FIRMS AS SETS OF CAPABILITIES: THEORY AND APPLICATIONS

JUN CHEN† AND MATTHEW ELLIOTT†

Abstract. We model firms as sets of scarce capabilities, where these capabilities constitute sources of competitive advantage. Markets are also represented by sets of capabilities, identifying those capabilities that are valuable in them. Doing so provides a new perspective on several industrial organization literatures. These include merger analysis, strategic alliances, investment and market dynamics. It also provides formal foundations for a prominent theory of strategic competition in the management literature. We apply our model to reconsider merger guidelines and to provide an alternative view on industry consolidation. We argue that merger analysis should be broader than is current practice by considering more markets, and provide an explanation for industry consolidation in terms of the equilibrium acquisition of capabilities through mergers.

1. Introduction

There are two predominant and competing theories of strategic competition in the management literature. With strong economic and industrial organization foundations, Porter (2008) argues that the industrial environment drives competitive advantage and that firms must exploit and manipulate this to make profits. In contrast, the resource-based view of the firm pioneered by Wernerfelt (1984) argues that firms should be viewed as sets of immutable and scarce resources and the firms derive their competitive advantage from these resources, broadly defined. This idea is simple and intuitive but does not have the mathematical and formal foundations of the other approach. As a result it suffers from critiques of tautology. Resources are defined to include everything that can generate a competitive advantage, while its objective is to provide a theory of competitive advantage (see Kraaijenbrink, Spender and Groen, 2010).

In this paper we provide formal foundations for the resource-based view of the firm. In doing so we resolve the tautology and build a framework that specifically ties firms’ capabilities to their relative competitiveness within markets. Our approach generalizes the industrial organization foundations of Porter (2008), and so also accommodates his theory of competitive advantage. Our unified theory delivers practical economic insights. We use the theory to argue that merger
analysis and regulations should be more holistic, considering markets beyond those in which the merging firms compete. We also explain industry consolidation as a profitable and possibly consumer surplus increasing change in equilibrium.

Our model is simple. Each firm has a finite set of capabilities and each market values a finite set of capabilities. The firms’ capabilities are their potential sources of comparative advantage, while markets’ capabilities reflect which of these potential sources of competitive advantage are useful in which markets. Firm and market capabilities are fixed in the short run, reflecting their scarcity. If one firm has more capabilities that are relevant for a given market than another it is a stronger competitor in that market and has a higher price-cost margin. We model this through reduced marginal costs, but it could equivalently be modeled through consumers’ higher willingness to pay. More precisely, we model Cournot competition in each market where the marginal cost of each firm in a given market is determined by the intersection of that firm’s capabilities and that market’s capabilities.

As firms are sets of capabilities and markets are also sets of capabilities, we let a firm hypergraph describe firms and a market hypergraph describe markets. In both hypergraphs the nodes of the graph are attributes/capabilities. In the firm hypergraph each firm is then represented by an edge which comprises of a subset of the available capabilities. Similarly, in the market hypergraph each market is represented by an edge, which is a subset of the available capabilities. See for example Figure 1 below.

In this example there are four underlying attributes. There are two markets shown in Panel (A) of Figure 1. In market $M_1$ attributes $a_1$ and $a_2$ are valued. In market $M_2$ attributes $a_1$, $a_3$ and $a_4$ are valued. There are four firms. Their capabilities are shown in Panel (B) of Figure 1. In the first market firms 1 and 4 have all the useful attributes and are strong competitors. Firm 2 is a weaker competitor with just one attribute and firm 3 is the weakest competitor with none.
of the useful attributes. In the second market firms 2, 3 and 4 all have (a different) two of the three useful capabilities and will be strong competitors. Firm 1 has only one of the three useful capabilities and is a weaker competitor.

We apply our model to address a couple of different, but related questions. First we use our model to reevaluate merger guidelines and analysis. We assume that a merged firm inherits the combined capabilities of merging firms but that it is costly to maintain additional capabilities and more costly per capability the more capabilities a firm has. This limits the profitability of mergers, particularly those that do not generate any relevant synergies by enhancing the competitiveness of the merged firm in some market. These costs might be associated with limitations on management time and expertise, or with compromising the ability of a firm to fine tune its corporate culture to efficiently maintain certain capabilities.

Our capability centric approach allows us to formalize (production) synergies. While this formalization is valuable in itself when considering the benefits of a merger within a market in which the merging firms compete, it also emphasizes the efficiency benefits mergers can have in other markets. If only one of the firms competes in a given market prior to the merger, the merger may result in a stronger competitor emerging in that market. We show that in equilibrium this always increases consumer surplus. It may also result in the merged firm entering new markets thereby increasing consumer surplus. Altogether these benefits can be substantial. Blocking a merger based on anticompetitive effects in one market (or prescribing remedies that render the merger unprofitable) can be socially inefficient. Moreover, in contrast to the holistic approach our model suggests, merger guidelines prescribe a market-by-market analysis:¹

“The Agencies will not challenge a merger if cognizable efficiencies are of a character and magnitude such that the merger is not likely to be anticompetitive in any relevant market. To make the requisite determination, the Agencies consider whether cognizable efficiencies likely would be sufficient to reverse the merger potential to harm customers in the relevant market, e.g., by preventing price increases in that market.”


¹See Farrell and Shapiro (2001) for an excellent discussion of the treatment of synergies in the merger guidelines within a market.
In our second application we show that markets can be stuck in an equilibrium in which there are too many low capability firms. Another equilibrium can also exist in which firms with different capabilities have merged to create fewer, more competitive firms. We provide an example in which the equilibrium with fewer but more competitive firms generates higher consumer surplus as well as higher profits. We can therefore explain waves of industry consolidation as an efficiency and profitability enhancing change in equilibrium, in which firms become more capable within the market through their acquisitions. A change in beliefs that others will merge is sufficient to trigger such a change in equilibrium, but exogenous events that reduce profitability can cause the fragmented market structure to no longer be an equilibrium and also trigger the consolidation.

1.1. Related Literature. Beyond the management literature we discuss above, which has no formal mathematical foundations, we are unaware of any economic paper that proposes modeling firms as sets of capabilities or formulates a hypergraph representation of firms or markets. There is one exception. Malamud and Rostek (2015) model financial exchanges, and competition across them, as a hypergraph. In their setting the set of nodes are the exchanges and the edges represent which traders are active on which of these exchanges. This is fundamentally different from the hypergraphs we consider where the nodes are attributes/capabilities.

There is a huge industrial organization literature that considers mergers. There are prominent papers in this literature that discuss synergies including defining different types of synergies (Chatterjee, 2002). Nevertheless, our formulation of synergies in capabilities/attributes space is novel to the best of our knowledge. Also, as argued by Jensen and Ruback (1983), mergers/acquisitions in general have been demonstrated to create economic value, and the gains created do not appear to come from the creation of market power. By explicitly modeling firms as sets of capabilities we are able to better identify sources of economic value created by mergers and acquisitions.

A related industrial organization literature examines the importance of synergies in motivating mergers, and how synergy realization depends on the similarity and complementarity of the merging firms. For example, Larsson and Finkelstein (1999) argue that the success of a merger or acquisition depends on the degree of synergies that are realized, and that not only the similarities across firms matter, but also their production and marketing complementarities. Our

\[^{2}\text{Dziubinski and Goyal (2013) also use hypergraphs, but in the very different setting of attack and defence networks.}\]
model provides a simple theoretical framework to capture how the complementarities between two firms generate synergy.

The literature has also documented a sharp change of the form of industrial firm in the United States over the course of the twentieth century (Davis and Diekmann, 1994). Following the conglomerate merger wave of the late 1960s and 1970s, Fligstein (1991) reports that by 1980 growth through diversification had become the most widely used corporate strategy among large firms. Fewer than 25 percent of Fortune 500 companies made all their sales within a single broadly-defined (2-digit SIC) industry. However, during the 1980s, a wave of “deconglomeration” restructured American industry, resulting in the largest US firms becoming considerably less diversified by 1990. Many papers seek to explain this transition in industrial structure (see Amihud and Lev, 1981, Fligstein, 1991, Bhagat et al., 1990, Davis and Stout, 1992, LeBaron and Speidell, 1987, Morck, Shleifer, and Vishny, 1990). Almost all suggest that an enormous “collective error” was made in the conglomeration period and represent the subsequent deconglomeration as a correction. Our model suggests an alternative explanation.

In our model, mergers that are followed by demergers can be efficiency enhancing if they re-distribute capabilities in a way that creates stronger competitors. Consider the markets shown below in Figure 2. There are two markets which are unrelated in the sense that they require disjoint capabilities. There are also initially two firms who are both weak competitors in both markets (Panel (A), Figure 3). A merger between these two firms creates a new conglomerate firm that competes strongly in both markets (Panel (B), Figure 3), but has to maintain an inefficiently large number of capabilities. A subsequent demerger generates two firms that are both strong competitors in their respective markets (Panel (C), Figure 3).

\[ \text{Figure 2. Market Hypergraph} \]

On industry consolidation, the literature most related to our paper is the work of Sutton (1991,2001).\(^3\) Sutton shows that exogenous changes in production technology or consumers’ willingness to pay, can trigger a period of industry consolidation. Endogenous investments in

\(^3\)See also Dasgupta and Stiglitz (1980).
sunk costs reduce marginal costs and the number of firms the market can support, causing some to exit. A classic example of such a change in market structure is the tire industry (Klepper and Simons, 2000). We offer a complementary but related mechanism through which consolidation can occur. Firms acquire scarce capabilities through mergers, thereby reducing marginal costs. This directly models consolidation as being achieved through mergers and provides a fuller explanation as to how marginal cost reductions are obtained. It also demonstrates that efficient consolidation can be triggered simply by changing expectations and the resulting change in the equilibrium being played. Conceptually, our results on stable industrial structures are also related to the networks literature (Jackson and Watts, 2001).

The organization of our paper is as follows: in Section 2, we introduce our model and prove the existence of Nash equilibrium given a fixed set of firms. In Section 3, we apply our framework to study merger guidelines and point out its lack of holistic view on mergers. In Section 4, we provide another application of our framework, and prove the existence of stable industrial structures and industry consolidation. We conclude in Section 5.

2. FIRMS AS SETS OF CAPABILITIES

There is a finite set of attributes \(A\), a finite set of firms \(\{1, \ldots, n\}\) and a finite set of markets \(\{1, \ldots, m\}\). Each firm \(i\) is associated with a set of capabilities \(F_i \subseteq A\). This represents the key competencies of firm \(i\). Each market \(j\) is also associated with a finite set of capabilities \(M_j \subseteq A\). This represents the attributes that are useful in market \(j\).

We sometimes represent this information in a hypergraph. A hypergraph is defined by a set of nodes, in this case the set of attributes \(A\), and a set of edges, each of which constitutes a subset of the nodes. The firm hypergraph is \(H_F = (A, \{F_1, F_2, \ldots, F_n\})\). Abusing notation, we sometimes write \(H_F = \{F_1, F_2, \ldots, F_n\}\). The market hypergraph is \(H_M = (A, \{M_1, M_2, \ldots, M_m\})\) and
similarly we abuse notation by writing $H_M = \{M_1, M_2, \ldots , M_m\}$. We let $\mathcal{H}(A)$ be the set of all possible firm hypergraphs fixing attributes $A$.

Firms that have more relevant capabilities for a given market are stronger competitors in that market. We assume that the marginal cost of firm $i$ in market $j$, $c_{ij}$, depends on how well $i$’s capabilities, $F_i$ match the capabilities associated with market $j$, $M_j$. For simplicity we define the variable $\theta_{ij} = |F_i \cap M_j|$ and assume that $i$’s marginal cost is a decreasing function of $\theta_{ij}$.

All firms simultaneously decide how much to produce in all markets. We take a zero output decision of firm $i$ in market $j$ to mean that the firm $i$ does not enter market $j$. The output choice of firm $i$ in market $j$ is given by $q_{ij}$ and the vector $q_i \in \mathbb{R}_+^m$ represents $i$’s entry and output choices in all markets.

We let $Q_j = \sum_{i=1}^m q_{ij}$ be the total output of all firms in market $j$, and let $P_j(Q_j)$ be the inverse demand function for market $j$. For simplicity we consider a linear inverse demand curve and set $P_j(Q_j) = \alpha_j - \beta_j Q_j$ with $\alpha_j, \beta_j > 0$. Firm $i$’s profits in market $j$ are:

$$\pi_{ij}(q_{ij}) = (\alpha_j - \beta_j Q_j - c_{ij}(\theta_{ij})) q_{ij}$$

Suppose firm $i$ operates in market $j$. Firm $i$ will then maximize its profits in market $j$ by selecting $q_{ij}$ such that:

$$q_{ij} = \frac{\alpha_j - \beta_j Q_j - c_{ij}(\theta_{ij})}{\beta_j}$$

Let $Q_{-ij} = Q_j - q_{ij}$ be the output of firms other than $i$ in market $j$. We denote the optimal output of firm $i$ in market $j$ by $q_{ij}^*(Q_{-ij}, \theta_{ij})$.

Firm $i$’s overall profitability is given by:

$$\pi_i(q_i) = \sum_{j=1}^m \pi_{ij}(q_{ij}) - \kappa(|F_i|),$$

where we assume that $\kappa$ is a weakly increasing and convex function. This captures the conglomerate costs associated with maintaining many unrelated capabilities. It may reflect the

4 Although we assume that the match of a firm’s capabilities with a market affects the firm’s profitability by reducing their marginal costs, our results would not change if the match instead increases consumers willingness to pay for $i$’s product. We let firms choose outputs and, as is standard Cournot models, all that matters is the price-cost margin.
scarcity of management time or inability of the firm to tailor their corporate culture towards maintaining a narrower set of capabilities.

Each firm’s overall profitability is the sum of its profitability over all markets, less a penalty for the number of capabilities it maintains. Thus, fixing capabilities, a firm’s output decision in each market can be made in isolation. Specifically, each firm will choose \( q_{ij} = \max\{q^*_ij(Q_{-ij}, \theta_{ij}), 0\} \).

2.1. Existence and uniqueness of Nash equilibrium. Standard results apply to our setting and fixing the capabilities of firms, there is a unique Nash equilibrium.

**Proposition 1.** There exists a unique Nash equilibrium. In equilibrium firms who are more capable in a given market produce more: If \( q_{ij} > q_{kj} \) then \( \theta_{ij} > \theta_{kj} \).

The proof is in Appendix A. It is a standard result that in the \( m = 1 \) case there is a unique equilibrium. This result is easily extended to our setting because firms’ profitability in each market is independent of decisions in other markets. It is also intuitive that more capable firms in a given market have higher output because they have lower marginal cost. An immediate implication is that the \( x \) firms who enter a given market are the \( x \) most capable with the \( x \) lowest marginal costs for that market.

2.2. Discussion. To the best of our knowledge, there are no other papers in the economics literature that represent firms and markets as sets of capabilities. Doing so enables us to provide a theory of competition\(^5\) in which different firms are better suited to compete in different markets. This enables firms acquisition strategies to be viewed in this light and provides a new perspective on corporate strategy. Such a perspective can be important for thinking about strategic alliances, mergers and firms’ investment decisions. It also provides valuable foundations for modeling the dynamic evolution of markets and undertaking empirical work. We explore some of these applications in the subsequent sections.

While the economics literature has not previously taken a capability centric view towards markets and firms, the management literature has long viewed firms in this way. Indeed, the resource based view of the firm is one of the preeminent theories of strategic competition. In relation to this literature our model provides simple formal foundations. In doing so it is able to address a long standing critique of tautology. While key competencies or capabilities are broadly defined

\(^5\)It does not, nor is it intended to, provide a theory of the firm. We do not fully explain firm boundaries or provide an explanation for what is made within firm and what is bought in from suppliers.
to include anything that yields a competitive advantage, they are also held up as determining competitive advantage. Modeling capabilities as a fixed set allows the separation of capabilities and actions of the firm in bringing these capabilities to bear in a given market. Although only modeled abstractly, this prevents including within the set of capabilities the ability to create or acquire new capabilities, and the infinite hierarchy that would otherwise result.

Our view of capabilities is perhaps a slightly narrower view than that originally envisaged, but one that provides separate role for managers in terms of capability acquisition and the decisions of which markets to enter. This provides a similar scope to the role for managers as in Porter (2008) which provides the main alternative theory of strategic competition. Moreover, by putting the resource based view of the firm into a standard IO setting, it can be combined with the IO foundations underlying Porter.

3. An Application: Mergers Guidelines

In this section we endogenize firms’ capabilities by allowing them to merge and demerge with each other. We do so by considering a two stage game. In stage 1 firms endogenously develop their capabilities through merging with each other and spinning off new companies. In stage 2 firms compete as described in the previous section. Representation of firms by sets of capabilities (or core competencies as they are referred to in the management literature) enables us to see how a merger can change which firms operate in which markets.

We begin by defining a merger. If firms \(i\) and \(k\) merge then these firms cease existing individually and a new firm \(l\) is created with attributes \(F_l = F_i \cup F_k\). On the hypergraph, it is equivalent to the union of the two hyperedges representing them. A demerger of a firm \(i\) into firms \(k\) and \(l\) is equivalent to the bi-partition of its set of capabilities so that \(F_i = F_k \cup F_l\) and \(F_k \cap F_l = \emptyset\). On the hypergraph, it is equivalent to the bi-partition of the hyperedge representing the initial firm.

We treat mergers and demergers asymmetrically. If two firms merge with overlapping capabilities the replicated capabilities are, in effect, lost in the newly created firm. Suppose, for example, that two firms both have good marketing departments before they merge and that these marketing teams constitute a key source of competitive advantage. After the merger only one marketing department is required and so the two departments are consolidated and some workers leave. If, after this consolidation has been completed, this firm then demerges to create two new firms, only one can inherit the strong marketing department. While in practice some attributes may be
inherited by both companies following a demerger, we rule this out for simplicity while noting that if we were to assume that all capabilities were inherited by both companies following a demerger there would exist a sequence of mergers and demergers that would result in all firms acquiring all capabilities. As capabilities are intended to represent sources of competitive advantage they need to be scarce and this modeling choice would be inconsistent with that.

We refer to the set of firms present in a given market as the market structure and collectively to these market structures across all markets as the industrial structure. Mergers can have many effects on the industrial structure. We document the changes that can occur when firm \(i\) merges with firm \(k\) to create firm \(l\). It is helpful to differentiate between: (i) overlapping markets in which both \(i\) and \(k\) compete (such that \(q_{ij} > 0\) and \(q_{kj} > 0\)); (ii) non-overlapping markets in which either \(i\) or \(k\) compete, but not both (such that \(q_{ij} > 0\) or \(q_{kj} > 0\) but \(\min\{q_{ij}, q_{kj}\} = 0\); and (iii) newly entered markets in which neither \(i\) nor \(k\) compete pre merger (such that \(q_{ij} = q_{kj} = 0\)).

**Concentration in overlapping markets:** In an overlapping market \(j\), firms \(i\) and \(k\) no longer compete against each other reducing competition.

**Synergies in overlapping markets:** In an overlapping market \(j\), if \((F_i \cap M_j) \subset (F_l \cap M_j)\) and \((F_k \cap M_j) \subset (F_l \cap M_j)\) then firm \(l\) is a stronger competitor in market \(j\) than either \(i\) or \(k\) was alone and we say that the merger has synergies in market \(j\).

**Synergies in non-overlapping markets:** In a non-overlapping market \(j\) with \(q_{ij} > 0\) pre-merger, if \(F_i \cap M_j \subset F_l \cap M_j\) such that \(l\) is a stronger competitor in market \(j\) than \(i\) was alone, then we say that the merger has synergies in market \(j\).

**Synergies in new markets:** There are synergies generated whenever the merged firm enters new markets. As \(q_{ij} > 0\) but \(q_{ij} = q_{kj} = 0\) by Proposition 1 we must have that \((F_i \cap M_j) \subset (F_l \cap M_j)\) and \((F_k \cap M_j) \subset (F_l \cap M_j)\).

**Economies of Scale:** If capabilities overlap sufficiently, the combined cost of maintaining capabilities can decrease. In particular, if \(F_i = F_k\) then combined maintenance costs will decrease from \(2\kappa(|F_i|)\) to \(\kappa(|F_i|)\) post merger.

**Diseconomies of Scope:** If capabilities do not overlap sufficiently, the combined cost of maintaining capabilities can increase. In particular, if \(F_i \cap F_k = \emptyset\) then by the convexity of \(\kappa\), combined maintenance costs will increase from \(\kappa(|F_i|) + \kappa(|F_k|)\) to \(\kappa(|F_i| + |F_k|)\) post merger.
These effects combine to determine whether a merger is profitable and whether it is socially efficient. The following result shows that consumer surplus always increases in all non-overlapping markets and all newly entered markets following a merger.

**Proposition 2.** Following a merger the equilibrium market price weakly decreases and consumer surplus weakly increases in all non-overlapping and all newly entered markets.

The proof is in Appendix A. Consider competition in a non-overlapping market in which a merger has synergies. After the merger the merged firm will be a stronger competitor and other firms may optimally exit the market reducing competition. Nevertheless consumer surplus always increases. The intuition is that for consumer surplus to be reduced the market price needs to increase following the merger, but in this case no firms would optimally exit.

While the competitive effects of a merger on newly entered and non-overlapping markets are unambiguously good, consumer surplus can increase or decrease in overlapping markets. This will depend on the magnitude of synergies in these markets versus increased concentration and the associated reduction in competition.

It is helpful to sometimes consider a special case of our model in which firms can only (effectively) compete in a market once they have all the attributes associated with that market. Under this *full cover* assumption, for a market $j$, firm $i$’s marginal cost is

$$c_{ij} = \begin{cases} c_j < \alpha_j & \text{if } M_j \subset F_i; \\ \infty & \text{if } M_j \not\subset F_i. \end{cases}$$

Thus a firm enters a market only if it has all the capabilities required by that market. The full cover assumption considerably simplifies the analysis. First, all firms entering a given market must make the same profits. Second, the effects of a merger are simpler. There can be no synergies in overlapping markets or non-overlapping markets so all synergies are confined to newly entered markets. Another implication is that a merger reduces consumer surplus in overlapping markets.

**Proposition 3.** Under the full cover assumption, following a merger the equilibrium market price increases and consumer surplus decreases in all overlapping markets.

In comparison with Proposition 3, Proposition 2 emphasizes the wider competitive benefits of mergers beyond the markets in which the merging firms overlap. Mergers will often improve
competition in some markets while reducing it in others. Merger policy typically focuses only on overlapping markets, placing little weight on the competitive benefits in other markets highlighted by Proposition 2. This can lead to the prohibition of mergers that are overall beneficial for society. While it is hard to evaluate the net effects on consumer surplus of mergers across all markets our model provides a framework for doing so as we show in Example 1 below.

**Example 1.** Suppose that there are attributes $\mathbf{A} = \{a_1, \ldots, a_6\}$. Let the market and firm hypergraphs be

$$H_M = \{M_1 = \{a_1, a_2\}, M_2 = \{a_3, a_4\}, M_3 = \{a_4, a_5\}, M_4 = \{a_5, a_6\}, M_5 = \{a_1, a_3, a_5\}, M_6 = \{a_2, a_4, a_6\}\}$$

$$H_F = \{F_1 = \{a_1, a_2, a_3, a_5\}, F_2 = \{a_3, a_4\}, F_3 = \{a_4, a_5\}, F_4 = \{a_5, a_6\}, F_5 = \{a_1, a_2, a_4, a_6\}\}$$

See Figure 4. We make the full cover assumption. If a merger between $F_1$ and $F_5$ occurs, then the number of firms in each market changes as below:

$$\begin{bmatrix}
M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\
Before merger: & 2 & 1 & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
After merger: & 1 & 2 & 2 & 1 & 1
\end{bmatrix}.$$

It is easy to see that after the merger market $M_1$ gets more concentrated while $M_2, M_3, M_4$ get less concentrated. Assuming that the inverse demand function in all markets is given by $P_j = \alpha - Q_j$ and that the marginal cost of producing is the same in all markets, overall consumers benefit considerably from the merger between $F_1$ and $F_5$.

Regulators in the US, EU and UK, when deciding whether to approve a merger, focus on anticompetitive effects in overlapping markets on a market by market basis. Based only on an analysis of overlapping markets, and with an objective of maximizing consumer surplus, the merger of $F_1$ and $F_5$ would be blocked. An intermediate action a regulatory authority might take in the above case is to approve the merger subject to a remedy requiring that a firm with attributes $a_1$ and $a_2$ be spun off prior to the merger to compete in market $M_1$. Although this would increase consumer surplus further conditional on the merger going through, it would also reduce the profitability of the merger and could make it unprofitable.
The policy implication of Example 1 is that all markets should be considered when evaluating the effects of a merger.

4. AN APPLICATION: STABLE INDUSTRIAL STRUCTURES AND CONSOLIDATION

4.1. Stable hypergraphs and their existence. In practice the industrial structure evolves through mergers and demergers. In this section, we are interested in finding a stable industrial structure in which there are no further pairwise profitable mergers or demergers. As a benchmark, we assume that firms can merge or demerge freely without regulation. In the game we consider, an industrial structure will be unstable if there exists a merger that is jointly profitable or a demerger that is jointly profitable. This is similar to concepts used when studying stable networks (see e.g. Jackson, 2008). One notable difference is that in our setting the set of decision makers changes following a merger or demerger. We also view each capability as a vertex of a hypergraph and each firm is represented by a hyperedge. So in our model, each decision maker is a hyperedge, instead of a node as is typical in network model.

Now we formally define the stability of a firm hypergraph, which by Proposition 1 uniquely pins down entry decisions into all markets and so also defines a stable industrial structure.

**Definition 1.** A firm hypergraph (distribution of firms over the set of capabilities $A$) $H_F \in \mathcal{H}(A)$ is stable if and only if

\[ \text{[Condition for stability]} \]
(1) [merger] there does not exist a pair $F_i, F_k \in H_F$ such that merging $i$ and $k$ to generate $F_l = F_i \cup F_k$ is strictly profitable:

$$\pi_l(H_F') > \pi_i(H_F) + \pi_k(H_F).$$

where $H_F'$ is the firm hypergraph following the merger.

(2) [demerger] For any $F_l \in H$ with $|F_l| > 1$, any demerger generating firms $F_i \subset F_l$ and $F_k = F_l \setminus F_i$ is strictly unprofitable:

$$\pi_l(H_F) > \pi_i(H_F') + \pi_k(H_F'),$$

where $H_F'$ is the firm hypergraph following the demerger.

A firm structure is stable if and only if there is no pair of firms which can earn higher profit after a merger than the total profits before it, and also there is no firm which can be demerged into two smaller firms such that the two demerged firms can earn higher total profits than the profit before the demerger.

To simplify our analysis, we maintain the full cover assumption throughout this section. To prove the existence of stable equilibria (at which there is no further merger or demerger), we need the following lemma.

**Lemma 1.** If there exists a sequence of profitable mergers/demergers generating firm hypergraphs $H_F^{(1)}, H_F^{(2)}, \ldots$ that only ends when there is no profitable merger/demerger, and a function $f : \mathcal{H}(A) \to \mathbb{R}$ that is weakly increasing such that $f(H_F^{(t)}) \leq f(H_F^{(t+1)})$, then either there exists a stable hypergraph or else there exists a sequence of profitable mergers and demergers that cycles and holds the function $f$ constant.

The proof is in Appendix A. It extends a similar and widely utilized result for networks in Jackson and Watts (2001). Next we use Lemma 1 to show that a stable hypergraph exists.

**Proposition 4.** Under the full cover assumption there exists a stable firm hypergraph.

The proof is in Appendix A. The key statistic we use to apply Lemma 1 is the sum over all firms of all their attributes. We show that in a profitable improving path this is weakly decreasing and that no cycle can exist holding this statistic constant. In principle there could exist a cycle of profitable mergers and demergers. Business stealing effects mean that negative externalities can be imposed on others at a given step in the cycle and in principle the sum of profits of
firms undertaking the mergers and demergers over the cycle can be strictly positive. The main challenge in proving Proposition 4 is showing that this cannot happen in our environment.

We now consider which industrial structures will be stable. First we show that if a firm’s set of capabilities can be partitioned such that each partition cell of capabilities is useful for disjoint sets of markets, then it is profitable for that firm to spin off several different firms through demergers according to the partition. We formalize this below.

**Proposition 5.** *If there are no lost synergies demergers are profitable.*

The proof is in Appendix A. It follows from the convexity of $\kappa$ which captures diseconomies of scope. Proposition 5 implies that if a firm can be split into two firms without losing any synergies in any market it operates in, then such a demerger is profitable. Intuitively, the demerged firms will be better able to focus their efforts around the markets they operate in than the conglomerate was able to do before the demerger.

4.2. **Stability and Efficiency.** We now turn to the questions of whether our stable industrial structures will be efficient, and if not why not, by considering an example. We start with the following definition.

**Definition 2.** *We say a firm structure is efficient if it maximizes the social surplus, i.e. the unweighted sum of firms’ profits and consumer surplus.*

**Example 2.** Suppose there are 6 attributes $\{a_1, \ldots, a_6\}$. Let the market hypergraph be

\[
H_M = \{M_1 = \{a_1, a_2\}, M_2 = \{a_2, a_3\}, M_3 = \{a_3, a_4\}, M_4 = \{a_4, a_5\}, M_5 = \{a_5, a_1\}, M_6 = \{a_1, a_2, a_3, a_4, a_5\}\,
\]

and consider the following three firm hypergraphs:

\[
H_1 = \{F_{11} = \{a_1, a_2\}, F_{12} = \{a_2, a_3\}, F_{13} = \{a_3, a_4\}, F_{14} = \{a_4, a_5\}, F_{15} = \{a_5, a_1\}\} ; \\
H_2 = \{F_{21} = \{a_1, a_2, a_3, a_4, a_5\}\} ; \\
H_3 = \{F_{31} = \{a_1, a_2, a_3, a_4, a_5\}, F_{32} = \{a_1, a_2, a_4, a_5\}\} .
\]

See Figure 5. We make the full cover assumption, and also assume that all markets have equivalent demand curves ($P_j = \alpha - Q_j$) and the marginal costs of production is the same in all markets. Then a firm’s profits in a given market depend only on the number of firms entering.
Abusing notation, we therefore let $\pi(k)$ be the profits a firm makes in a market with $k$ firms. We assume that $\pi(1) > \kappa(2)$ so that a monopolist can cover their fixed cost when serving a market $M_k$, for $k = 1, \ldots, 5$. Consider first the firm hypergraph $H_1$ (corresponding to subfigure (B) of Figure 5). Any demergers would prevent the firms in question from competing in any market and are unprofitable as $\pi(1) > \kappa(2)$, so this industrial structure is stable if and only if no two adjacent firms want to merge and no two other firms want to merge. This requires that:

$$\kappa(3) > 2\kappa(2)$$

(1)

and

$$\pi(2) < \kappa(4) - 2\kappa(2).$$

(2)

Consider now the second firm hypergraph $H_2$ (corresponding to subfigure (C) of Figure 5). No mergers are possible as there is a single firm, so this hypergraph is stable if and only if it is not profitable for $F_{21}$ to demerge: (i) into two firms with attributes $(a_1, a_2, a_3)$ and $(a_4, a_5)$; (ii) into two firms with attributes $(a_1, a_2, a_3, a_4)$ and $(a_5)$; (iii) into three firms with attributes $(a_1, a_2, a_3, a_4)$ and $(a_5)$; or (iv) into three firms with attributes $(a_1, a_2, a_3)$, $(a_4)$ and $(a_5)$. Although there are many possible demergers they are all weakly less profitable than one of these four demergers. Thus industrial structure $H_2$ is stable if and only if

$$3\pi(1) > \max\{\kappa(5) - \kappa(2) - \kappa(3), \kappa(5) - \kappa(4) - \kappa(1)\}$$

(3)

and

$$4\pi(1) > \max\{\kappa(5) - 2\kappa(2) - \kappa(1), \kappa(5) - \kappa(3) - 2\kappa(1)\}.$$  

(4)

Firm hypergraph $H_3$ (corresponding to subfigure (D) of Figure 5) is not stable. A merger between firms $F_{32}$ and $F_{31}$ reduces capability maintenance costs by $\kappa(4)$ while increasing profits in markets $M_1$, $M_4$ and $M_5$.

Consumer surplus in a market with $k$ firms is $\frac{k^2}{2}\pi(k)$. Thus the industrial structure represented by $H_3$ is more efficient than $H_1$ and $H_2$ respectively if

$$12\pi(2) - 3\pi(1) > \kappa(5) + \kappa(4) - 5\kappa(2).$$

(5)

and

$$12\pi(2) - \frac{9}{2}\pi(1) > \kappa(4).$$

(6)

We now identify parameter values for which the industrial structures $H_1$ and $H_2$ are simultaneously stable, while industrial structure $H_3$ is more efficient than both but unstable. This requires satisfying inequalities 1–6. Let $\kappa(q) = \phi(q - 1)^2$ where $\phi > 0$ and $q \geq 1$. In the above
hypergraphs, $\kappa(1) = 0$ implies that a firm with a single capability who is therefore unable to enter any market is able to freely dispose their capability and make 0 net profits. We also let $\pi(1) = 12\phi$. Since there is Cournot competition in the second stage of our game with linear demand, $\pi(2) = (4/9)\pi(1)$, and so $\pi(2) = (16/3)\phi$. It is straightforward to verify that for these parameter values inequalities 1–6 are satisfied.

Example 2 shows that for the same parameter values there can be stable industrial structures that are too concentrated and not concentrated enough while a more efficient intermediate industrial structure is unstable. There could be too many small firms or too few big firms in equilibrium from a social planner’s perspective. This motivates regulations that restrict mergers, as is currently practiced, but also potentially intervention to encourage mergers when the industrial structure is fragmented.

4.3. Industry Consolidation. Example 2 demonstrates that industrial structures can be both too fragmented or too concentrated. To gain further insight into stable industrial structures it is informative to consider when mergers are complementary and when they are substitutes: When does a given merger/demerger create stronger or weaker incentives for additional mergers/demergers?
The additional net profits generated by a merger are higher when the markets entered post merger are more concentrated—profits increase by more in each market while the efficiency losses in terms of capability maintenance costs are fixed. Conversely, the additional net profits generated by a demerger are higher when the markets entered pre merger are less concentrated—lost profits are smaller while the reduction in capability maintenance costs is fixed. Both mergers and demergers can increase competition in all markets, decrease competition in all markets or increase competition in some markets while decreasing competition in other markets. Broadly, a merger/demerger that decreases competition will incentivize additional mergers and disincentivize demergers while a merger/demerger that increases competition will disincentivize additional mergers and incentivize demergers.

When there is a single relevant market, mergers reduce the number of competitors and are complements. As a result there can be multiple stable market structures that have very different levels of concentration. Moreover, one of these market structures may be considerably more efficient than the others. We show this in the simple example below. In this example a very fragmented market structure is stable, in which no firms are especially strong or capable competitors, and a much more concentrated market structure with strong and more capable competitors is also stable. The less concentrated hypergraph generates lower aggregate profits and lower consumer surplus. Moreover, starting form this fragmented industry structure, rapid consolidation can occur if there is a change in beliefs about whether other firms are going to merge.

**Example 3.** Suppose there are $u$ attributes $\{a_1, \ldots, a_u\}$ and for simplicity we suppose there is a single market with inverse demand curve $P = \alpha - Q$. The market hypergraph is

$$H_M = \{M_1 = \{a_1, \ldots, a_u\}\}.$$ 

Consider now the following two firm hypergraphs:

$$H_1 = \left( F_{si} = \{a_i\} \right)_{s=1,\ldots,k, i=1,\ldots,u};$$

$$H_2 = \left( F_s = \{a_1, \ldots, a_u\} \right)_{s=1,\ldots,k'}.$$ 

The firm hypergraph $H_1$ consists of $ku$ small firms each with a single capability and the firm hypergraph $H_2$ consists of $k$ firms each with $u$ capabilities. Firms’ marginal costs for the unique market $M_1$ are decreasing in their capabilities. For simplicity let $F_i$’s marginal cost be $c_i =$
\( \alpha / (|F| + u) \) and let the fixed cost of maintaining \( q \) capabilities be \( \kappa(q) = (q - 1)^e \) for \( q \geq 1 \) and some \( e > 1 \).

We assume that merger regulations prevent any further mergers when the firm hypergraph is \( H_2 \). Then for \( e \) close to 1 and \( k \geq 3u \), the following can be established:

(i) \( H_1 \) is stable;
(ii) \( H_2 \) is stable;
(iii) \( H_2 \) generates higher consumer surplus than \( H_1 \).

The calculations are in Appendix B but some intuition follows. Consider first the fragmented hypergraph. Suppose two firms with the same capabilities merge. As production costs remain unchanged, the combined profits of the firms after the merger will be the same as their combined profits were one of them to exit the market. If one exits the other’s profits increase a little, but not much because there are still many firms present, and the exiting firm loses the profits they would otherwise have received. When the market is sufficiently fragmented the net effect is negative. Mergers between firms with different capabilities induce some additional profits through lowering marginal cost, but also require maintaining additional capabilities which is costly. Overall these mergers are not profitable either.

Consider now hypergraph \( H_2 \). By assumption additional mergers are prohibited. So the hypergraph will be stable as long as demergers are unprofitable. This is the case because demerged firms are weaker competitors (have higher marginal costs) in the market than the other firms, and collectively extract lower profits. The demerger will reduce combined fixed costs, but when \( e \) is close to 1 these effects are small and the former effect will dominate making demergers unprofitable. Finally, the consumer surplus generated by the two hypergraphs is

\[
CS(H_1) = \frac{1}{2} \left( \frac{ku}{ku + 1} \right)^2 \left( 1 - \frac{1}{1 + u} \right)^2 \alpha^2,
\]

\[
CS(H_2) = \frac{1}{2} \left( \frac{k}{k + 1} \right)^2 \left( 1 - \frac{1}{2u} \right)^2 \alpha^2.
\]

As \( u > 1 \), when \( k \) is sufficiently large \( CS(H_1) < CS(H_2) \) and \( H_2 \) generates more consumer surplus than \( H_1 \). Total profits are also higher for \( H_2 \) than \( H_1 \).

As both \( H_1 \) and \( H_2 \) are stable, sudden transitions are possible. Starting form \( H_2 \), if firms anticipate that the industry will consolidate through a sequence of mergers these beliefs are self
fulfilling and the mergers anticipated will be profitable. In this way, through rapid industry consolidation we can move from hypergraph $H_1$ to hypergraph $H_2$. Much empirical evidence documents waves of industry consolidation. Viewing firms as sets of capabilities provides a simple and natural formal explanation for this.

Example 3 shows that viewing firms as sets of capabilities can help explain waves of industry consolidation as the transition between multiple equilibria in which firms become more capable competitors through their acquisitions. This is related to the work of Sutton (1991, 2001). The main difference is that we do not require exogenous changes in technology or demand to precipitate the changes in industrial structure and that additional entry is limited by the scarcity of capabilities rather than through the accumulation of sunk costs as a barrier to entry. Further, in our model mergers lower marginal costs incentivizing further mergers, while in Sutton’s work sunk cost accumulation lowers marginal cost and incentivizes mergers.

5. Conclusions

In this paper we combine the two main theories of competitive advantage from the management literature into a single framework and provide mathematical foundations. We do so by modeling firms and markets as sets of capabilities where a firm’s marginal cost in a given market is decreasing in the intersection of its capabilities with those that would be useful in the market. Given these heterogeneous marginal costs firms decide which markets to enter and how much to produce in each market. The equilibrium is unique pinning down market prices and firm profits. This representation of firms and markets as hypergraphs helps formalize the notion of synergies obtained through mergers. By linking firms’ marginal costs in each market to their relevant capabilities we also model natural entry barriers: Firms need to acquire a suitable set of capabilities before they can compete effectively in a given market. Our approach to merger analysis emphasizes the potential efficiency gains that be obtained in non-overlapping markets and calls for a more holistic analysis than is currently set out in the US merger guidelines and widely practiced. Also, by defining what it means for an industrial structure to be stable and in equilibrium, we explain industry consolidation as the transition from a less profitable but stable market structure to a more profitable (and possibly higher consumer surplus) market structure with fewer but more capable firms.

Our model is simple and intended as a first step. On the theoretical side there is much that can be generalized. Firms’ competitiveness can more generally be defined by the set of relevant
capabilities they have for a market rather than the number. Such an approach would lead to a partial ordering (in set inclusion with respect to capabilities) of the competitiveness of firms in each market. A more dynamic approach than has been taken in this paper is also clearly called for. Markets evolve over time in terms of the capabilities they value and in terms of their size. Explicitly modeling this would further emphasize the role of management in terms of acquiring and developing the capabilities of firms. On the empirical side there is also much to do. Are mergers that generate more complementary sets of capabilities more profitable in the long run? Do newly merged firms use their new found capabilities to enter new markets? Do firms become stronger competitors after mergers in non-overlapping markets?

The model can also be applied to study many problems we have not mentioned. For example, government procurement contracts can also be viewed as demanding a set of capabilities. In some cases no one firm has all the required capabilities and bidding for the contract occurs through endogenously formed consortia of firms. An interesting theoretical and empirical question is then what can be said about the competitiveness of these bidding processes? Viewing firms as sets of capabilities might help address questions like this.

REFERENCES


A.1. Proof of Proposition 1.

Proof. Firm $i$’s total profits are given by:

$$\pi_i(q_i) = \sum_{j=1}^{m} \pi_{ij}(q_{ij}) - \kappa(|F_i|).$$

As $F_i$ is fixed, there is nothing that links firm $i$’s decisions across markets and for all markets $j$,

$$\frac{d\pi_i(q_i)}{dq_{ij}} = \frac{d\pi_{ij}(q_{ij})}{dq_{ij}}.$$ 

Thus $i$’s profit maximizing output decision in market $j$ is independent of $i$’s output decisions in other markets. This implies that $i$’s output decision in each market can be considered in isolation and it is sufficient to show that there is a unique Nash equilibrium in a given market. Although this result is standard, we prove it below for completeness.

As we are considering a fixed market we suppress the market index $j$. Without loss of generality, we order the marginal cost of all firms: $c_1 \leq c_2 \leq \ldots \leq c_n$. Now we will show that there exists a unique Nash equilibrium $(q_i^*)_{1 \leq i \leq n}$ such that:

1. if $\alpha < c_1$, then $q_i^* = 0$ for all $1 \leq i \leq n$.
2. if $\alpha > c_1$, then
   $$q_i^* = \begin{cases} \frac{p - c_i}{\beta}, & \text{if } i \leq i^*; \\ 0, & \text{if } i > i^*, \end{cases}$$
   where $i^* \in \{1, \ldots, n\}$ is uniquely identified by the marginal costs$^6$, and $p$ is the market equilibrium price:
   $$p = \frac{\alpha + c_1 + \ldots + c_{i^*}}{i^* + 1}. \tag{8}$$

Linear inverse demand function implies that each firm has a quadratic (necessarily concave) profit function:

$$\pi_i = (\alpha - \beta Q - c_i)q_i = -\beta q_i^2 + (\alpha - \beta Q - c_i)q_i.$$

$^6$Let $C(i) = \frac{c_1 + \ldots + c_i}{i+1}$. It is easy to prove: (1) if $C(i) > c_i$, then $C(k) > c_k$ for any $k < i$; (2) if $C(i) \leq c_i$, then $C(k) \leq c_k$ for any $k > i$. Then this implies that there exists a unique $i^* \in \{1, \ldots, n\}$ such that either $1 \leq i^* \leq n - 1$ satisfies $C(i^*) > c_i$, and $C(i^* + 1) \leq c_{i^* + 1}$, or $i^* = n$ satisfies $C(i^*) > c_i$. 

If \((q^*_i)_{1 \leq i \leq n}\) is a Nash equilibrium, then \(q^*_i > 0\) if and only if \(\pi_i(q^*) > 0\). Moreover, at \((q^*_i)_{1 \leq i \leq n}\), it follows from the first order conditions that

\[
q^*_i > 0 \quad \text{implies} \quad \left. \frac{d\pi_i}{dq_i} \right|_{q_i=q^*_i} = -\beta q_i + \alpha - \beta Q - c_i = 0, \quad (9)
\]

\[
q^*_i > 0 \quad \text{if and only if} \quad \left. \frac{d\pi_i}{dq_i} \right|_{q_i=q^*_i} = -\beta q_i + \alpha - \beta Q - c_i > 0, \quad (10)
\]

and

\[
q^*_i = 0 \quad \text{if and only if} \quad \left. \frac{d\pi_i}{dq_i} \right|_{q_i=0} = -\beta q_i + \alpha - \beta Q - c_i \leq 0. \quad (11)
\]

If \(\alpha \leq c_1\), then \(\left. \frac{d\pi_i}{dq_i} \right|_{q_i=0} \leq \alpha - c_i \leq 0\). By (11), \(q^*_i = 0\) for any \(i\). Therefore, this is also the unique Nash equilibrium.

Let us now consider the case \(\alpha > c_1\). First it is easy to check that (7) is a Nash equilibrium. Indeed, for \(i \leq i^*\), (10) holds, and hence \(q^*_i = (p - c_i)/\beta\) is a best response; for \(i \geq i^*\), (11) holds, and hence \(q^*_i = 0\) is a best response. Next we need to prove that (7) is the unique Nash equilibrium. By (9), we can write any Nash equilibrium as \(q_i = \max\{0, (p - c_i)/\beta\}\). It is easy to see that for a given \(p\), there exists a unique \((q_i)_{1 \leq i \leq n}\). Now suppose there are two equilibria \((q^1_i)_{1 \leq i \leq n}\) and \((q^2_i)_{1 \leq i \leq n}\) corresponding to \(p^1\) and \(p^2\) respectively. Without loss of generality, we can assume \(p^1 > p^2\). Then

\[
q^1_i = \max \left\{ 0, \frac{p^1 - c_i}{\beta} \right\} \geq \max \left\{ 0, \frac{p^2 - c_i}{\beta} \right\} = q^2_i, \quad \forall i.
\]

Therefore, \(Q^1 = \sum_{i=1}^{n} q^1_i \geq \sum_{i=1}^{n} q^2_i = Q^2\). This implies \(p^1 \leq p^2\), which is a contradiction. \(\square\)


Proof. We first prove a key lemma. For a given market \(j\) consider what happens when the relevant capabilities of some firms improve. Recall that \(\theta_{ij} = |F_i \cap M_j|\). We say there is an improvement in the capabilities of the firms in market \(j\) if \(\theta_{ij}\) weakly increases for all \(i\).

Lemma 2. For market \(j\), if there is an improvement in the capabilities of the firms in it, then the market price weakly decreases.

Proof. As \(c_{ij}\) is a decreasing function of \(\theta_{ij}\), firm \(i\)'s marginal cost decreases when \(\theta_{ij}\) increases. Let \(c_{ij}\) be \(i\)'s marginal cost before the improvement in \(i\)'s capabilities and let \(c'_{ij}\) be \(i\)'s marginal cost afterwards. Since we consider a fixed market, we can suppress the market index \(j\). Then holding entry fixed at a set of firms \(\{1, \ldots, i^*\}\), the new market price is
Thus holding entry fixed the market price decreases:

\[
\frac{\alpha + c_1' + \ldots + c_i'}{i^* + 1} - \frac{\alpha + c_1 + \ldots + c_i^*}{i^* + 1} = \sum_{i=1}^{i^*} \frac{c_i' - c_i}{i^* + 1} < 0.
\]

However, entry may not remain fixed in equilibrium. There are two cases. First there may exist a firm who entered the market before the change and chooses not to enter after the change. Let \(i'\) be such a firm. By definition, before the improvement in capabilities, we have that \(p > c_{i'}\). If \(i'\) no longer finds it profitable to enter then we must have \(p > c_{i'} \geq c_i' > p'\) and so the market price decreases, increasing consumer surplus.

The second case is the one in which entry increases in the strong set order, so that new firms participate in the market as well as all those that previously participated. Let \(\hat{i} > i^*\) be the new marginal firm. We then have that

\[
p' = \frac{\alpha + c_1' + \ldots + c_i'}{\hat{i} + 1} = \frac{\alpha + c_1' + \ldots + c_{i^*}' + \left(\frac{i^* + 1}{\hat{i} + 1}\right) c_{i^*+1}' + \ldots + c_i'}{\hat{i} + 1}
\]

Thus \(p'\) is a weighted average of the market price \(\hat{p}\) that would obtain if entry was held fixed and the average marginal cost of the new firms that enter. As all of these firms must have marginal costs below \(p'\) (for entry to be profitable), if follows that \(\hat{p} > p'\). Thus as \(p > \hat{p}\) we conclude that \(p > p'\) and the market price must decrease. Thus, regardless of whether entry changes the market price decreases following an improvement in the capabilities of firms. This also increases consumer surplus.

The proof of Proposition 2 follows almost immediately from Lemma 2. A merger weakly improves the merged firm’s set of capabilities in both a newly entered and non-overlapping market. Then
by Lemma 2 the market price will weakly decrease and consumer surplus weakly increases regardless of whether there is exit. □


Proof. Suppose firms $i$ and $k$ merge to firm $l$. Consider an overlapping market $j$. Under the full cover assumption for any firm $x$ competing in market $j$, $F_x \cap M_j = M_j$. Moreover, in equilibrium all firms $x$ with capabilities $F_x \supseteq M_j$ will enter market $j$. Thus, if pre merger there are $y$ firms competing in market $j$ and these $y$ firms have the same marginal cost, post merger there will be $y - 1$ firms competing in market $j$ with the same marginal cost as before. Thus the market price will increase and consumer surplus will be reduced. □


Proof. We use an idea similar to the energy function in physics. As there is a finite set of attributes, there is also a finite set of possible hypergraphs. Since $f$ is weakly increasing along the sequence of hypergraphs, $f(H^{(t)})$ must converge to a constant in finite steps. When the convergence happens, there are two possibilities. Either the sequence ends or it cycles. Suppose it ends at step $\tau$. Then there can be no profitable merger or demerger and the hypergraph $H^{(\tau)}_F$ is stable. Alternatively the sequence includes a cycle. There exists a sequence of profitable mergers and demergers that cycles. Moreover, as $f$ is weakly increasing each step of the cycle must hold $f$ constant. □


Proof. Our proof consists of two steps:

1. Step 1: we identify a function that is weakly increasing at each step in any sequence of profitable mergers and demergers and applying Lemma 1 argue that either a stable hypergraph exists or else a cycle of profitable mergers/demergers exists.

2. Step 2: we show that such a cycle cannot exist.

Define a function

$$\Phi_t = \sum_{F_i \in H^{(1)}_F} |F_i| - \sum_{F_i \in H^{(t)}_F} |F_i|.$$  \hfill (12)

Thus $\Phi_t$ is the total number of capabilities summed over all firms in period 1 less the total number of capabilities summed over all firms in period $t$. Notice that $\Phi_t$ weakly increases
for any sequence of mergers or demergers. If a merger between two firms with overlapping capabilities occurs, $\Phi_t$ strictly decreases and, by definition, a demerger doesn’t change $\Phi_t$. We can therefore apply Lemma 1 to conclude that either there exists a stable hypergraph or else there exists a sequence of profitable mergers and demergers that cycles such that $\Phi_t = \Phi_{t+1}$ for all $t$. Moreover, in this case

(i) there must be both mergers and demergers in the cycle.

(ii) in any merger in this cycle, the merging firms must have disjoint set of capabilities.

We now prove Step 2 by contradiction. Suppose that the set of firms reaches a cycle with length $c$ by step $t$ such that $H^{(c+t)}_F = H^{(t)}_F$. We denote by $N_j^{(t)}$ the number of firms in market $j$ at step $\tau$. Let $N^{(\tau)} = (N_1^{(\tau)}, \ldots, N_m^{(\tau)})$. Observe that $N^{(\tau)}$ has the following properties for $\tau \geq t$:

(1) if it is a merger that causes the transition from step $\tau$ to step $\tau+1$, then $N_j^{(\tau+1)} - N_j^{(\tau)} = 1$ if and only if the new merged firm enters market $j$;

(2) if it is a demerger that causes the transition from step $\tau$ to step $\tau+1$, then $N_j^{(\tau+1)} - N_j^{(\tau)} = -1$ if and only if neither of the demerged firms continue to participate in market $j$.

Let $\Delta C_\tau$ be the set of coordinates (markets) over which $N^{(\tau)}$ is different from $N^{(\tau+1)}$, i.e. $\Delta C_\tau = \{j|N_j^{(\tau)} \neq N_j^{(\tau+1)}\}$. Let $\Delta \kappa_\tau$ be the fixed cost change of either a merger or demerger from period $\tau$ to $\tau+1$. As $\Phi_\tau = \Phi_{\tau+1}$ and $\kappa$ is convex, $\Delta \kappa_\tau$ is positive for a merger and negative for a demerger. Let $T_M \subset \{t, \ldots, t+c\}$ be the set of steps $\tau$ such that there is a merger from $\tau$ to $\tau+1$, and $T_D \subset \{t, \ldots, t+c\}$ be the set of steps $\tau$ such that there is a demerger from $\tau$ to $\tau+1$.

As in the cycle all mergers are between firms with disjoint capabilities, a merger is profitable if and only if the profits obtained from entering new markets is greater than the increase in fixed costs. That is,

$$\Delta \kappa_\tau < \sum_{j \in \Delta C_\tau} \pi_j(N_j^{(\tau+1)}), \quad \text{for all } \tau \in T_M,$$

(13)

where $\pi_j(N_j^{(\tau+1)}$ is the profit each firm earned in market $j$ when there are $N_j^{(\tau+1)}$ firms in that market. Notice that in (13), we have suppressed the index of the firms since under the full cover assumption the profit of a firm only depends on the total number of firms in a market.
Similarly, for a demerger, the decreased fixed cost is greater than profit loss of exit. That is,

$$-\Delta \kappa_\tau > \sum_{j \in \Delta C_\tau} \pi_j(N_j^{(\tau)}), \quad \text{for all } \tau \in T_D. \quad (14)$$

Summing up (13) over all mergers and (14) over all demergers in a cycle, we have that total change of (increased) fixed cost following mergers is less than total profits of entry:

$$\sum_{\tau \in T_M} \Delta \kappa_\tau < \sum_{\tau \in T_M} \sum_{j \in \Delta C_\tau} \pi_j(N_j^{(\tau+1)}) \quad (15)$$

and total change of (decreased) fixed cost following demergers is greater than total profit loss of exit:

$$-\sum_{\tau \in T_D} \Delta \kappa_\tau > \sum_{\tau \in T_D} \sum_{j \in \Delta C_\tau} \pi_j(N_j^{(\tau)}) \quad (16)$$

Since the total change of fixed cost following mergers on a cycle is equal to the total change of fixed cost following demergers:

$$\sum_{\tau \in T_M} \Delta \kappa_\tau + \sum_{\tau \in T_D} \Delta \kappa_\tau = 0,$$

by (15) and (16), the total profit loss of exit is less than total profits of entry:

$$\Psi = \sum_{\tau \in T_M} \sum_{j \in \Delta C_\tau} \pi_j(N_j^{(\tau+1)}) - \sum_{\tau \in T_D} \sum_{j \in \Delta C_\tau} \pi_j(N_j^{(\tau)}) > 0. \quad (17)$$

Next, we will prove that total profit loss of exit is actually equal to total profits of entry, and then this gives us a desired contradiction.

To prove that total profit loss of exit is equal to total profits of entry, we compute their difference:

$$\Psi = \sum_{j=1}^m \left[ \sum_{\tau:N_j^{(\tau+1)}=N_j^{(\tau)}+1} \pi_j(N_j^{(\tau+1)}) - \sum_{\tau:N_j^{(\tau+1)}=N_j^{(\tau)}-1} \pi_j(N_j^{(\tau)}) \right] = 0. \quad (18)$$

The second equality is because, for a fixed market $k$, $N_j^{(c+t)} = N_j^{(t)}$ and the values of $N_j^{(\tau+1)} - N_j^{(\tau)}$ are taken from a set $\{-1, 0, 1\}$. For each firm that realizes a profit from entering the market $k$ with $N$ firms following a merger, there is some firm that realizes an equal loss through a demerger that reduces the number of firms in market $k$ from $N + 1$ to $N$. Equation (18) contradicts equation (17). We thus conclude that there cannot exist a cycle of profitable mergers and demergers. Lemma 1 therefore implies that there exists a stable hypergraph. □

Proof. We seek to show that if there exists a non-trivial partition of $F_i$ into $F_1, F_2, \ldots, F_s$ ($F_k \cap F_l = \emptyset$ for any $1 \leq k \neq l \leq s$ and $\bigcup_{k=1}^{s} F_k = F_i$) such that $(F_i \cap M_j) = (F_k \cap M_j)$ for some $k = 1, \ldots, s$ and all markets $M_j$ entered, then it is profitable for $i$ to demerge into $s$ smaller firms with sets of capabilities $F_1, \ldots, F_s$ respectively.

The spin-off does not change the number of industries the firm operates, and hence its profit. Then the proof trivially follows from the reduction of fixed costs: $\kappa(|F_i|) \geq \kappa(|F_1|) + \ldots + \kappa(|F_s|)$.

□

Appendix B. Calculations Supporting Example 3

Any two firms $F_{si}, F_{ti} \in H_1$ do not merge because they lose revenue (profits without counting fixed cost) when $ku$ is large (the business stealing effect is large), and they do not save fixed cost ($\kappa(|F_{si}|) = \kappa(|F_{ti}|) = 0$). Any two firms $F_{si}, F_{tj} \in H_1 (i \neq j)$ do not merge because the merger loses profits for large $u$ (the business stealing effect is large, the synergy is small and the increased fixed costs from merger is large). So $H_1$ is stable.

To prove $H_2$ is stable, we just need to check that any firm does not want to demerge. Suppose that a firm of $H_2$ demerges to two smaller firms with $u_1$ and $u_2 = u - u_1$ number of attributes respectively. Then the profit of the two demerged firms without counting fixed costs (denoted by $P$) can be calculated:

$$P = \left(1 + \frac{(k-1)\frac{1}{u_1} + \frac{1}{u_1+u_1} + \frac{1}{u_2+u}}{k+2} - \frac{1}{u_1+u} \right)^2 + \left(1 + \frac{(k-1)\frac{1}{u_2} + \frac{1}{u_1+u} + \frac{1}{u_2+u}}{k+2} - \frac{1}{u_2+u} \right)^2.$$

Then we can estimate:

$$P \leq \left( \frac{2 + (k-1)\frac{1}{u} - k \left[ \frac{1}{u_1+u} + \frac{1}{u_2+u} \right]}{k+2} \right)^2 \leq \left( \frac{2 - (\frac{k}{3} + 1)\frac{1}{u}}{k+2} \right)^2.$$

Further, for $k \geq 3u$,

$$P \leq \left( \frac{2 - (\frac{k}{3} + 1)\frac{1}{u}}{k+2} \right)^2 \alpha^2 < \left( \frac{1 - \frac{1}{2u}}{k+1} \right)^2 \alpha^2.$$

This implies that without counting fixed costs, the aggregated profit of demerged firms is less than the firm’s profit pre demerger. When $e$ is close to 1, the demerger cannot save much fixed cost. Therefore, the net effect of the demerger is negative and it is not profitable for the firm to demerge. This proves that $H_2$ is stable.