1k+1}$ for all k, and each outside option link can benefit only the buyer or the seller, if the highest quality buyer does not want to form an outside option link to the second highest quality seller, no buyer wants to form any outside option link. By equivalent reasoning, if the highest quality seller does not want to form an outside option link to the second highest quality buyer no seller wants to form any outside option link. Thus the efficient network will be formed if

$$\max\{(1-\beta)(a_{12} - a_{22}), \beta(a_{21} - a_{11})\} \leq c$$

□

A.9. **Proof of Proposition SA-8.**

Proof. It will be shown that at both extreme points of the core, buyers' payoffs are greater in $\mathbf{L}$ than in $\mathbf{L} \setminus \{j'\}$, while sellers' payoffs are greater in $\mathbf{L} \setminus \{j'\}$ than in $\mathbf{L}$.

First, consider the initial payoffs in $\mathbf{L}$ if sellers receive their outside option. These payoffs could have been found by running the multi-unit auction mechanism analyzed by Démange, Gale, and Sotomayor (1986) (with sellers' prices initialized at 0). In this mechanism, the price of sellers with whom trade is overdemanded is bid up. Suppose now that j' is removed from the network. Any price vector at which trade with a seller was previously overdemanded, at any point in the multi-unit auction mechanism, would still result in trade with that seller being overdemanded. Thus the same prices can be reached at an interim stage of this mechanism. Continuing the multi-unit auction mechanism, sellers prices can only be bid up further.

To find buyers' outside options, the multi-unit auction mechanism could be used with the role of buyers and sellers reversed: Sellers bid down their prices, competing for buyers whom many of them would like to supply at the current prices. Suppose now a seller j' is removed from the network. At current prices, trade with no buyer would be overdemanded and so buyers' outside options must be weakly lower. Furthermore, if the multi-unit auction were run again, the removal of j' may result in trade with a buyer not being overdemanded when it was previously, and ultimately the price paid by the buyer not being bid down as far.

It has been shown that in network $\mathbf{L} \setminus \{j'\}$, sellers' $(j \neq j')$ outside options and their seller-optimal core payoffs are both higher than in $\mathbf{L}$. It therefore follows that for a given β, sellers' payoffs are higher in $\mathbf{L} \setminus \{j'\}$ than in $\mathbf{L}$. Conversely in $\mathbf{L} \setminus \{j'\}$ buyers' outside options and their buyer-optimal core payoffs are both lower than in $\mathbf{L}$, so that for a given β, buyers' payoffs are higher in $\mathbf{L}$ than in $\mathbf{L} \setminus \{j'\}$. □