A NETWORK APPROACH TO PUBLIC GOODS

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Abstract. Suppose each of several agents can exert costly effort that creates nonrival, heterogeneous benefits for some of the others. How do negotiated outcomes depend on the heterogeneities? To study this question, we construct a matrix—or a weighted, directed network—that describes the marginal benefits agents can confer on one another. We first show that an outcome is Pareto efficient if and only if the largest eigenvalue of the marginal benefits matrix evaluated at that outcome is equal to 1. A corollary describes the players whose participation is essential for any Pareto improvement. We then show that an agent's contribution in any Lindahl equilibrium corresponds to his eigenvector centrality in the benefits network. This provides a new market foundation and interpretation for widely-used network statistics, and, conversely, a network perspective on price equilibria. Finally, we discuss strategic foundations for Lindahl outcomes in our setting, explaining when negotiations will result in contributions that correspond to network centralities.

When economic agents produce public goods, mitigate public bads, or more generally create externalities, the incidence of those externalities is often nonuniform. A nation's economic policies—e.g., implementing a fiscal stimulus, legislating environmental regulations, or reducing trade barriers—benefit foreign economies differently. Investments by a firm in research yield different spillovers for various producers and consumers. Cities' mitigation of pollution matters most for neighbors sharing the same environmental resources. And within a firm, an employee's efforts (e.g., toward team production) will benefit other employees to different degrees. How do such asymmetries affect different agents' levels of effort? Whose effort is particularly critical?

The analysis of such questions will, of course, depend on the type of economic outcome that is considered—in other words, on the solution concept. An active research program has focused on one-shot Nash equilibria of public goods games where agents unilaterally choose how much effort to put forth; see, e.g., Ballester, Calvó-Armengol, and Zenou

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(2006), Bramoullé, Kranton, and d'Amours (2013), and Allouch (2015, 2013). These works model nonuniform externalities via particular functional forms in which a network is a set of parameters. Links describe the pairs of players who directly affect each other's payoffs or incentives, as when two people collaborate on a project. The main results then characterize equilibrium effort levels via certain network statistics. Since these statistics are major subjects of study in their own right, the connection yields a rich set of intuitions, as well as analytical techniques for comparative statics, identifying "key players," and various other policy analyses.[1]

The static Nash equilibrium is a useful benchmark, relevant in cases where decisions are unilateral, with limited scope for repetition or commitment. Under this solution concept, agents do not internalize the externalities of their effort. Indeed, in a public goods game, players free-ride on the contributions of others, leading to a classic "tragedy of the commons" problem. The resulting inefficiencies can be substantial; in the context of problems like climate change, some argue they are disastrous. In cases where large gains can be realized by improving on the unilateral benchmark, institutions arise precisely to foster multilateral cooperation. Global summits,[2] the World Trade Organization, research consortia, and corporate team-building practices all aim to mitigate free-riding by facilitating commitment. Therefore, rather than working with the static Nash equilibrium, this paper focuses on the complementary benchmark of *Pareto efficient* public goods provision in the presence of nonuniform externalities.

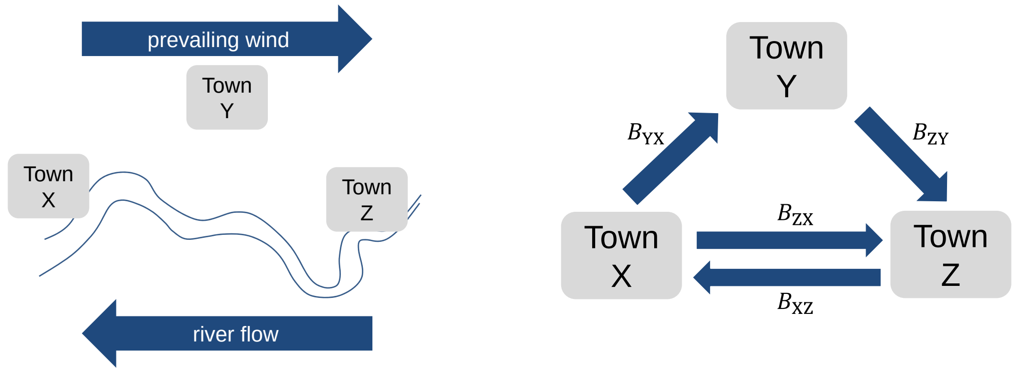
Our contribution is to show that taking a network perspective on the system of externalities sheds new light on efficient outcomes and the scope for efficient cooperation. This approach yields two kinds of results. First, it provides a new kind of characterization of when Pareto improvements are possible, along with new intuitions. Second, it examines which efficient outcomes might be reached through negotiations and how agents' contributions at these outcomes depend on the network of externalities. The insights that the analysis generates can help address questions such as who should be given a seat at the negotiating table or admitted to a team. In contrast to the previous work mentioned above, our characterizations are non-parametric: A "network" representation of externality incidence arises naturally from general utility functions. Finally, we provide new economic foundations and intuitions for statistics that are widely used to measure the centrality of agents in a network by relating these statistics to concepts such as Pareto weights and market prices.

1. Example and Roadmap

We now present the essentials of the model in a simplified example. Section 2 defines all the primitives formally in the general case. Each agent has a one-dimensional effort/action choice, $a_i \geq 0$; it is costly for an agent to provide effort, which yields positive, non-rival externalities for (some) others. For a concrete example, suppose there are three towns: X,

[1] There are many empirical applications of these results. See, for example, Calvó-Armengol, Patacchini, and Zenou (2009) and Acemoglu, García-Jimeno, and Robinson (2014). Other theoretical papers that examine different issues related to the provision of public goods on networks include Bramoullé and Kranton (2007) and Galeotti and Goyal (2010).

[2] For example, it was at the Rio Earth Summit that the first international treaty on climate change was hammered out. There have been several other summits and associated climate change agreements since.



(A) Town locations. (B) Benefits network.

FIGURE 1. In this illustration of the framework, towns benefit from each other' pollution reduction. Town i benefits from j's pollution reduction if pollution travels from j to i, which can happen via the wind or via the flow of the river. Let $B_{ij} = \partial u_i / \partial a_j$ be the marginal benefit to i from j's reduction (per unit of j's marginal cost, which is normalized to be 1). These numbers may vary with the action profile, (a_X, a_Y, a_Z).

Y and Z, located as shown in Figure 1a, each generating air and water pollution during production. Because of the direction of the prevailing wind, the air pollution of a town affects only those east of it. A river flows westward, so Z's water pollution affects X but not Y, which is located away from the river.

Town i can forgo $a_i \geq 0$ units of production at a net cost of a dollar per unit, reducing its pollution and creating positive externalities for others affected by that pollution. The important part of this assumption is that the value of forgone production outweighs private environmental benefits; this assumes that the net private benefits of increasing effort have already been exhausted if they were present. Let $u_i(a_{\mathrm{X}}, a_{\mathrm{Y}}, a_{\mathrm{Z}})$ denote i's payoff.

Suppose the leaders of the towns attend a summit to try to agree on improvements that will benefit all of them. We focus on like-for-like agreements, in which agents trade favors by providing the public good of effort to each other, which is a relevant case for many practical negotiations.[3] We begin by studying the set of all outcomes that are Pareto efficient and how they can be characterized in terms of the structure of externalities.

The conceptual platform for this—and for the rest of the paper—is to analyze a matrix whose entries record the marginal benefits per unit of marginal cost that each agent can confer on each other, for a given action profile. In our example, the entries of this matrix are $B_{ij}(\mathbf{a}) = \frac{\partial u_i}{\partial a_j}(\mathbf{a}) / \left(-\frac{\partial u_i}{\partial a_i}(\mathbf{a}) \right) = \frac{\partial u_i}{\partial a_j}(\mathbf{a})$ for $i \neq j$, since we have normalized all marginal costs of effort to be 1. The diagonal terms of the matrix are set to 0, so that it records only the externalities between players, and not their own costs. This *benefits matrix* can be equivalently represented as a (weighted, directed) network, where a link from i to j

[3]This also parallels the above-mentioned papers regarding games on networks, which study one-dimensional contributions. In Section OA2 of the Online Appendix we consider transfers: the very simple benchmark of quasi-linear preferences, as well as the general case, where our main results have natural analogues.

represents that i's effort affects j's welfare (see Figure 1b). That network is the key object whose statistics we will relate to economic outcomes.

Our first result shows that *an interior action profile* $\mathbf{a}$ *is Pareto efficient if and only if* 1 *is a largest eigenvalue of* $\mathbf{B}(\mathbf{a})$. The reason for this is as follows: The matrix $\mathbf{B}(\mathbf{a})$ is a linear system describing how investments translate into returns at the margin. Consider a particular sequence of investments: In Figure 1b, Z can increase its action slightly and provide a marginal benefit to X. Then X, in turn, can "pass forward" some of the resulting increase in its utility, investing costly effort to help Z and Y. Finally, Y can also pass forward some of the increase in his utility by increasing his action, creating further benefits for Z. If they can all receive back more than they invest in such a multilateral adjustment, then the starting point is not Pareto efficient. It is in such cases that the linear system $\mathbf{B}(\mathbf{a})$ is "expansive": There is scope for everyone to get more out of it than they put in. And an expansive system is characterized by having a largest eigenvalue exceeding 1. If the largest eigenvalue of $\mathbf{B}(\mathbf{a})$ is less than 1, then everyone can be made better off by reducing investment. As a result, the interior Pareto efficient outcomes have a benefits matrix with a largest eigenvalue exactly equal to 1. Section 3.1 makes this discussion rigorous (see Proposition 1). Section 3.2 develops some of its interpretations and applications. It fleshes out the idea, already suggested by the informal argument, that cycles in the benefits network are critical for Pareto improvements and, correspondingly, that they determine the size of the largest eigenvalue. Lastly, it discusses a simple algorithm to find the players who are essential to a negotiation—in the sense that without their participation, there is no Pareto improvement on the status quo. They are the ones whose removal causes a large disruption of cycles in the benefits network, as measured by the decrease in its largest eigenvalue.

One point on the Pareto frontier that is of particular interest is the classic Lindahl solution that completes the "missing markets" for externalities. If all externalities were instead tradable goods, we could consider the Walrasian outcome and identify the set of prices at which the market clears. If personalized taxes and subsidies equivalent to these prices could be charged in our public goods setting, then the same efficient outcome would obtain. Such an allocation is called a Lindahl outcome.[4] Our second main result characterizes the Lindahl outcomes in terms of the *eigenvector centralities* of nodes in the marginal benefits network.

Eigenvector centrality is a way to impute importance to nodes in a network. Given a network $\mathbf{G}$, the eigenvector centrality of node i satisfies:[5]

$$c_i \propto \textstyle\sum_j G_{ij} c_j. \tag{1}$$

This equation says that i's centrality is proportional to a weighted sum of its neighbors' centralities. Thus the definition is a fixed-point condition and, in vector notation, becomes $\mathbf{c} = \mathbf{G}\mathbf{c}$, so that the centralities of players are a right-hand eigenvector of the network $\mathbf{G}$. The measure captures the idea that central agents are those with strong connections to other central agents; equation (1) is simply a linear version of this statement. The notion of eigenvector centrality recurs in a large variety of applications in various disciplines,

[4]A formal definition of Lindahl outcomes appears in Section 4.

[5]Under a network connectedness condition, these equations pin down relative centralities uniquely.

and our main conceptual contribution is to relate it in a simple and general way to price equilibria. At the end of this section we expand briefly on this point.

In our setting, we define the eigenvector centrality action profiles as

$$\mathbf{a} = \mathbf{B}(\mathbf{a})\mathbf{a}. \tag{2}$$

Theorem 1 (in Section 4) establishes the equivalence between the eigenvector centrality action profiles and the Lindahl outcomes. One way to interpret condition (2) is that i contributes in proportion to a weighted sum of others' contributions a_j; the weights are i's marginal valuations of the efforts of other agents.

Section 5 shows that the eigenvector condition (2) can be expressed in terms of walks in the benefits network, with the more central agents being those who sit at the locus of larger direct and indirect incoming marginal benefit flows. This relates price-based outcomes to the structure of the network. Building on this interpretation, Theorem 1 is applied to study a problem in which a team has to decide which new member to admit. As another application, we study cases in which we can calculate the eigenvector centrality action profiles explicitly. This, in turn, is used this to give several important network centrality measures an economic microfoundation and interpretation in terms of price equilibria.[6] This exercise echoes the general conceptual message of Theorem 1—that there is a close connection between markets and network centrality—but for a wider range of network statistics and in a case where centralities can be computed explicitly in terms of exogenous parameters.

The Lindahl equilibria are of interest on more than just normative grounds. In Section 6 we provide strategic foundations for this solution concept by building on existing results. First, using ideas from the literature on Walrasian bargaining (especially Dávila, Eeckhout, and Martinelli (2009) and Penta (2011)), we consider a model of multilateral negotiations that selects the Lindahl outcomes from the Pareto frontier. We then apply ideas of Hurwicz (1979a,b) on implementation theory to show that the Lindahl equilibria are those selected by all mechanisms that are optimal in a certain sense. Finally, we note that, in our setting, Lindahl outcomes are robust to coalitional deviations—i.e., are in an appropriately defined core.

We close by putting our work in a broader context of research on networks and centrality, beyond the most closely related papers on externalities and public goods. The interdependence of economic interactions is a defining feature of economies. When a firm does more business it might employ more workers, who then have more income to spend on other goods, and so on. Eigenvector centrality (equation 1) loosely captures this idea. While in broad terms prior results suggest a connection between eigenvector centrality and economic outcomes, those results' reliance on parametric assumptions leaves open the possibility that the connection exists only in special cases, and is heavily dependent on the functional forms. Our contribution is to show that the connection between centrality and markets goes deeper by formalizing it in a simple model without parametric assumptions. In doing this, we give a new economic angle on a concept that has been the subject of

[6]Relatedly, Du, Lehrer, and Pauzner (2015) show how a ranking problem for locations on an unweighted graph can be studied via an associated perfectly competitive exchange economy in which agents have Cobb–Douglas utility functions. We discuss the connection in more detail in Section 5.2.

much study. In sociology, key contributions on eigenvector-type centrality measures include Katz (1953), Bonacich (1987), and Friedkin (1991). For a survey of applications and results on network centrality from computer science and applied mathematics, especially for ranking problems, see Langville and Meyer (2012).[7] Other applications include identifying those sectors in the macroeconomy that contribute the most to aggregate volatility via a network of intersectoral linkages (Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2012); analyzing communication in teams (Prat, de Martí, and Calvó-Armengol, 2015); and the measurement of intellectual influence (Palacios-Huerta and Volij, 2004). The last paper discusses axiomatic foundations of eigenvector centrality; other work taking an axiomatic approach includes Altman and Tennenholtz (2005) and Dequiedt and Zenou (2014).

We discuss other closely related literature in more detail at those points where we expect the comparisons to be most helpful. Omitted proofs and some supporting analyses are deferred to appendices.

2. Framework

2.1. **The Environment.** There is a set of agents or players, $N = \{1, 2, \ldots, n\}$. The *outcome* is determined by specifying an action, $a_i \in \mathbb{R}_+$, for each agent i.[8] Taking a higher action may be interpreted as doing more of something that helps the other agents—for instance, mitigating pollution. Agent i has a utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}$, where u_i is concave and continuously differentiable; agent i's payoff when the action profile $\mathbf{a}$ is played is written $u_i(\mathbf{a})$.

The assumption of a single dimension of effort per agent is relaxed in Section OA1 of the Online Appendix, while the implicit assumption of no transfers (except through the actions) is relaxed in Section OA2.

2.2. **Main Assumptions.** The following four assumptions are maintained in all results of the paper, unless a result explicitly states a different set of assumptions.

Assumption 1 (Costly Actions)**.** Each player finds it costly to invest effort, holding others' actions fixed: $\frac{\partial u_i}{\partial a_i}(\mathbf{a}) < 0$ for any $\mathbf{a} \in \mathbb{R}^n_+$ and $i \in N$.

Assumption 2 (Positive Externalities)**.** Increasing any player's action level weakly benefits all other players: $\frac{\partial u_i}{\partial a_j}(\mathbf{a}) \geq 0$ for any $\mathbf{a} \in \mathbb{R}^n_+$ whenever $j \neq i$.

Because the externalities are positive and nonrival, this is a public goods environment. Together, the two assumptions above make the setting a potential tragedy of the commons. The unique Nash equilibrium of a game in which players choose their actions entails that

[7]Perhaps the most famous application of eigenvector centrality is the PageRank measure introduced as a part of Google's early algorithms to rank search results (Brin and Page, 1998). For early antecedents of using eigenvectors as a way to "value" or rank nodes, see Wei (1952) and Kendall (1955).

[8]We use $\mathbb{R}_+$ (respectively, $\mathbb{R}_{++}$) to denote the set of nonnegative (respectively, positive) real numbers. We write $\mathbb{R}^n_+$ (respectively, $\mathbb{R}^n_{++}$) for the set of vectors $\mathbf{v}$ with n entries such that each entry is in $\mathbb{R}_+$ (respectively, $\mathbb{R}_{++}$). When we write an inequality between vectors, e.g., $\mathbf{v} > \mathbf{w}$, that means the inequality holds coordinate by coordinate, i.e., $v_i > w_i$ for each $i \in N$.

everyone contributes nothing, $a_i = 0$ for each i, even though other outcomes may Pareto dominate this one.[9]

Two additional technical assumptions are useful:

Assumption 3 (Connectedness of Benefits)**.** For all $\mathbf{a} \in \mathbb{R}^n_+$, if M is a nonempty proper subset of N, then there exist $i \in M$ and $j \notin M$ (which may depend on $\mathbf{a}$) such that $\frac{\partial u_i}{\partial a_j}(\mathbf{a}) > 0$.

This posits that it is not possible to find an outcome and partition society into two nonempty groups such that, at that outcome, one group does not derive any marginal benefit from the effort of the other group.[10]

Finally, we assume that the set of points where everybody wants to scale up all effort levels is bounded. To state this, we introduce a few definitions. Under a utility profile $\mathbf{u}$, action profile $\mathbf{a}' \in \mathbb{R}^n_+$ *Pareto dominates* another profile $\mathbf{a} \in \mathbb{R}^n_+$ if $u_i(\mathbf{a}') \geq u_i(\mathbf{a})$ for all $i \in N$, and the inequality is strict for some i. We say $\mathbf{a}'$ *strictly Pareto dominates* $\mathbf{a}$ if $u_i(\mathbf{a}') > u_i(\mathbf{a})$ for all $i \in N$ and that $\mathbf{a}$ is *Pareto efficient* (or simply *efficient*) if no other action profile Pareto dominates it.

Assumption 4 (Bounded Improvements)**.** The set

$$\{\mathbf{a} \in \mathbb{R}^n_+ : \text{there is an } s > 1 \text{ so that } s\mathbf{a} \text{ strictly Pareto dominates } \mathbf{a}\}$$

is bounded.[11]

This assumption is necessary to keep the problem well-behaved and ensure the existence of a Pareto frontier, as well as of solutions to a bargaining problem we will study.[12]

2.3. **Key Notions.** We write $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ for a profile of utility functions. The Jacobian, $\mathbf{J}(\mathbf{a}; \mathbf{u})$, is the n-by-n matrix whose (i, j) entry is $J_{ij}(\mathbf{a}; \mathbf{u}) = \partial u_i(\mathbf{a})/\partial a_j$. The *benefits matrix* $\mathbf{B}(\mathbf{a}; \mathbf{u})$ is then defined as follows:

$$B_{ij}(\mathbf{a}; \mathbf{u}) = \begin{cases} \frac{J_{ij}(\mathbf{a};\mathbf{u})}{-J_{ii}(\mathbf{a};\mathbf{u})} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

As discussed in the roadmap, when $i \neq j$, the quantity $B_{ij}(\mathbf{a}; \mathbf{u})$ is i's marginal rate of substitution between decreasing own effort and receiving help from j. In other words, it is how much i values the help of j, measured in the number of units of effort that i would be willing to put forth in order to receive one unit of j's effort.

Suppose $\mathbf{u}$ satisfies the assumptions of Section 2.2. Since $J_{ii}(\mathbf{a}; \mathbf{u}) < 0$ by Assumption 1, the benefits matrix is well-defined. Assumptions 1 and 2 imply that it is entrywise

[9]While we model positive externalities, our environment is equivalent to one with negative externalities in which it is costly to decrease actions. Although we do not include environments that have both positive and negative externalities, we can relax Assumptions 1 and 2 to some extent. The key technical result we need to apply is a suitable generalization of the Perron–Frobenius theorem (see, for example, Johnson and Tarazaga (2004) and Noutsos (2006)). We consider the more restrictive environment for simplicity.

[10]See Section OA5 of the Online Appendix for a discussion of extending the analysis when this assumption does not hold.

[11]This condition is weaker than assuming that the set of Pareto efficient outcomes is bounded.

[12]For details, see Section 6 and particularly the proof of Proposition 2.

nonnegative. Assumption 3 is equivalent to the statement that this matrix is irreducible[13] at every $\mathbf{a}$.

In discussing both the Jacobian and the benefits matrix, when there is no ambiguity about what $\mathbf{u}$ is, we suppress it.

For any nonnegative matrix $\mathbf{M}$, we define $r(\mathbf{M})$ as the maximum of the magnitudes of the eigenvalues of $\mathbf{M}$, also called the *spectral radius.* That is,

$$r(\mathbf{M}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{M}\},$$

where $|\lambda|$ denotes the absolute value of the complex number λ. By the Perron–Frobenius Theorem (see Appendix A for a statement), any such matrix has a *real*, *positive* eigenvalue equal to $r(\mathbf{M})$. Thus, we may equivalently think of $r(\mathbf{M})$ as the largest eigenvalue of $\mathbf{M}$ on the real line.

This quantity can be interpreted as a single measure of how expansive a matrix is as a linear operator—how much it can scale up vectors that it acts on. When applied to the benefits matrix $\mathbf{B}$, it will identify the scope for Pareto improvements.

3. Efficiency and the Spectral Radius

The thesis of this paper is that we can gain insight about efficient solutions to public goods problems by constructing, for any action profile $\mathbf{a}$ under consideration, a network in which the agents are nodes and the weighted links among them measure the marginal benefits available by increasing actions. The adjacency matrix of this network is $\mathbf{B}(\mathbf{a})$.

This section offers support for the thesis by showing that an important statistic of this network—the size of the largest eigenvalue—can be used to diagnose whether an outcome is Pareto efficient (Section 3.1). After presenting this general result, we discuss interpretations (especially in terms of the structure of the network) and applications.

3.1. A Characterization of Pareto Efficiency.

Our main result on efficiency is the following.

Proposition **1.**

(i) Under Assumptions 1, 2, and 3, an interior action profile $\mathbf{a} \in \mathbb{R}^n_{++}$ is Pareto efficient if and only if the spectral radius of $\mathbf{B}(\mathbf{a})$ is 1.

(ii) Under Assumptions 1 and 2, the outcome $\mathbf{0}$ is Pareto efficient if and only if $r(\mathbf{B}(\mathbf{0})) \leq 1$.

An intuition for part (i) was presented in the roadmap. When the spectral radius is greater than 1 we can obtain a Pareto improvement by one agent increasing his action, generating benefits for others, and then other agents repeatedly passing forward some of the benefits they receive. When the spectral radius is less than 1, the same intuition holds but agents can obtain the Pareto improvement by all reducing their actions. Of course, at the $\mathbf{0}$ action profile reduction is not possible. Part (ii) shows that in this case the boundary solution of everyone taking the null action is Pareto efficient.

The proof of Proposition 1 is in Appendix C. There are two key steps. The first step takes the first-order conditions from the social planner's problem, with Pareto weights $\boldsymbol{\theta}$

[13] A matrix $\mathbf{M}$ is *irreducible* if it is not possible to find a nonempty proper subset S of indices so that $M_{ij} = 0$ for every $i \in S$ and $j \notin S$.

($\boldsymbol{\theta}\mathbf{J}(\mathbf{a}^*) = \mathbf{0}$), and transforms them into a particular eigenvector equation, $\boldsymbol{\theta}\mathbf{B}(\mathbf{a}^*) = \boldsymbol{\theta}$. On its own this algebraic manipulation is not especially enlightening. It entails that if $\mathbf{a}^*$ is efficient then $\mathbf{B}(\mathbf{a}^*)$ has 1 as an eigenvalue. But the converse does not hold, so more is needed to characterize Pareto efficiency. The second key step is to recognize that the conditions necessary to apply the Perron–Frobenius Theorem hold. The Pareto weights are non-negative, and by construction $\mathbf{B}(\mathbf{a}^*)$ contains only weakly positive elements and is irreducible. Applying the Perron–Frobenius Theorem we can then conclude from $\boldsymbol{\theta}\mathbf{B}(\mathbf{a}^*) = \boldsymbol{\theta}$ that the largest eigenvalue of $\mathbf{B}(\mathbf{a}^*)$ must be 1 for any Pareto efficient actions $\mathbf{a}^*$. Conversely, by the same theorem, whenever the largest eigenvalue of $\mathbf{B}(\mathbf{a}^*)$ is 1, there exist non-negative Pareto weights such that $\mathbf{a}^*$ solves the planner's problem.

Proposition 1 shows that we can diagnose whether an outcome is Pareto efficient using just the spectral radius of the benefits matrix, and dispensing with the artificial construct of Pareto weights. Moreover, the spectral radius provides more than just qualitative information; it can also be interpreted as a quantitative measure of the size of the inefficiency. In particular, the spectral radius measures the best return on investment in public goods per unit of cost that can simultaneously be achieved for all agents. Details on this can be found in Section B. We axiomatize the spectral radius of the benefits matrix as a measure of marginal (in)efficiency in a sister paper, Elliott and Golub (2015).

The condition that the spectral radius of $\mathbf{B}(\mathbf{a})$ is 1 is independent of how different players' cardinal utilities are measured—as, of course, it must be, since Pareto efficiency is an ordinal notion. To see how the benefits matrix changes under reparameterizations of cardinal utility, suppose we define, for each $i \in N$, new utility functions $\widehat{u}_i(\mathbf{a}) = f_i(u_i(\mathbf{a}))$ for some differentiable, strictly increasing functions f_i. If we let $\widehat{\mathbf{B}}$ be the benefits matrix obtained from these new utility functions, then $\mathbf{B}(\mathbf{a}) = \widehat{\mathbf{B}}(\mathbf{a})$; this follows by applying the chain rule to the numerator and denominator in the definition of the benefits matrix.

3.2. **Interpretations and Applications.** To gain intuition for our characterization of Pareto efficiency in terms of the spectral radius, it is useful to consider what structural features of the benefits network this quantity measures. In this section, we first suggest via a simple example that the spectral radius captures the strength of *cycles* of marginal benefits; we then present a formal result to that effect. Appendix B elaborates on these ideas and explores using the spectral radius as a measure of inefficiency.

3.2.1. *A Simple Example.* Suppose $N = \{1, 2, 3\}$ and

$$\mathbf{B}(\mathbf{0}) = \begin{bmatrix} 0 & 0 & 7 \\ 5 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix}.$$

This network is depicted in Figure 2. It can be computed that $r(\mathbf{B}(\mathbf{0})) = (5 \cdot 6 \cdot 7)^{1/3} \approx 5.94$. Thus, there is substantial potential for cooperation at the status quo of $\mathbf{0}$. The spectral radius of $\mathbf{B}(\mathbf{0})$ is the geometric mean of weights in the benefits network along the cycle—the product of these weights to the power $1/\ell$, where ℓ is the cycle's length. We state without proof that this is always the case when there is just one cycle in the network.

3.2.2. *Interpreting the Spectral Radius: Cycles of Mutual Benefit.* As noted in the roadmap, if we can find a cycle of players such that each can help the next, then that creates scope for cooperation. The larger the benefits that can be passed along a cycle, the greater

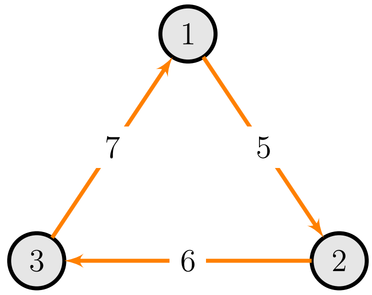


Figure 2. The network of the example. An arrow from i to j indicates that $B_{ji} > 0$ (i.e. benefits flow from i to j) and the link is labeled by the weight B_{ji}.

that scope. This section formalizes the idea that the spectral radius of a matrix can be interpreted generally as measuring the intensity of such cycles.

A (directed) *cycle of length ℓ in the matrix* $\mathbf{M}$ is a sequence

$$(c(1), c(2), \ldots, c(\ell), c(\ell+1))$$

of elements of N (players), so that: the cycle starts and ends at the same node ($c(\ell+1) = c(1)$); and $M_{c(t)c(t+1)} > 0$ for each $t \in \{1, \ldots, \ell\}$.[14] Let $\mathcal{C}(\ell; \mathbf{M})$ be the set of all cycles of length ℓ in matrix $\mathbf{M}$. For any nonnegative matrix $\mathbf{M}$, define the *value* of a cycle $c \in \mathcal{C}(\ell; \mathbf{M})$ as

$$v(c; \mathbf{M}) = \prod_{t=1}^{\ell} M_{c(t)c(t+1)}. \tag{3}$$

This is the number obtained by taking the product of all weights along the cycle c in the weighted directed graph defined by $\mathbf{M}$. Define

$$V(\ell; \mathbf{M}) = \sum_{c \in \mathcal{C}(\ell; \mathbf{M})} v(c; \mathbf{M}).$$

This is the sum of the values of all cycles of length ℓ in matrix $\mathbf{M}$.

The total value of long cycles provides an asymptotically exact estimate of the spectral radius in the following sense:[15]

Fact **1.**

(i) For any nonnegative matrix $\mathbf{M}$, $r(\mathbf{M}) = \limsup_{\ell \to \infty} V(\ell; \mathbf{M})^{1/\ell}$.

(ii) In particular, if $\mathbf{B}$ and $\widehat{\mathbf{B}}$ are two nonnegative matrices such that $\mathbf{B} \geq \widehat{\mathbf{B}}$, then $r(\mathbf{B}) \geq r(\widehat{\mathbf{B}})$.

Intuitively, benefits networks with an imbalanced structure, in which it is rare for the beneficiaries of one agent's effort to be able to directly or indirectly "give back," will have a lower spectral radius and there will be less scope for cooperation. Section 3.2.4 gives a concrete example where this insight is key to determining which agents are essential.

[14] Any node in this sequence may be repeated arbitrarily many times.

[15] Note that $V(\ell; \mathbf{M}) = \text{trace}\left(\mathbf{M}^{\ell}\right)$. With this replacement, the fact is standard—see, e.g., Milnor (2001).

3.2.3. *How Essential Is a Player?* The efficiency results of Section 3.1 provide a simple way of characterizing how essential any given player is to the negotiations. Suppose for a moment that a given player exogenously may or may not be able to participate in an institution to negotiate an outcome that Pareto dominates the status quo. If he is not able, then his action is set to the status quo level of $a_i = 0$. How much does such an exclusion hurt the prospects for cooperation by the other agents?

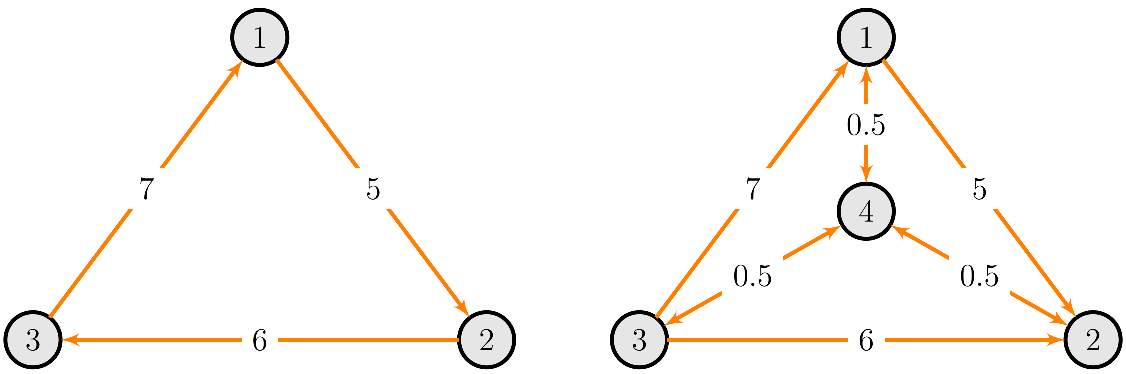
Without player i, the benefits matrix at the status quo of $\mathbf{0}$ is equal to the original $\mathbf{B}(\mathbf{0})$ without row and column i; equivalently, each entry in that row and column may be set to 0. Call a matrix constructed that way $\mathbf{B}^{[-i]}(\mathbf{0})$. By Fact 1(ii), the spectral radius of $\mathbf{B}^{[-i]}(\mathbf{0})$ is no greater than that of $\mathbf{B}(\mathbf{0})$. In terms of consequences for efficiency, the most dramatic case is one in which the spectral radius of $\mathbf{B}(\mathbf{0})$ exceeds 1 but the spectral radius of $\mathbf{B}^{[-i]}(\mathbf{0})$ is less than 1. Then by Proposition 1(ii), a Pareto improvement on $\mathbf{0}$ exists when i is present but not when i is absent.

This argument shows that *player i's participation is essential to achieving any Pareto improvement on the status quo precisely when his removal changes the spectral radius of the benefits matrix at the status quo from being greater than* 1 *to being less than* 1.

3.2.4. *A More Elaborate Example: Who Is Essential?* We now build on the example of Section 3.2.1 to illustrate some conditions under which a player is essential. Suppose $N = \{1, 2, 3, 4\}$ and

$$\mathbf{B}(\mathbf{0}) = \begin{bmatrix} 0 & 0 & 7 & 0.5 \\ 5 & 0 & 6 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}. \tag{4}$$

See Figure 3b for a graphical depiction of this benefits matrix and a comparison with the example of Section 3.2.1.



(A) The example of Section 3.2.1.

(B) The present example (equation 4).

FIGURE 3. The benefit flows in (b) differ from those of (a) in two ways. First, the arrow from 2 to 3 has been flipped, destroying the directed cycle $1 \to 2 \to 3$. Second, a new player (#4) has been added, with bilateral links of weight 0.5 in the benefits matrix to all other players.

The import of the example is that player 4, even though he confers the smallest marginal benefits, is the only essential player. Without him, there are no cycles at all and the spectral radius of the corresponding benefits matrix $\mathbf{B}^{[-4]}(\mathbf{0})$ is 0 (by Fact 1). On the other hand, when he is present but any one other player ($i \neq 4$) is absent, then there is a cycle whose edges multiply to more than 1, and the spectral radius of $\mathbf{B}^{[-i]}(\mathbf{0})$ exceeds 1 (again by Fact 1(i)). Thus, the participation of a seemingly "small" player in negotiations can make an essential difference to the ability to improve on the status quo when that player completes cycles in the benefits network.

4. Lindahl Outcomes

In this section, we focus attention on a particular class of Pareto efficient solutions, the Lindahl outcomes. Our main result relates these outcomes to network properties of the marginal benefits matrix. Informally, a Lindahl outcome is an analogue, in a public goods setting, of a Walrasian equilibrium allocation where all the externalities are priced and the market clears. Instead of standard prices for private goods, the prices are personalized taxes and subsidies: Each player pays a personalized tax for every public good he enjoys (in proportion to how much of that public good is produced), and receives a personalized subsidy (financed by others' taxes) per unit of effort he invests in the public good he provides. If negotiations could somehow create the "missing markets"—if they could emulate a situation in which all externalities are priced competitively—Lindahl outcomes are what would result. The main result in this section, Theorem 1, relates agents' contributions in Lindahl outcomes to how "central" they are in the network of externalities.

Definition 1. An action profile $\mathbf{a}^*$ is a *Lindahl outcome* for a preference profile $\mathbf{u}$ if there is an n-by-n matrix (of prices) $\mathbf{P}$ so that the following conditions hold for every i:

(i) The inequality

$$\sum_{j:j\neq i} P_{ij} a_j \leq a_i \sum_{j:j\neq i} P_{ji} \qquad (\mathrm{BB}_i(\mathbf{P}))$$

is satisfied when $\mathbf{a} = \mathbf{a}^*$;

(ii) for any $\mathbf{a}$ such that the inequality $\mathrm{BB}_i(\mathbf{P})$ is satisfied, we have $u_i(\mathbf{a}^*) \geq u_i(\mathbf{a})$.

So, given prices $\mathbf{P}$, agent i optimally produces a_i^* units of output, generating an income of $a_i^* \sum_{j:j\neq i} P_{ji}$, which he uses to buy a_j^* units of good j, for each $j \neq i$. Thus $\mathbf{a}^*$ is everyone's most preferred action profile. Note that, as outputs are non-rivalrous, each person $j \neq i$ can simultaneously buy a_i^* units of i's output.[16]

The Lindahl solution assumes, as a conceptual device, personalized competitive prices for all the externalities. Such prices are not set through the equilibrating forces operating in competitive Walrasian markets (Samuelson, 1954). In Section 6, we review game-theoretic microfoundations for the Lindahl concept in our setting, explaining what sorts

[16]There need not be any transferable private commodity in which these prices are denominated. We can think of each player having access to artificial tokens, facing prices for the public goods denominated in these tokens, and being able to choose any outcome subject to not using more tokens than he receives from others.

of negotiations can lead to Lindahl outcomes, and what design considerations can make such negotiations desirable.

Our characterization of the Lindahl outcomes of the economy we study is in terms of *eigenvector centrality action profiles.*

DEFINITION **2.** An action profile $\mathbf{a} \in \mathbb{R}^n_+$ is an *eigenvector centrality action profile* if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{B}(\mathbf{a})\mathbf{a} = \mathbf{a}$.

Note that $\mathbf{a}$ is, according to the condition, a right-hand eigenvector of $\mathbf{B}(\mathbf{a})$ with eigenvalue 1. Because actions are nonnegative, the Perron–Frobenius Theorem implies that such an $\mathbf{a}$ is the Perron, or principal, eigenvector—the one associated to the largest eigenvalue of the matrix. The entries of this vector have been studied as measures of the importance or centrality of nodes in a network, explaining the name. An important aspect of the definition is that central nodes are those that are connected—in a sense that depends on the application—to other central nodes. This can be seen by writing

$$a_i = \sum_{j \in N} B_{ij} a_j. \tag{5}$$

Equation (5) asserts that each player's contribution is a weighted sum of the other players' contributions, where the weight on a_j is proportional to the marginal benefits that j confers on i.[17]

THEOREM **1.** The following are equivalent for a nonzero $\mathbf{a} \in \mathbb{R}^n_+$:
(i) $\mathbf{B}(\mathbf{a})\mathbf{a} = \mathbf{a}$, i.e., $\mathbf{a}$ is an eigenvector centrality action profile;
(ii) $\mathbf{a}$ is a Lindahl outcome.

We can also establish that any nonzero Lindahl outcome is interior (Lemma 1, Appendix C). An outline of the proof of Theorem 1 is below and the complete proof appears in Appendix C. However, before presenting the argument, it is worth remarking on some simple consequences of Theorem 1. First, at any interior Lindahl outcome $\mathbf{a}$, the matrix $\mathbf{B}(\mathbf{a})$ has a nonnegative right eigenvector $\mathbf{a}$ with eigenvalue 1, and therefore, by the Perron–Frobenius Theorem, a spectral radius of 1. Proposition 1 then implies the Pareto efficiency of $\mathbf{a}$, providing an alternative proof of the First Welfare Theorem.[18]

Second, the condition $\mathbf{B}(\mathbf{a})\mathbf{a} = \mathbf{a}$ is a system of n equations in n unknowns (the coordinates of $\mathbf{a}$). By a standard argument (see, e.g., Shannon, 2008), this entails that for generic utility functions satisfying our assumptions, the set of solutions will be of dimension 0 in $\mathbb{R}^n_+$. Therefore, the set of Lindahl outcomes is typically "small," as is usually the case with sets of market equilibria.

Finally, the equivalence between Lindahl outcomes and eigenvector centrality action profiles allows us to establish the existence of a Lindahl equilibrium in our setting, where standard proofs do not go through because of their boundedness requirements:

PROPOSITION **2.** Either $\mathbf{a} = \mathbf{0}$ is Pareto efficient or there is an eigenvector centrality action profile in which all actions are strictly positive.

[17]We discussed in Section 3.1 that the set of Pareto efficient action profiles is invariant to rescaling the utility functions, because such rescalings do not affect the benefits matrix. The same argument implies that the eigenvector centrality action profiles are also invariant to such rescalings.

[18]A standard proof can be found in, e.g., Foley (1970).

The proof of Proposition 2 is in Appendix C. We can also show that the profile $\mathbf{0}$ is a Lindahl outcome if and only if it is Pareto efficient (Proposition 8, Section D).

4.1. **An Outline of the Proof of Theorem 1.** It will be convenient to introduce scaling-indifferent action profiles. From the definition of the benefits matrix, these are easily verified to be equivalent to the eigenvector centrality action profiles, and we will use the two notions interchangeably.

DEFINITION **3.** An action profile $\mathbf{a} \in \mathbb{R}^n_+$ satisfies *scaling-indifference*[19] (or *is scaling-indifferent*) if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0}$.

We will show that a profile is a Lindahl outcome if and only if it is an eigenvector centrality action profile. The more difficult part is the "if" part. The key fact is that the system of equations $\mathbf{B}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{a}^*$ allows us to extract Pareto weights that support the outcome $\mathbf{a}^*$ as efficient, and using those Pareto weights and the Jacobian, we can construct prices that support $\mathbf{a}^*$ as a Lindahl outcome.

Now in more detail: Suppose we have a nonzero $\mathbf{a}^*$ so that $\mathbf{B}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{a}^*$. As we noted in the previous section, the profile $\mathbf{a}^*$ is then interior and Pareto efficient. So there exist Pareto weights $\boldsymbol{\theta}$ such that $\mathbf{a}^*$ solves $\mathcal{P}(\boldsymbol{\theta})$.[20] That implies that the first-order conditions of the Pareto problem $\mathcal{P}(\boldsymbol{\theta})$ defined in the proof of Proposition 1(i) are satisfied for these weights: for each j, we have

$$\sum_{i \in N} \theta_j J_{ij}(\mathbf{a}^*) = 0. \tag{6}$$

Let us guess Lindahl prices $P_{ij} = \theta_j J_{ij}(\mathbf{a}^*)$ for $i \neq j$. Let us also define, for notational convenience, $P_{ii} = \theta_i J_{ii}$. Note that, by definition of the prices and (6), we have

$$P_{ii} = -\sum_{j : j \neq i} P_{ji}. \tag{7}$$

We claim that

$$\sum_{j \in N} P_{ij} a_j^* = \theta_i \sum_{j \in N} J_{ij}(\mathbf{a}^*) a_j^* = 0.$$

The first equality is by construction of the prices, and the second is by the fact $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$ (recall that eigenvector centrality action profiles and scaling-indifferent profiles are equivalent). In view of (7), this says that each agent is exhausting his budget. It remains only to see that each agent is optimizing at prices $\mathbf{P}$. The essential reason for this is that price ratios are equal to marginal rates of substitution by construction. Indeed, when all the denominators involved are nonzero, we may write:

$$\frac{P_{ij}}{P_{ik}} = \frac{\theta_i J_{ij}(\mathbf{a}^*)}{\theta_i J_{ik}(\mathbf{a}^*)} = \frac{J_{ij}(\mathbf{a}^*)}{J_{ik}(\mathbf{a}^*)}. \tag{8}$$

[19]To see the reason for the name, note that, to a first-order approximation, $\mathbf{u}(\mathbf{a} + \varepsilon \mathbf{v}) \approx \mathbf{u}(\mathbf{a}) + \varepsilon \mathbf{J}(\mathbf{a})\mathbf{v}$. Suppose now that actions $\mathbf{a}$ are scaled by $1 + \varepsilon$, for some small real number ε; this corresponds to setting $\mathbf{v} = \mathbf{a}$. If $\mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0}$, then all players are indifferent, in the first-order sense, to this small proportional perturbation in everyone's actions.

[20]This is the point where the Perron–Frobenius Theorem plays a key role—recall the discussion that follows Proposition 1(i).

The key intuition ends there, but for the interested reader we offer a brief comment—which is not essential to anything that follows—on the form of the prices. According to the formula, $P_{ij} = \theta_i J_{ij}(\mathbf{a}^*)$, the price that i pays for j's effort, is proportional to the marginal value to i of j's effort, and also proportional to i's Pareto weight θ_i. Why do prices take this form?

First, note that in the constrained optimization problem of maximizing $u_i(\mathbf{a})$ subject to $\mathrm{BB}_i(\mathbf{P})$, the first-order conditions boil down to $P_{ij} = \mu_i^{-1} \cdot J_{ij}(\mathbf{a})$, where μ_i is the Lagrange multiplier on the constraint $\mathrm{BB}_i(\mathbf{P})$—i.e., the marginal utility of relaxing the constraint $\mathrm{BB}_i(\mathbf{P})$, or the marginal utility of income to i.

Next, consider the planner who puts weight θ_i on player i. At any optimal outcome, it must be that $\mu_i\theta_i$, which is the planner's marginal utility of giving additional income to any agent i, does not depend on i. Otherwise, the social planner would want to "reallocate income"—i.e., have some agents increase their actions so others have more to spend. Now, this equality of $\mu_i\theta_i$ across i implies that Pareto weights are inversely proportional to the marginal utilities of income.

Combining these two observations, we deduce that $P_{ij} = \theta_i J_{ij}(\mathbf{a})$, which is the guess we made above.[21]

Consider now the converse implication—that if $\mathbf{a}^*$ is a nonzero Lindahl outcome, then $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$. A nonzero Lindahl outcome $\mathbf{a}^*$ can be shown to be interior. (This is Lemma 1 in Appendix C.) Given this, and that agents are optimizing given prices, we have

$$\frac{P_{ij}}{P_{ik}} = \frac{J_{ij}(\mathbf{a}^*)}{J_{ik}(\mathbf{a}^*)},$$

which echoes (8) above. In other words, each row of $\mathbf{P}$ is a scaling of the same row of $\mathbf{J}(\mathbf{a}^*)$. Therefore, the condition that each agent is exhausting his budget[22], which can be succinctly written as $\mathbf{P}\mathbf{a}^* = \mathbf{0}$, implies that $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$.

5. Applications

In this section we present two applications of our general results. First we show how the above analysis can be used to predict who will be admitted to a team. Second, we use special cases of our results our results to provide market interpretations of several measures of network centrality that have been utilized in a huge variety of settings, both within economics and especially in other fields. To address both applications, it will be helpful to think about eigenvector centralities in terms of walks on the network.

In Section 3.2.2, we saw that the spectral radius of the benefits matrix could be interpreted through the values of long cycles. A related interpretation applies to eigenvector centrality action profiles. A *walk of length ℓ in the matrix* $\mathbf{M}$ is a sequence $(w(1), w(2), \ldots, w(\ell+1))$ of elements of N (player indices) such that $M_{w(t)w(t+1)} > 0$ for each $t \in \{1, 2, \ldots, \ell\}$.[23] Let $\mathcal{W}_i^{\downarrow}(\ell; \mathbf{M})$ be the set of all walks of length ℓ in $\mathbf{M}$ ending

[21]We thank Phil Reny for this insight.

[22]This follows because each agent is optimizing given prices, and by Assumption 3 there is always some contribution each agent wishes to purchase.

[23]As with cycles, defined in Section 3.2.2, nodes can be repeated in this sequence. Note also that a cycle is a special kind of walk.

at i (in our notation, such that $w(\ell+1)=i$). For a matrix $\mathbf{M}$, define the value of a walk w of length ℓ as the product of all matrix entries (i.e., link weights) along the walk:

$$v(w;\mathbf{M}) = \prod_{t=1}^{\ell} M_{w(t)w(t+1)}.$$

Note that such walks can repeat nodes—for example, they may cover the same cycle many times. Then we have the following:

PROPOSITION **3.** Let $\mathbf{M} = \mathbf{B}(\mathbf{a})^\mathsf{T}$ and assume this matrix is aperiodic.[24] Then $\mathbf{a}$ is an eigenvector centrality action profile if and only if, for every i and j,

$$\frac{a_i}{a_j} = \lim_{\ell\to\infty} \frac{\displaystyle\sum_{w\in\mathcal{W}_i^{\downarrow}(\ell;\mathbf{M})} v(w;\mathbf{M})}{\displaystyle\sum_{w\in\mathcal{W}_j^{\downarrow}(\ell;\mathbf{M})} v(w;\mathbf{M})}.$$

A walk in $\mathbf{B}(\mathbf{a})^\mathsf{T}$ ending at i can be thought of as a chain of benefit flows: e.g., k helps j, who helps i. The value of such a walk is the product of the marginal benefits along its links. According to Proposition 3, a player at an eigenvector centrality action profile (and hence a Lindahl outcome) contributes in proportion to the total value of such benefit chains that end with him.[25]

An implication of this analysis is that if the benefits i receives from j decrease at all action profiles, i.e., $B_{ij}(\mathbf{a})$ decreases for all $\mathbf{a}$, then i's eigenvector centrality action level relative to all other agents will decrease. Thus, it is the benefits i receives, rather than the benefits i confers on others, which really matter for i's eigenvector centrality. If, for example, there is an agent who can very efficiently provide benefits to the other agents, and eigenvector centrality action profiles are played, then there can be high returns from increasing the marginal benefits that this agent receives from others (and particularly those others with high eigenvector centrality). This has important implications, which we now discuss.

5.1. **Application: Admitting a New Team Member.** Suppose agents $1,\ldots,8$ currently form a team. These team members must first decide whom, if anyone, to admit as a new member of their team. They have three options: admit nobody; admit candidate 9; or admit candidate 10. Afterwards the team collectively decides how much effort each of them should exert. We assume that these negotiations result in the Lindahl actions being played (see Section 6.1).

Who can provide benefits to whom in the initial team is described by the unweighted, directed graph $\mathbf{G}$ (with entries in $\{0,1\}$), illustrated in Figure 4a. Once the decision

[24]A *simple cycle* is one that has no repeated nodes except the initial/final one. A matrix is said to be *aperiodic* if the greatest common divisor of the lengths of all simple cycles in that matrix is 1.

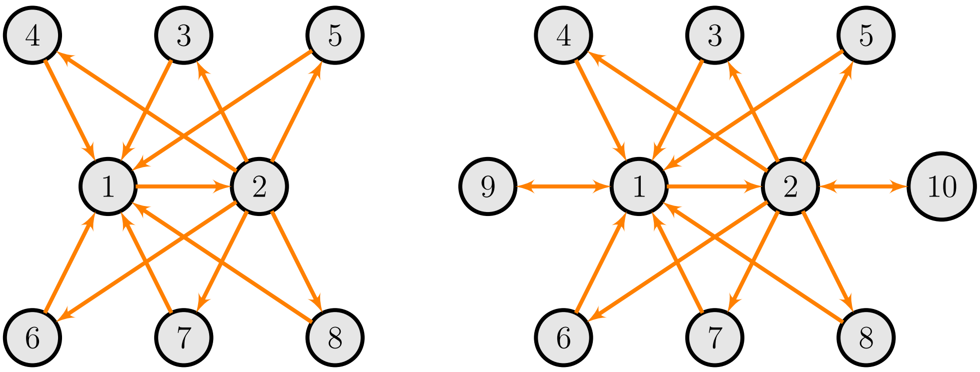
[25]The formula of the proposition would also hold if had we defined $\mathbf{M} = \mathbf{B}(\mathbf{a})$ and replaced $\mathcal{W}_i^{\downarrow}(\ell;\mathbf{M})$ by $\mathcal{W}_i^{\uparrow}(\ell;\mathbf{M})$, which is the set of walks of length ℓ in $\mathbf{M}$ that *start* at i. The convention we use above is in keeping with thinking of a walk in $\mathbf{B}(\mathbf{a})^\mathsf{T}$ capturing the direction in which benefits flow; recall the discussion in Section 3.2.2.

about team composition has been made, G_{ij} is set to 0 if either i or j is not in the team. The utility function of i is then:

$$u_i(\mathbf{a}) = \sum_{j \in N} G_{ij} \log(1 + a_j) - a_i.$$

Agents not on the team will choose to exert no effort[26] and will receive a payoff of 0. The additional benefits agents 9 and 10 can provide to the existing team members are shown in Figure 4b. Whom, if anyone should the team admit? Will the team be able to agree on the best agent to admit?

A quick inspection of Figure 4 reveals the following about the original team: Agent 1 receives benefits from many other agents and so should have a high eigenvector centrality action. Agent 2, on the other hand, receives benefits only from agent 1, but his effort provides benefits to many others. As agent 2's eigenvector centrality action depends on the benefits he receives, and not on those that he can confer, he will take a relatively low action. The returns from agent 2 increasing his action should therefore be higher than the returns from agent 1 increasing his action. This observation suggests that overall the team might be better off admitting agent 10.



(A) Initial Team (B) Prospective New Members

FIGURE 4. There are 8 current members of a team and they must decide whether to admit an additional member or not. The two additional members they can choose between are agents 9 and 10.

We now formalize the above intuition using the tools we have developed. By Proposition 4, agents' Lindahl actions are given by their eigenvector centralities. Applying the scaling-indifference characterization of these actions, $\mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0}$, we find that the eigenvector centrality action of agent i is characterized by $a_i = \sum_j (G_{ij} a_j)/(1 + a_j)$. The unique[27] eigenvector centrality actions if no new team members are admitted are $\mathbf{a}$; if agent 9 is added they are $\mathbf{a}'$; and if agent 10 is instead added they are $\mathbf{a}''$, where the last entry of

[26]For such an agent i, $u_i(\mathbf{a}) = -a_i$.

[27]Uniqueness is established by first noting that each agent's preferences satisfy the gross substitutes property, and therefore the Lindahl outcome is unique (McKenzie, 1959).

the vector corresponds to the action taken by the new team member:

$$\mathbf{a} = \begin{pmatrix} .166 \\ .625 \\ .385 \\ \vdots \\ .385 \\ - \end{pmatrix} \qquad \mathbf{a}' = \begin{pmatrix} .214 \\ .681 \\ .405 \\ \vdots \\ .405 \\ .681 \end{pmatrix} \qquad \mathbf{a}'' = \begin{pmatrix} 2 \\ 1 \\ .5 \\ \vdots \\ .5 \\ .5 \end{pmatrix}.$$

If added, agent 9 will take a higher action than agent 10. However, agent 10 induces a bigger increase in the action taken by agent 2, who provides benefits to many of the other agents. The utility vectors for the initial 8 team members when the eigenvector centrality action profiles are played are shown below for the three options of admitting nobody, admitting agent 9, and admitting agent 10:

$$\mathbf{u}(\mathbf{a}) = \begin{pmatrix} 0.286 \\ 0.356 \\ 0.101 \\ \vdots \\ 0.101 \end{pmatrix} \qquad \mathbf{u}(\mathbf{a}') = \begin{pmatrix} 0.425 \\ 0.462 \\ 0.114 \\ \vdots \\ 0.114 \end{pmatrix} \qquad \mathbf{u}(\mathbf{a}'') = \begin{pmatrix} 0.433 \\ 0.504 \\ 0.193 \\ \vdots \\ 0.193 \end{pmatrix}.$$

From the perspective of each initial team member, admitting agent 10 dominates admitting agent 9. Even agent 1, who directly benefits from admitting agent 9 but not 10, prefers to admit agent 10. This is because agent 10 provides benefits to agent 2, who provides benefits to many other agents, inducing these agents to take higher actions that benefit agent 1. These indirect benefit flows outweigh the direct benefit flows agent 1 would receive from agent 9 in equilibrium.

While in general the incentives of agents will not be aligned when deciding whom to admit to a team or whom to invite to join negotiations, studying the network structure of the externalities can help us understand some possible implications of changing team compositions. One lesson is that agents who have the potential to provide benefits to many others realize this potential when others are included in the team who provide many benefits to them.

5.2. **Explicit Formulas for Lindahl Outcomes.** The general characterization of Lindahl outcomes in terms of network centrality (Theorem 1) makes no parametric assumptions on preferences. The cost of this generality is that the characterization is implicit: An eigenvector centrality action profile $\mathbf{a}$ is defined with respect to a matrix, $\mathbf{B}(\mathbf{a})$, that depends on $\mathbf{a}$ itself. However, by choosing a certain parametrization of utilities we can eliminate this dependence.

The preferences we consider are:

$$u_i(\mathbf{a}) = -a_i + \sum_j \left[\alpha G_{ij} a_j + H_{ij} \log a_j\right], \tag{9}$$

where $\mathbf{G}$ and $\mathbf{H}$ are nonnegative matrices (networks) with zeros on the diagonal (no self-links) and $\alpha < 1/r(\mathbf{G})$. Let $h_i = \sum_j H_{ij}$. For any preferences in this family, the condition for $\mathbf{a}$ to be an eigenvector centrality action profile ($\mathbf{a} = \mathbf{B}(\mathbf{a})\mathbf{a}$) boils down to $\mathbf{a} = \mathbf{h} + \alpha \mathbf{G}\mathbf{a}$.

There are several special cases worth considering. If $\alpha = 0$, then $a_i = h_i$ and i's Lindahl action is equal to the number of i's neighbors in $\mathbf{H}$. This measure of i's centrality in the network $\mathbf{H}$ is known as i's *degree centrality.* If, instead, $h_i = 1$ for all i, then agents' Lindahl actions are $\mathbf{a} = [\mathbf{I} - \alpha\mathbf{G}]^{-1}\mathbf{1}$. The right-hand side is a different measure of agents' centralities in the network $\mathbf{G}$, known as their *Bonacich centralities.* Like degree centrality, it depends on the number of i's neighbors, but also depends on longer-range paths.[28] Finally, in this setting, as α approaches 1, agents' actions become proportional to their normalized eigenvector centralities in $\mathbf{G}$. These results are further discussed in Section OA6 of the Online Appendix.

These results do more than illustrate our results with particular calculations. Because Lindahl outcomes are defined in terms of prices, the formulas we have presented may be viewed as microfoundations for certain network centrality measures in terms of price equilibria. Viewed in this light, each result says that for particular preferences, the allocations defined by Lindahl are equal to centralities according to a particular measure. Such a connection permits a new interpretation of well-known centrality measures, clarifying their connection to classical ideas about markets. In addition, this connection may permit new analytical techniques inspired by price equilibria.

In emphasizing the correspondence between centrality and outcomes of a market, our perspective is related to independent work by Du et al. (2015), who microfound eigenvector centrality via an exchange economy with Cobb-Douglas preferences. The parametric forms required to recover the "standard" centrality measures differ in the two models, but both share the perspective that centrality and markets are closely related and each concept can be used to shed light on the other. An advantage of the public goods economy we study is that our characterizations above are a special case of an eigenvector characterization that applies without parametric assumptions. We believe these projects taken together offer hope for a fairly rich theory of connections between market outcomes and network centrality.

6. Foundations for the Lindahl Solution

In Section 4, we presented the definition of the Lindahl solution as a conceptual device for emulating missing markets for externalities, but deferred discussion of how it can be implemented in actual negotiations over public goods. In this section, we discuss several game-theoretic approaches to bargaining that provide foundations for the Lindahl solution. In view of Theorem 1 in Section 4, these are equivalently strategic foundations for the class of eigenvector centrality action profiles. Our discussions here adapt existing results on Walrasian bargaining, implementation theory, and the core—though, in each case, we have to adjust existing ideas somewhat to work in our setting with unbounded action spaces.

6.1. **A Group Bargaining Game.**

We consider a bargaining game closely related to those studied by Dávila, Eeckhout, and Martinelli (2009) and Penta (2011). These papers are part of a broader literature that seeks multilateral bargaining foundations for Walrasian outcomes.[29]

[28] It can also be characterized via the equation $\mathbf{a} = \alpha\mathbf{G}\mathbf{a} + \mathbf{1}$, which resembles the condition defining eigenvector centrality. For more background and discussion, see Ballester, Calvó-Armengol, and Zenou (2006, Section 3) and (Jackson, 2008, Section 2.2.4).

[29] See also Yildiz (2003) and Dávila and Eeckhout (2008). An alternative path to implementing efficient outcomes might be to generalize the charity auctions introduced by Conitzer and Sandholm (2004). In

The game begins in state s_0, and the timing of the game within a period is:

(i) A new state s is selected. If the state at the beginning of the period is s', then state s is selected with probability $p(s', s)$. Each state s identifies a proposer $\nu(s) \in N$.
(ii) The proposer $\nu(s)$ selects a direction $\mathbf{d} \in \Delta^n$, where Δ^n is the simplex in $\mathbb{R}^n$.
(iii) All agents then simultaneously respond. Each may vote "no" *or* may specify a maximum scaling of the proposed direction by selecting $\lambda_i \in \mathbb{R}_+$.
(iv) If any agent votes "no", we begin the next period by returning to step (i); i.e., the proposal is deemed to be rejected and someone else is selected to propose a direction.
(v) If nobody votes "no", then actions $\mathbf{a} = (\min_i \lambda_i)\mathbf{d}$ are implemented.

The game can go on for infinitely many periods. Until an agreement is reached and actions are taken, players receive their status quo payoffs $u_i(\mathbf{0}) = \mathbf{0}$ each period; afterward they receive the payoffs of the implemented action forever. Players evaluate streams of payoffs according to the expectation of a discounted sum of period payoffs. We fix a common discount factor $\delta \in (0, 1)$.

We will show that efficient outcomes are obtainable in equilibrium and we will characterize this set. More precisely, we will find the set of efficient perfect equilibrium outcomes in this game—i.e., ones resulting in paths of play not Pareto dominated by any other path of play.[30] Let $A(\delta)$ be the set of nonzero action profiles $\mathbf{a}$ played in some efficient perfect equilibrium for discount factor δ.

Proposition 4. Suppose actions $\mathbf{a} = \mathbf{0}$ are Pareto inefficient, that utilities are strictly concave, and that the assumptions of Section 2.2 hold. Then $A(\delta)$ is the set of Lindahl outcomes—or, equivalently, the set of eigenvector centrality action profiles.

We relegate the formal proof to the Appendix and only outline the main argument here.[31] First, note that Pareto efficiency requires that, in every state, the same deterministic action profile be agreed on during the first round of negotiations.[32] Delay is

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that work, people condition their charitable contributions on others' contributions (and so choose action vectors that, in some ways, resemble the directions chosen in the bargaining game we are about to define formally). A paper taking this approach is Ghosh and Mahdian (2008). They locate people on a social network and assume they benefit linearly from their neighbors' contributions. There is an equilibrium of their game that achieves the maximum possible feasible contributions (subject to individual rationality), and this involves positive contributions being made if and only if the largest eigenvalue of the (fixed) network is greater than one.

[30]There will also be many inefficient equilibria. For example, for any direction, it is an equilibrium in the second stage of the game for all agents to select the zero action profile, as none of them will be pivotal when they do so. Requiring efficiency rules out these equilibria, but perhaps more reasonable equilibria too.

[31]Penta (2011) has a similar result in which the equilibria of games without externalities converge to the Walrasian equilibria as players become patient. As we saw in Section 4, Walrasian equilibria are closely related to our eigenvector centrality condition. Nevertheless, the settings are quite different. Penta (2011) considers an endowment economy, and it is important for his results that, whenever outcomes are Pareto inefficient, there is a pair of agents that can find a profitable pairwise trade. This does not hold in our framework.

[32]As defined above, our notion of Pareto efficiency requires that no sequence of action profiles can be found that yields a Pareto improvement from the *ex ante* perspective. This notion is quite strong, as the first step in the argument demonstrates. We could instead require only that, in each period, a Pareto efficient action profile be played. Under this weaker condition, it is possible to prove a version of Proposition 4 holding in the limit as $\delta \to 1$.

inefficient as there is discounting, and the strict concavity of utility functions means that it is also inefficient for different actions to be played with positive probability—it would be a Pareto improvement to play a convex combination of those actions instead. Consider now which deterministic actions can be played. Intuitively, the structure of the game can be interpreted as giving all agents veto power over how far actions are scaled up in the proposed direction. This constrains the possible equilibrium outcomes to those in which no agent would want to scale down actions. Next, we show that if there are some agents who strictly prefer to scale *up* actions at the margin, while all other agents are (first-order) indifferent, the current action profile is Pareto inefficient. The set of action profiles that remain as candidate efficient equilibrium outcomes are those in which all agents are indifferent to scaling the actions up *or* down at the margin. Recalling Definition 3 in Section 4.1, these are the eigenvector centrality action profiles. This is why only eigenvector centrality action profiles can occur in an efficient perfect equilibrium. The proof is completed by constructing such an equilibrium for any eigenvector centrality action profile.

A recent literature has found a connection between the Nash equilibria of one-shot games in networks and centrality measures in those networks. Key papers in this literature include Ballester, Calvó-Armengol, and Zenou (2006) on skill investment with externalities, and Bramoullé, Kranton, and d'Amours (2013) on local public goods.[33] Although Proposition 4 resembles results from this literature in the sense of characterizing particular equilibria in terms of centrality, the economics and mathematics underlying it are quite different. Economically, the Nash equilibria studied in the previous literature are characterized by inefficiencies due to a free-riding problem, while we study a bargaining game designed to overcome these problems and achieve an efficient solution. Mathematically, the action spaces in the games are different (players propose a direction that actions should be increased in instead of independently choosing their own actions). Finally, our results do not require any functional form assumptions on utilities, instead relying only on concavity.

6.2. **Implementation Theory: The Lindahl Outcome as a Robust Selection.** We have shown that the efficient, perfect equilibria of a specific bargaining game correspond to the eigenvector centrality action profiles. An alternative approach, based on implementation theory, places a more stringent requirement on the game—requiring *all* equilibria to yield efficient improvements on the status quo—and studies what can be said about the class of all mechanisms achieving this desideratum. It will turn out that Lindahl outcomes play a distinguished role from this perspective as well.

In more detail: following Hurwicz (1979a), consider a designer of an institution who can leave future participants with a game to play. She does not know what preferences they will have. She believes they will play an equilibrium, but does not know which equilibrium. She would like to design the game so that all its equilibria have some desirable properties.

[33]These papers offer more comprehensive surveys of the literature. Most recently, Allouch (2015) has studied a network version of the setting introduced by Bergstrom, Blume, and Varian (1986) on the voluntary (static Nash) private provision of public goods. Generalizing results of Bramoullé, Kranton, and d'Amours (2013), he derives comparative statics of public goods provision using network centrality tools. For a different approach to public goods problems on networks, see Bramoullé and Kranton (2007).

In this section we informally describe an implementation-theoretic foundation of Lindahl equilibrium based on this setting, relegating the formal treatment to Sections OA3 and OA4 of the Online Appendix. The key choice of the designer is a game form or mechanism. Specifying a game form involves choosing strategy spaces for all the agents and defining an outcome function that maps strategy profiles into outcomes or allocations—in our case, action profiles $\mathbf{a} \in \mathbb{R}^n$. The designer assumes that the profile of players' preferences, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, comes from some set $\mathcal{U}$, but she does not know exactly what preferences they will have. She also assumes that players will end up playing a complete information Nash equilibrium of her game, but she has no control over which equilibrium. We look for games the designer can create in which, for all preference profiles, all Nash equilibria satisfy some desirable properties. The two basic normative criteria we impose are *Pareto efficiency* and *individual rationality.* Pareto efficiency requires that any action profile resulting from equilibrium play of the game is Pareto efficient. Individual rationality ensures that every player is no worse off than at the status quo.

We also impose an upper hemicontinuity condition on the mechanism: For every sequence of preference profiles $(\mathbf{u}^{(k)})$ converging[34] to $\mathbf{u}$, and every sequence of outcomes $\left(\mathbf{a}^{(k)}\right)$ resulting from equilibrium play, if $\mathbf{a}^{(k)} \to \mathbf{a}$, then $\mathbf{a}$ must be an outcome that results from equilibrium play for preferences $\mathbf{u}$. This condition is imposed for technical reasons, but also has some normative appeal. It prevents arbitrarily small changes in preferences having large effects on the action profiles that are implemented.

If the designer may choose any game form subject to these desiderata, what can be said about what will be played in equilibrium? Certainly, the set of equilibria will vary as the designer varies the game form. Nevertheless, it turns out that there are certain outcomes that occur as equilibrium outcomes for *every* game form satisfying the desiderata we have outlined above. This set of outcomes[35] is called the set of *robustly attainable* outcomes. And the argument in favor of the Lindahl selection is the following proposition, which asserts that, under suitable assumptions, this set is exactly equal to the set of Lindahl outcomes.

PROPOSITION **5.** Suppose the possible preference profiles $\mathcal{U}$ are all those satisfying the assumptions of Section 2.2, and the number of players n is at least 3. Then the robustly attainable actions are the Lindahl outcomes—which, assuming they are nonzero, are the eigenvector centrality action profiles.

A corollary of this result is the following. Suppose multiplicity of equilibria is considered a drawback of a mechanism—perhaps because this renders it less effective at coordinating the players on one efficient outcome. In that case, mechanisms implementing just the eigenvector centrality action profiles—which exist—do the best job of avoiding multiplicity. In particular, such mechanisms result in a single equilibrium outcome when there is a unique eigenvector centrality action profile.

[34]Formally, what is assumed is compact convergence: The $(\mathbf{u}^{(k)})$ converge uniformly on every compact set.

[35]Formally, the intersection of equilibrium sets across game forms satisfying our desiderata.

A formal statement and proof of Proposition 5, along with a formalization of the discussion above, can be found in Sections OA3 and OA4 of the Online Appendix.[36]

6.3. **Coalitional Deviations: A Core Property.** As we are modeling negotiations, a natural question is whether some subset of the agents could do better by breaking up the negotiations and coming to some other agreement among themselves. Although this is outside the scope of actions available to the agents as modeled, the Lindahl outcomes are robust to coalitional deviations, if we assume that following a deviation, negotiations collapse and the non-deviating players choose their individually optimal or status quo actions. Formally, we make the following definition.

Definition. An action profile $\mathbf{a}$ is *robust to coalitional deviations* if there is no nonempty coalition $M \subseteq N$ and no other action profile $\mathbf{a}'$ so that:

(i) $a'_i = 0$ for all $i \notin M$;
(ii) each $i \in M$ weakly prefers $\mathbf{a}'$ to $\mathbf{a}$;
(iii) some $i \in M$ strictly prefers $\mathbf{a}'$ to $\mathbf{a}$.

For any actions agents other than i can take, holding constant these actions $\mathbf{a}_{-i}$, agent i's payoff is maximized by i selecting $a_i = 0$. In our setting, this also minimizes the payoffs of any group of deviating players, taking the deviators' actions as given. Thus, the response by the complementary coalition is both individually optimal for the punishers, and maximally harsh to the punished.[37]

PROPOSITION **6.** If $\mathbf{a} \in \mathbb{R}^n_+$ is an eigenvector centrality action profile, then $\mathbf{a}$ is robust to coalitional deviations.

As by Theorem 1 the eigenvector centrality action profiles are the same as the Lindahl outcomes, it suffices to explain why the Lindahl outcomes are robust to coalitional deviations.

We do this by reducing the question to the core property of standard competitive markets. In the Lindahl exercise we suppose all externalities are priced and look for a market equilibrium. It is useful to imagine a related artificial economy with many tradeable private goods mimicking the externalities. In particular, if agent i provides some positive benefits to agent j from taking an action $a_i > 0$, we create a good ij to represent these externalities. The good ij is produced only by i and valued only by j. If agent i takes action a_i, he produces a_i units of good ij for each such j, and j enjoys the benefits of i's effort according to how much of that good j buys. Consider a market

[36]The above proposition is an analogue of Theorem 3 of Hurwicz (1979a). Because the environment studied in that paper—with assumptions such as nonzero endowments of all private goods—is not readily adapted to ours, we prove the result separately, using Hurwicz's insights combined with Maskin's Theorem.

[37]Action profiles robust to coalitional deviations correspond to those that are in the β-core, which in this environment are the same as those in the α-core. The α-core is defined by a deviating coalition first choosing their actions to maximize their payoffs and then the other players choosing actions to punish the deviating coalition given what has happened. The β-core is defined by the non-deviating players first choosing their actions to punish the deviating coalition, and then the deviators choosing actions given that (Aumann and Peleg, 1960). In our setting, as action levels of 0 for the non-deviating players always minimize the payoffs of each member of a deviating coalition, the order of the moves does not matter.

equilibrium of this artificial economy. By the core property of market equilibria, there are no profitable coalitional deviations in such an equilibrium.

But how does this relate to the profitability of coalitional deviations in the original setting, where there are externalities and not tradeable goods? The remarkable yet simple insight of Shapley and Shubik (1969) is that the standard core of the artificial economy can be identified with the set of action profiles that are robust to coalitional deviations in our setting. In defining the core of the economy with tradeable externalities, we think of a deviating coalition ceasing trade with players outside of it. When externalities are not tradeable, we define outcomes robust to coalitional deviations by positing that a deviating coalition is punished by players outside the coalition reverting to the zero action level, i.e., the action level at which the deviating coalition receives no benefits from the rest of society. Both coalitional deviations yield the same payoffs, so the same action profiles are robust to coalitional deviations in both settings. Thus, the Lindahl outcomes, as defined in Section 6.3, are robust to coalitional deviations.

6.4. **Commitment and Information.** The foundations for Lindahl outcomes that we have presented in this section have two key features: (i) some amount of commitment over actions; (ii) complete information among the negotiating agents.

The assumption of commitment is standard in implementation theory, but nevertheless crucial for overcoming the free-riding problem. Some amount of commitment is necessary to contemplate efficient solutions—whether that commitment is obtained through repeated interaction, or modeled via exogenous rules of the game, as in Sections 6.1 and 6.2. To what extent such enforcement can obtain in particular public goods problems is an important question. Our contribution is to examine, in the benchmark case where there is commitment, how the network of externalities affects an important class of efficient solutions.

In terms of information, we assume that while the designer of the game or mechanism may be ignorant of everything but the basic structure of the environment, the players interact in an environment of complete information about each other's preferences. Indeed, when transferable utility is not assumed—i.e., when Vickrey–Clarke–Groves pivot mechanisms are not available—mechanism design with interim uncertainty in environments such as ours is not well-understood.[38] Versions of our model with asymmetric information are certainly worth studying. We would expect the connections we identify between favor-trading games and networks to be relevant for that analysis.

7. Concluding Discussion

Many practical problems, such as preventing harmful climate change, entail a tragedy of the commons. It is in each agent's interest to free-ride on the efforts of others. A question at the heart of economics, and of intense public interest, is the extent to which negotiations can overcome such problems and lead to outcomes different from, and better than, the outcomes under static, unilateral decisions. Our thesis is that, in addressing this problem it is informative to study the properties of a network of externalities.

Cycles in this network are necessary for there to be any scope for a Pareto improvement, and summing these cycles in a certain way identifies whether a Pareto improvement is

[38]See, e.g., Garratt and Pycia (2015) for recent work.

possible or not. We can use this insight to identify which agents, or sets of agents, are essential to a negotiation in the sense that their participation is necessary for achieving a Pareto improvement on the status quo.

Moreover, a measure of how central agents are in this network—eigenvector centrality—tells us what actions agents would take under the Lindahl solution. In our environment, the Lindahl solution is more than just a hypothetical construct describing what we could expect if missing markets were somehow completed. The Lindahl outcomes correspond to the efficient equilibria of a bargaining game. Moreover, an implementation-theoretic analysis selects the Lindahl solutions as ones that are particularly robust to the specification of the negotiation game.

From the eigenvector centrality characterization of Lindahl outcomes, we can see that agents' actions are determined by a weighted sum of the marginal benefits they receive, as opposed to the marginal benefits they can provide to others.[39] This has implications for the design of negotiations. If there is an agent who is in a particularly strong position to provide direct and indirect benefits to others, it will be especially important to include others in the negotiation who can help this agent. Our results formalize this intuition and quantify the associated tradeoffs in the formation of a team.

The analysis in this paper also suggests novel directions for statistical work. Suppose an econometrician has imperfect observations or estimates of marginal costs and benefits and wants to test the hypothesis that an observed outcome is Pareto efficient.[40] There are many possible ways to formulate this hypothesis as a restriction on the data. We suggest one based on the spectral radius, which has the virtue of yielding test statistics with clear economic interpretations.

We first describe a naive approach, and then discuss an alternative one based on our results that has some advantages over the naive one. Recall from Section 3.1 that the first-order conditions for Pareto efficiency state that $\boldsymbol{\theta}\mathbf{J}(\mathbf{a}) = \mathbf{0}$ for some $\boldsymbol{\theta}$ in Δ^n, the simplex in $\mathbb{R}^n_+$. If one wanted to test the hypothesis that a given outcome is Pareto efficient, one could define the function $v(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\mathbf{J}(\mathbf{a})\|$ for some norm, and then define $v_{\min}(\mathbf{J}(\mathbf{a}))$ as its minimum under the data as $\boldsymbol{\theta}$ ranges over Δ^n. We would then formulate the efficiency hypothesis as

$$v_{\min}(\mathbf{J}(\mathbf{a})) = 0. \tag{10}$$

This would be a perfectly legitimate test, but the characterization of Pareto efficiency in Proposition 1(i) suggests a different formulation of the efficiency hypothesis, namely the statement that the spectral radius of the benefits matrix is equal to 1:

$$r(\mathbf{B}(\mathbf{a})) = 1. \tag{11}$$

This approach, unlike the restriction of (10), uses a test statistic that has a clear economic meaning, making rejections of the efficiency hypothesis more interpretable. For instance, if $r(\mathbf{B}(\mathbf{a}))$ is estimated to be large, that suggests underinvestment rather than overinvestment in public goods. (Recall Fact 1 in Section 3, and see also Appendix B.)

[39] The most explicit version of this statement occurs in a parametric case treated in Section OA6.3 of the Online Appendix, in which the weights of incoming walks according to an exogenous network fully determine equilibrium efforts.

[40] This discussion uses only the Pareto efficiency analysis in Section 3, without relying on any particular refinement of the Pareto frontier.

It is not clear how one would extract such information from the approach based on (10). Moreover, the spectral radius is a function with nice regularity and monotonicity properties.[41] Therefore, even when $\mathbf{B}(\mathbf{a})$ is noisily observed, the distribution of the test statistic of (11) may be particularly amenable to characterization, at least asymptotically. We believe it would be worthwhile to examine these and other related issues in the context of a fleshed-out econometric model.

[41]See Fact 1 in Section 3 for the monotonicity. Cohen (1978) discusses the behavior of the derivatives of the spectral radius $r(\mathbf{M})$ in the entries of the matrix $\mathbf{M}$, which is relevant for econometric calculations.

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Appendix A. The Perron–Frobenius Theorem

The key mathematical tool we use is the Perron–Frobenius Theorem. We state it here for ease of reference and so that we can refer to the different parts that we rely on at different points in the paper.[42]

Theorem (Perron–Frobenius)**.** Let $\mathbf{M}$ be an irreducible, square matrix with no negative entries and spectral radius $r(\mathbf{M})$. Then:

(i) The real number $r(\mathbf{M})$ is an eigenvalue of $\mathbf{M}$.
(ii) There is a vector $\mathbf{p}$ (called a Perron vector) with only positive entries such that $\mathbf{Mp} = r(\mathbf{M})\mathbf{p}$.
(iii) If $\mathbf{v}$ is a nonzero vector with nonnegative entries such that $\mathbf{Mv} = q\mathbf{v}$ for some $q \in \mathbb{R}$, then $\mathbf{v}$ is a positive scalar multiple of $\mathbf{p}$, and $q = r(\mathbf{M})$.

Note that because a matrix has exactly the same eigenvalues as its transpose, all the same statements are true, with the same eigenvalue $r(\mathbf{M}) = r(\mathbf{M}^\mathsf{T})$, when we replace $\mathbf{M}$ by its transpose $\mathbf{M}^\mathsf{T}$. This observation yields a *left*-hand Perron eigenvector of $\mathbf{M}$, i.e., a row vector $\mathbf{w}$ such that $\mathbf{wM} = r(\mathbf{M})\mathbf{w}$. For non-symmetric matrices, it is typically the case that $\mathbf{w}^\mathsf{T} \neq \mathbf{p}$. The analogue of property (iii) in the theorem holds for $\mathbf{w}$.

Appendix B. Egalitarian Pareto Improvements

This section serves two purposes. First it presents a result that is of interest in its own right, clarifying the sense in which the spectral radius of the benefits matrix measures the magnitude of inefficiency rather than merely diagnosing it. Second, it introduces some terminology and results that will be useful in subsequent proofs, particularly the proof of Proposition 2, which establishes the existence of an eigenvector centrality action profile.

Let Δ_n denote the simplex in $\mathbb{R}^n_+$ defined by $\Delta_n = \{\mathbf{d} \in \mathbb{R}^n_+ : \sum_i d_i = 1\}$.

Definition 4. The *bang for the buck* vector $\mathbf{b}(\mathbf{a}, \mathbf{d})$ at an action profile $\mathbf{a}$ along a direction $\mathbf{d} \in \Delta_n$ is defined by

$$b_i(\mathbf{a}, \mathbf{d}) = \frac{\sum_{j:j\neq i} J_{ij}(\mathbf{a})d_j}{-J_{ii}(\mathbf{a})d_i}.$$

This is the ratio

$$\frac{i\text{'s marginal benefit}}{i\text{'s marginal cost}}$$

evaluated at $\mathbf{a}$, when everyone increases actions slightly in the direction $\mathbf{d}$. We say a direction $\mathbf{d} \in \Delta_n$ is *egalitarian at* $\mathbf{a}$ if all the entries of $b_i(\mathbf{a}, \mathbf{d})$ are equal.

Proposition 7. At any $\mathbf{a}$, there is a unique egalitarian direction $\mathbf{d}^{\mathrm{eg}}(\mathbf{a})$. Every entry of $\mathbf{b}(\mathbf{a}, \mathbf{d}^{\mathrm{eg}}(\mathbf{a}))$ is equal to the spectral radius of $\mathbf{B}(\mathbf{a})$.

Proposition 7 shows that for any action profile $\mathbf{a}$, there is a unique "egalitarian" direction in which actions can be changed at $\mathbf{a}$ to equalize the marginal benefits per unit of marginal cost accruing to each agent and that this benefit-to-cost ratio will be equal to the spectral radius of $\mathbf{B}(\mathbf{a})$. Thus, the spectral radius of $\mathbf{B}(\mathbf{a})$, when it exceeds 1, can

[42]Meyer (2000, Section 8.3) has a comprehensive exposition of this theorem, its proof, and related results. Conventions vary regarding whether the Perron–Frobenius Theorem encompasses all the parts of our statement below or just (i).

be thought of as a measure of the size of Pareto improvements available by increasing actions. (A corresponding interpretation applies when the spectral radius is less than 1.)

Proof. Fix $\mathbf{a}$ and denote by r the spectral radius of $\mathbf{B}(\mathbf{a})$. Since $\mathbf{B}(\mathbf{a})$ is nonnegative and irreducible, the Perron–Frobenius Theorem guarantees that $\mathbf{B}(\mathbf{a})$ has a right-hand eigenvector $\mathbf{d}$ such that

$$\mathbf{B}(\mathbf{a})\mathbf{d} = r\mathbf{d}. \tag{12}$$

This is equivalent to $\mathbf{b}(\mathbf{a}, \mathbf{d}) = r\mathbf{1}$, where $\mathbf{1}$ is the column vector of ones. Therefore, there is an egalitarian direction that generates a bang for the buck of r (the spectral radius of $\mathbf{B}(\mathbf{a})$) for everyone.

Now suppose $\widetilde{\mathbf{d}} \in \Delta_n$ is any egalitarian direction, i.e., for some b we have

$$\mathbf{b}(\mathbf{a}, \widetilde{\mathbf{d}}) = b\mathbf{1}.$$

This implies

$$\mathbf{B}(\mathbf{a})\widetilde{\mathbf{d}} = b\widetilde{\mathbf{d}}. \tag{13}$$

By the Perron–Frobenius Theorem (statement (iii)), the only real number b and vector $\widetilde{\mathbf{d}} \in \Delta_n$ satisfying (13) are $b = r$ and $\widetilde{\mathbf{d}} = \mathbf{d}$.

Thus, $\mathbf{d}^{\text{eg}}(\mathbf{a}) = \mathbf{d}$ has all the properties claimed in the proposition's statement. □

Appendix C. Omitted Proofs

Proof of Proposition 1:

We first prove part (i). For any nonzero $\boldsymbol{\theta} \in \mathbb{R}^n_+$, define $\mathcal{P}(\boldsymbol{\theta})$, the Pareto problem with Pareto weights $\boldsymbol{\theta}$, as:

$$\text{maximize} \sum_{i \in N} \theta_i u_i(\mathbf{a}) \text{ subject to } \mathbf{a} \in \mathbb{R}^n_+.$$

Suppose that an interior action profile $\mathbf{a}^*$ is Pareto efficient. Assumption 1 guarantees that $J_{ii}(\mathbf{a}^*)$ is strictly negative. We may multiply utility functions by positive constants to achieve the normalization $J_{ii}(\mathbf{a}^*) = -1$ for each i. This is without loss of generality: It clearly does not affect Pareto efficiency, and it easy to see that scaling utility functions does not affect $\mathbf{B}(\mathbf{a}^*)$. Since $\mathbf{a}^*$ is Pareto efficient, it solves $\mathcal{P}(\boldsymbol{\theta})$ for some nonzero $\boldsymbol{\theta} \in \mathbb{R}^n_+$ (this is a standard fact for concave problems). And therefore $\mathbf{a}^*$ satisfies $\mathcal{P}(\boldsymbol{\theta})$'s system of first-order conditions: $\boldsymbol{\theta}\mathbf{J}(\mathbf{a}^*) = \mathbf{0}$. By our normalization, $\mathbf{J}(\mathbf{a}) = \mathbf{B}(\mathbf{a}) - \mathbf{I}$, where $\mathbf{I}$ is the n-by-n identity matrix, so the system of first-order conditions is equivalent to $\boldsymbol{\theta}\mathbf{B}(\mathbf{a}^*) = \boldsymbol{\theta}$.

This equation says that $\mathbf{B}(\mathbf{a}^*)$ has an eigenvalue of 1 with corresponding left-hand eigenvector $\boldsymbol{\theta}$. Since $\mathbf{B}(\mathbf{a}^*)$ is a nonnegative matrix, and irreducible by Assumption 3, the Perron–Frobenius Theorem applies to it. That theorem says that the only eigenvalue of $\mathbf{B}(\mathbf{a}^*)$ that can be associated with the nonnegative eigenvector $\boldsymbol{\theta}$ is the spectral radius itself.[43] Thus, the spectral radius of $\mathbf{B}(\mathbf{a}^*)$ must be 1.

Conversely, suppose that $\mathbf{B}(\mathbf{a}^*)$ has a largest eigenvalue of 1, and again normalize each i's utility function so that $J_{ii}(\mathbf{a}^*) = -1$. The Perron–Frobenius Theorem guarantees the existence of a nonnegative left-hand eigenvector $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}\mathbf{B}(\mathbf{a}^*) = \boldsymbol{\theta}$. Consequently,

[43] See part (iii) of the statement of the theorem in Section A.

the first-order conditions of the Pareto problem $\mathcal{P}(\boldsymbol{\theta})$ are satisfied (using the manipulation of the first-order conditions we used above). By the assumption of concave utilities, it follows that $\mathbf{a}^*$ solves the Pareto problem for weights $\boldsymbol{\theta}$ (i.e., the first-order conditions are sufficient for optimality), and so $\mathbf{a}^*$ is Pareto efficient.

We now prove part (ii), starting with the case in which $\mathbf{B}(\mathbf{0})$ is irreducible.

If $r(\mathbf{B}(\mathbf{0})) > 1$, then Proposition 7 in Section B yields an egalitarian direction at $\mathbf{0}$ with bang for the buck exceeding 1; this is a Pareto improvement at $\mathbf{0}$.

If $\mathbf{0}$ is not Pareto efficient, there is an $\mathbf{a}' \in \mathbb{R}_+$ such that $u_i(\mathbf{a}') \geq u_i(\mathbf{0})$ for each i, with strict inequality for some i. Using Assumption 3, namely the irreducibility of $\mathbf{B}(\mathbf{a}')$, as well as the continuity of the u_i, we can find[44] an $\mathbf{a}''$ with all positive entries so that $u_i(\mathbf{a}'') > u_i(\mathbf{0})$ for all i. Let $\mathbf{v}$ denote the derivative of $\mathbf{u}(\zeta\mathbf{a}'')$ in ζ evaluated at $\zeta = 0$. This derivative is strictly positive in every entry, since (by convexity of the u_i) the entry v_i must exceed $[u_i(\mathbf{a}'') - u_i(\mathbf{0})]/a''_i$. By the chain rule, $\mathbf{v} = \mathbf{J}(\mathbf{0})\mathbf{a}''$. From the fact that $\mathbf{v}$ is positive, we deduce via simple algebraic manipulation that there is a positive vector $\mathbf{w}$ so that $\mathbf{B}(\mathbf{0})\mathbf{w} > \mathbf{w}$. And from this it follows by the Collatz–Wielandt formula (Meyer, 2000, equation 8.3.3) that the spectral radius of $\mathbf{B}(\mathbf{0})$ exceeds 1.

Now assume $\mathbf{B}(\mathbf{0})$ is reducible.

First, suppose $r(\mathbf{B}(\mathbf{0})) > 1$. Then the same is true when $\mathbf{B}(\mathbf{0})$ is replaced by one of its irreducible blocks, and in that case a Pareto improvement on $\mathbf{0}$ (involving only the agents in the irreducible block taking positive effort) is found as above. So $\mathbf{0}$ is not Pareto efficient.

Conversely, suppose $\mathbf{0}$ is not Pareto efficient. There is an $\mathbf{a}' \in \mathbb{R}_+$ such that $u_i(\mathbf{a}') \geq u_i(\mathbf{0})$ for each i, with strict inequality for some i. Let $P = \{i : a'_i > 0\}$ be the set of agents taking positive actions at $\mathbf{a}'$. And let $\widehat{\mathbf{B}}(\mathbf{0})$ be obtained by restricting $\mathbf{B}(\mathbf{0})$ to P (i.e. by throwing away rows and columns not corresponding to indices in P). For each $i \in P$, there is a $j \in P$ such that $\widehat{B}_{ij}(\mathbf{0}) > 0$; otherwise, i would be worse off than at $\mathbf{0}$. Therefore, each $i \in P$ is on a cycle[45] in $\widehat{\mathbf{B}}(\mathbf{0})$. And it follows that for each $i \in P$ there is a set $P_i \subseteq P$ such that $\widehat{\mathbf{B}}(\mathbf{0})$ is irreducible when restricted to P_i. Next, applying the argument of footnote 44 to each such P_i separately, we can find $\mathbf{a}''$ such that $u_i(\mathbf{a}'') > u_i(\mathbf{0})$ for each $i \in P$. From this point we can argue as above[46] to conclude that $r(\widehat{\mathbf{B}}(\mathbf{0})) > 1$. Since $\widehat{\mathbf{B}}(\mathbf{0})$ is a submatrix of $\mathbf{B}(\mathbf{0})$, by Fact 1, $r(\mathbf{B}(\mathbf{0})) > 1$. ∎

Proof of Theorem 1: We first prove the following Lemma.

Lemma 1. If $\mathbf{a}^* \neq \mathbf{0}$ is a Lindahl outcome for preference profile $\mathbf{u}$, then $\mathbf{a}^* \in \mathbb{R}^n_{++}$.

Proof. Assume, toward a contradiction, that $\mathbf{a}^*$ has some entries equal to 0. Let $\mathbf{P}$ be the matrix of prices that support $\mathbf{a}^*$ as a Lindahl outcome. Let S be the set of i so that

[44]Suppose otherwise and let $\mathbf{a}''$ be chosen so that $\mathbf{u}(\mathbf{a}'') - \mathbf{u}(\mathbf{0}) \geq \mathbf{0}$ (note this is possible, since $\mathbf{a}'' = \mathbf{a}'$ satisfies this inequality) and so that the number of 0 entries in $\mathbf{u}(\mathbf{a}'') - \mathbf{u}(\mathbf{0})$ is as small as possible. Let S be the set of i for which $u_i(\mathbf{a}'') - u_i(\mathbf{a}) > 0$. Then by irreducibility of benefits, we can find $j \in S$ and $k \notin S$ such that $J_{kj}(\mathbf{0}) > 0$. Define $a'''_j = a''_j + \varepsilon$ and $a'''_i = a''_i$ for all $i \neq j$. If $\varepsilon > 0$ is chosen small enough, then by continuity of the u_i we have $u_i(\mathbf{a}''') - u_i(\mathbf{a}) > 0$ for all $i \in S$, but also $u_k(\mathbf{a}''') - u_k(\mathbf{a}) > 0$, contradicting the choice of $\mathbf{a}''$.

[45]Recall the definition in Section 3.2.2.

[46]The Collatz–Wielandt formula does not assume irreducibility.

$a_i^* = 0$, which is a proper subset of N since $\mathbf{a}^* \neq \mathbf{0}$. By Assumption 3 (connectedness of benefit flows), there is an $i \in S$ and a $j \notin S$ so that $J_{ij}(\mathbf{a}^*) > 0$. We will argue that this implies

$$P_{ij} > 0.$$

If this were not true, then an $\mathbf{a} \neq \mathbf{a}^*$ in which only j increases his action slightly relative to $\mathbf{a}^*$ would satisfy $\mathrm{BB}_i(\mathbf{P})$ in Definition 1 and be preferred by i to the outcome $\mathbf{a}^*$, contradicting the definition of a Lindahl outcome.

Now consider $\mathrm{BB}_i(\mathbf{P})$, the budget balance condition of agent i, at the outcome $\mathbf{a}^*$:

$$\sum_{k:k \neq i} P_{ik} a_k^* \leq a_i^* \sum_{k:k \neq i} P_{ki}.$$

Since $a_i^* = 0$, the right-hand side of this is 0. But $P_{ij} > 0$, and $a_j^* > 0$ (since $j \notin S$), so the left-hand side is positive. That is a contradiction. ☐

It will now be convenient to use an equivalent definition of Lindahl outcomes:

Definition 5. An action profile $\mathbf{a}^*$ is a *Lindahl outcome* for a preference profile $\mathbf{u}$ if there exists an n-by-n matrix $\mathbf{P}$ with each column summing to 0, so that the following conditions hold for every i:

(i) The inequality

$$(\widehat{\mathrm{BB}}_i(\mathbf{P})) \qquad \sum_{j \in N} P_{ij} a_j \leq 0$$

is satisfied when $\mathbf{a} = \mathbf{a}^*$;

(ii) for any $\mathbf{a}$ such that $\widehat{\mathrm{BB}}_i(\mathbf{P})$ is satisfied, we have $u_i(\mathbf{a}^*) \geq u_i(\mathbf{a})$.

Given a Lindahl outcome defined as in Definition 1, set $P_{ii} = -\sum_{j:j \neq i} P_{ji}$ to find prices satisfying the new definition.[47] Conversely, the prices of Definition 5 work in Definition 1 without modification, since the original definition does not involve the diagonal terms of $\mathbf{P}$ at all.

We now show (ii) implies (i). Suppose $\mathbf{a}^* \in \mathbb{R}_+^n$ is a nonzero Lindahl outcome. Lemma 1 implies that $\mathbf{a}^* \in \mathbb{R}_{++}^n$, or in other words that $\mathbf{a}^*$ has only positive entries. Let $\mathbf{P}$ be the matrix of prices satisfying the conditions of Definition 5. Consider the following program for each $i \in N$, denoted by $\Pi_i(\mathbf{P})$:

$$\text{maximize } u_i(\mathbf{a}) \text{ subject to } \mathbf{a} \in \mathbb{R}_+^n \text{ and } \widehat{\mathrm{BB}}_i(\mathbf{P}).$$

By definition of a Lindahl outcome, $\mathbf{a}^*$ solves $\Pi_i(\mathbf{P})$. By Assumption 3, there is some agent $j \neq i$ such that increases in his action a_j would make i better off. Therefore, the budget balance constraint $\widehat{\mathrm{BB}}_i(\mathbf{P})$ is satisfied with equality, so that $\mathbf{P}\mathbf{a}^* = \mathbf{0}$. Because $\mathbf{a}^*$ is interior, the gradient of the maximand u_i must be orthogonal to the constraint set given by $\widehat{\mathrm{BB}}_i(\mathbf{P})$. In other words, row i of $\mathbf{J}(\mathbf{a}^*)$ is parallel to row i of $\mathbf{P}$. These facts together imply $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$ and so $\mathbf{B}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{a}^*$ (see Section 6.1).

We now show that (i) implies (ii). Since $\mathbf{a}^*$ is a nonnegative right-hand eigenvector of $\mathbf{B}(\mathbf{a}^*)$, the Perron–Frobenius Theorem guarantees that 1 is a largest eigenvalue of $\mathbf{B}(\mathbf{a}^*)$.

[47]In essence, $-P_{ii}$ is the total subsidy agent i receives per unit of effort, equal to the sum of personalized taxes paid by other people to him for his effort.

Arguing as in the proof of Proposition 1(i), we deduce that there is a nonzero vector $\boldsymbol{\theta}$ for which $\boldsymbol{\theta}\mathbf{J}(\mathbf{a}^*) = \mathbf{0}$. We need to find prices supporting $\mathbf{a}^*$ as a Lindahl outcome. Define the matrix $\mathbf{P}$ by $P_{ij} = \theta_i J_{ij}(\mathbf{a}^*)$ and note that for all $j \in N$ we have

$$\sum_{i \in N} P_{ij} = \sum_{i \in N} \theta_i J_{ij}(\mathbf{a}^*) = [\boldsymbol{\theta}\mathbf{J}(\mathbf{a}^*)]_j = 0, \tag{14}$$

where $[\boldsymbol{\theta}\mathbf{J}(\mathbf{a}^*)]_j$ refers to entry j of the vector $\boldsymbol{\theta}\mathbf{J}(\mathbf{a}^*)$.

Note that $\mathbf{B}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{a}^*$ implies $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$ and each row of $\mathbf{P}$ is just a scaling of the corresponding row of $\mathbf{J}(\mathbf{a}^*)$. We therefore have:

$$\mathbf{P}\mathbf{a}^* = \mathbf{0}, \tag{15}$$

and these prices satisfy budget balance.

We claim that, for each i, the vector $\mathbf{a}^*$ solves $\Pi_i(\mathbf{P})$. This is because the gradient of u_i at $\mathbf{a}^*$, which is row i of $\mathbf{J}(\mathbf{a}^*)$, is normal to the constraint set by construction of $\mathbf{P}$. Moreover, by (15) above, $\mathbf{a}^*$ satisfies the constraint $\widehat{\mathrm{BB}}_i(\mathbf{P})$. The claim then follows by the concavity of u_i. ∎

Proof of Proposition 2: We will use the Kakutani Fixed Point Theorem to find an eigenvector centrality action profile. Define $Y = \{\mathbf{a} \in \mathbb{R}^n_+ : \min_i [\mathbf{J}(\mathbf{a})\mathbf{a}]_i > 0\}$, the set of action profiles $\mathbf{a}$ at which everyone has positive gains from scaling $\mathbf{a}$ up. It is easy to check that Y is convex.[48] Also, Y is bounded by Assumption 4. Thus, $\overline{Y}$, the closure of Y, is compact.[49]

Define the correspondence $F : \overline{Y} \setminus \{\mathbf{0}\} \rightrightarrows \overline{Y}$ by

$$F(\mathbf{a}) = \{\lambda \mathbf{a} \in \overline{Y} : \lambda \geq 0 \text{ and} \min_i [\mathbf{J}(\lambda\mathbf{a})\mathbf{a}]_i \leq 0\}.$$

This correspondence, given the argument $\mathbf{a}$, returns all actions $\lambda\mathbf{a}$ (i.e., on the same ray as $\mathbf{a}$) such that, at $\lambda\mathbf{a}$, at least one agent does not want to further scale up actions. Finally, recalling the definition of $\mathbf{d}^{\mathrm{eg}}(\mathbf{a})$ from Appendix B, define the correspondence $G : \overline{Y} \rightrightarrows \overline{Y}$ by

$$G(\mathbf{a}) = F(\mathbf{d}^{\mathrm{eg}}(\mathbf{a})).$$

Note that $\mathbf{d}^{\mathrm{eg}}(\mathbf{a})$ is always nonzero, so that the argument of F is in its domain.[50] The function $\mathbf{d}^{\mathrm{eg}}$ is continuous (Wilkinson, 1965, pp. 66–67), and F is clearly upper hemicontinuous, so it follows that G is upper hemicontinuous. Finally, from the definitions of Y

[48]Given $\mathbf{a}, \mathbf{a}' \in Y$ and $\lambda \in [0,1]$, define $\mathbf{a}'' = \lambda\mathbf{a} + (1-\lambda)\mathbf{a}'$. Note that for all $i \in N$ and $\varepsilon \geq -1$

$$u_i((1+\varepsilon)\mathbf{a}'') \geq \lambda u_i((1+\varepsilon)\mathbf{a}) + (1-\lambda)u_i((1+\varepsilon)\mathbf{a}')$$

by concavity of the u_i. Differentiating in ε at $\varepsilon = 0$ yields the result.

[49]It is tempting to define $Y = \{\mathbf{a} \in \mathbb{R}^n_+ : \min_i [\mathbf{J}(\mathbf{a})\mathbf{a}]_i \geq 0\}$ instead and avoid having to take closures; but this set can be unbounded even when $\overline{Y}$ as we defined it above *is* bounded. For example, our assumptions do not exclude the existence of an (infinite) ray along which $\min_i [\mathbf{J}(\mathbf{a})\mathbf{a}]_i = 0$.

[50]Even though the domain of F is not a compact set, G is a correspondence from a compact set into itself.

and F it follows that F is nonempty-valued.[51] Since $\overline{Y}$ is a compact and convex set, the Kakutani Fixed Point Theorem implies that there is an $\mathbf{a} \in \overline{Y}$ such that $\mathbf{a} \in F(\mathbf{d}^{\text{eg}}(\mathbf{a}))$. Writing $\widehat{\mathbf{a}} = \mathbf{d}^{\text{eg}}(\mathbf{a})$, this means that there is some $\lambda \geq 0$ such that $\min_i[\mathbf{J}(\lambda\widehat{\mathbf{a}})\widehat{\mathbf{a}}]_i \leq 0$. Let $\mathbf{a}^* = \lambda\widehat{\mathbf{a}}$. We will argue that $\mathbf{a}^*$ satisfies scaling-indifference (and is therefore an eigenvector centrality action profile).

Suppose $\mathbf{a}^* \neq \mathbf{0}$. Then by continuity of the function $\lambda \mapsto \mathbf{J}(\lambda\widehat{\mathbf{a}})\widehat{\mathbf{a}}$, there is some i for which we have $[\mathbf{J}(\mathbf{a}^*)\widehat{\mathbf{a}}]_i = 0$, so that some player's marginal benefit to scaling is equal to his marginal cost. Since $\widehat{\mathbf{a}}$ is an egalitarian direction at the action profile $\mathbf{a}^*$, the equation $[\mathbf{J}(\mathbf{a}^*)\widehat{\mathbf{a}}]_i = 0$ must hold for *all* i, and therefore $\mathbf{J}(\mathbf{a}^*)\widehat{\mathbf{a}} = \mathbf{0}$. Since $\widehat{\mathbf{a}}$ and $\mathbf{a}^*$ are parallel, we deduce $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$. The condition $\mathbf{J}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$ and Assumption 3—connectedness of benefit flows—imply that $\mathbf{a}^* \in \mathbb{R}^n_{++}$.

If $\mathbf{a}^* = \mathbf{0}$, consider the bang for the buck vector $\mathbf{b}(\mathbf{0}, \widehat{\mathbf{a}})$, which corresponds to starting at $\mathbf{0}$ and moving in the egalitarian direction $\widehat{\mathbf{a}}$. Because $\widehat{\mathbf{a}}$ is egalitarian, we can write $\mathbf{b}(\mathbf{0}, \widehat{\mathbf{a}}) = b\mathbf{1}$ for some b. And we can deduce that b is no greater than 1—otherwise, $F(\widehat{\mathbf{a}})$ would not contain $\mathbf{a}^* = \mathbf{0}$. By Proposition 7, it follows that $r(\mathbf{B}(\mathbf{0})) \leq 1$. Then $\mathbf{0}$ is Pareto efficient by Proposition 1(ii). ∎

Proof of Proposition 3: Let $\mathcal{W}_i^{\uparrow}(\ell; \mathbf{M})$ be the set of all walks of length ℓ in a matrix $\mathbf{M}$ starting at i, so that $w(1) = i$. The proof follows immediately from the following observation.

Fact **2.** For any irreducible, nonnegative matrix $\mathbf{Q}$, and any i, j

$$\frac{p_i}{p_j} = \lim_{\ell \to \infty} \frac{\sum_{w \in \mathcal{W}_i^{\uparrow}(\ell; \mathbf{Q})} v(w; \mathbf{Q})}{\sum_{w \in \mathcal{W}_j^{\uparrow}(\ell; \mathbf{Q})} v(w; \mathbf{Q})},$$

where $\mathbf{p}$ is any nonnegative right-hand eigenvector of $\mathbf{Q}$ (i.e. a right-hand Perron vector in the terminology of Section A).

Proof. Note that the formula above is equivalent to

$$\frac{p_i}{p_j} = \lim_{\ell \to \infty} \frac{\sum_k \left[\mathbf{Q}^\ell\right]_{ik}}{\sum_k \left[\mathbf{Q}^\ell\right]_{jk}}, \tag{16}$$

where $\left[\mathbf{Q}^\ell\right]_{ik}$ denotes the entry in the (i, k) position of the matrix $\mathbf{Q}^\ell$. To prove (16), let $\rho = r(\mathbf{Q})$ and note that

$$\lim_{\ell \to \infty} (\mathbf{Q}/\rho)^\ell = \mathbf{w}^\mathsf{T}\mathbf{p}, \tag{17}$$

where $\mathbf{w}$ is a left-hand Perron vector of $\mathbf{Q}$, and $\mathbf{p}$ is a right-hand Perron vector (recall Section A). This is statement (8.3.13) in Meyer (2000); the hypothesis that $\mathbf{Q}$ is primitive

[51]Toward a contradiction, take a nonzero $\mathbf{a}$ such that $F(\mathbf{a})$ is empty. Let $\overline{\lambda}$ be the maximum λ such that $\lambda\mathbf{a} \in \overline{Y}$; such a $\overline{\lambda}$ exists because $\mathbf{a}$ is nonzero and $\overline{Y}$ is compact. Since $\mathbf{J}(\overline{\lambda}\mathbf{a})\mathbf{a} > \mathbf{0}$ it follows that for all i,

$$\left.\frac{du_i((1+\varepsilon)\overline{\lambda}\mathbf{a})}{d\varepsilon}\right|_{\varepsilon=0} > 0,$$

from which it follows that $(\overline{\lambda} + \delta)\mathbf{a} \in Y$ for small enough δ. This contradicts the choice of $\overline{\lambda}$ (recalling the definition of Y).

in that statement follows from the assumed aperiodicity of $\mathbf{Q}$ (see Theorems 1 and 2 of Perkins (1961)). To conclude, observe that (17) directly implies (16). $\square$

To prove the proposition from Fact 2, set $\mathbf{Q} = \mathbf{B}(\mathbf{a}) = \mathbf{M}^\mathsf{T}$ and note that then the right-hand side of the equation in Fact 2 is equal to the right-hand side of the equation in Proposition 3. The statement that $\mathbf{a}$ is an eigenvector centrality action profile is equivalent to the statement that $\mathbf{a}$ is a right-hand Perron eigenvector of $\mathbf{Q} = \mathbf{B}(\mathbf{a})$.

Proof of Proposition 4: We begin by showing that in all Pareto efficient perfect equilibria, an eigenvector centrality action profile must be played.

Pareto efficiency requires two things. First, as there is discounting ($\delta < 1$), it requires that that agreement be reached at the first round of negotiations. Second, Pareto efficiency requires that, almost surely, some particular action profile be played on the equilibrium path, regardless of the state reached in the first period. Toward a contradiction, suppose there is immediate agreement but that different agreements are reached in different states that occur with positive probability. Let $\mathbf{a}(s)$ be the actions played in equilibrium in state s. The probability of being in state s for the first round of negotiations is $p(s_0, s)$. As utility functions are strictly concave, a Pareto improvement can be obtained by the players choosing strategies that result in the deterministic action profile $\overline{\mathbf{a}} = \sum_{s \in S} p(s_0, s)\mathbf{a}(s)$ being played in all states.

So let $\mathbf{a}$ be the nonrandom Pareto efficient action profile on which players immediately agree in some efficient perfect equilibrium of the game. We will show it is an eigenvector centrality action profile. If $\mathbf{J}(\mathbf{a})\mathbf{a}$ has a negative entry, say i, then player i did not best-respond in stage (iii) of the game, in which a scaling was selected. By choosing a smaller λ_i (for example, the largest λ_i such that $[\mathbf{J}(\lambda_i \mathbf{d})\mathbf{d}]_i \geq 0$), that player would have secured a strictly higher payoff.

Therefore, $\mathbf{J}(\mathbf{a})\mathbf{a} \geq \mathbf{0}$. We claim this holds with equality. Suppose, by way of contradiction, that it does not. Then

$$\mathbf{D}(\mathbf{a})^{-1}\mathbf{J}(\mathbf{a})\mathbf{a} \gneqq \mathbf{0},$$

where $\mathbf{D}(\mathbf{a})$ is a diagonal matrix with $D_{ii}(\mathbf{a}) = -J_{ii}(\mathbf{a})$ and zeros off the diagonal. We then have $\left(\mathbf{D}(\mathbf{a})^{-1}\mathbf{J}(\mathbf{a}) + \mathbf{I}\right)\mathbf{a} = \mathbf{B}(\mathbf{a})\mathbf{a} \gneqq \mathbf{a}$. By irreducibility of $\mathbf{B}(\mathbf{a})$, there then exists an $\mathbf{a}'$ such that $\mathbf{B}(\mathbf{a})\mathbf{a}' > \mathbf{a}'$ (with strict inequalities in each entry). The Collatz–Wielandt formula (Meyer, 2000, equation 8.3.3) says that $r(\mathbf{B}(\mathbf{a}))$ is given by:

$$\min_{a'_i} \frac{[\mathbf{B}(\mathbf{a})\mathbf{a}']_i}{a'_i}.$$

Thus, $r(\mathbf{B}(\mathbf{a})) > 1$ and, by Proposition 1, $\mathbf{a}$ is Pareto inefficient, which is a contradiction.

Thus, we have established that $\mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0}$. Because the action profile $\mathbf{0}$ is not Pareto efficient by assumption, we deduce that $\mathbf{a}$ is nonzero, and therefore it is scaling-indifferent. Applying the definition of $\mathbf{B}$, we conclude $\mathbf{a}$ is an eigenvector centrality action profile.

To finish the proof, it remains only to show that for any eigenvector centrality action profile $\mathbf{a}$, we can find a perfect equilibrium that supports it. The strategies are as follows: Any player, when proposing a direction, suggests $\mathbf{d} = \mathbf{a}/\sum_i a_i$, i.e., the normalization of $\mathbf{a}$. When responding to proposals, every player vetoes any direction other than $\mathbf{d}$. On the other hand, if $\mathbf{d}$ is proposed, then player i sets

$$\lambda_i = \min\{\lambda : [\mathbf{J}(\lambda \mathbf{d})\mathbf{d}]_i \leq 0\}. \tag{18}$$

This is well-defined because for $\lambda = \sum_i a_i$, we have $\mathbf{J}(\lambda \mathbf{d})\mathbf{d} = \lambda^{-1}\mathbf{J}(\mathbf{a})\mathbf{a}$, all of whose entries are 0 because $\mathbf{a}$ is scaling-indifferent. Indeed, by strict concavity of the utility functions, $[\mathbf{J}(\lambda \mathbf{d})\mathbf{d}]_i$ is decreasing in λ and so $\lambda_i = \sum_i a_i$ for all i. Thus direction $\mathbf{d}$ is proposed and actions $\lambda \mathbf{d} = \mathbf{a}$ are selected under this strategy profile.

The proof that this is an equilibrium is straightforward. Consider i's incentives. Given that the other players respond to proposals as specified in this strategy profile, the only outcomes that can ever be implemented are in the set $P = \{\mu \mathbf{d} : 0 \leq \mu \leq \max_{j \neq i} \lambda_j\}$. Consider a subgame where someone has proposed direction $\mathbf{d}$. Voting "no" can yield only some action in P at a later period (or no agreement forever). By definition of λ_i, responding with λ_i yields maximum utility among all points in P; thus, players have incentives to follow the strategy profile when responding to a proposal of direction $\mathbf{d}$. The same argument shows that proposing a direction other than $\mathbf{d}$ cannot be a profitable deviation—it will result in rejection and the implementation of something in P later—whereas by playing the proposed equilibrium, i could obtain the payoff of $\mathbf{a}$ now. Finally, when a direction other than $\mathbf{d}$ is proposed, players are indifferent between voting "yes" and voting "no", because the proposal will be rejected by the votes of the others. ∎

Proof of Proposition 6: Applying Theorem 1, we will work with the Lindahl outcomes rather than the eigenvector centrality action profiles. Let $\mathbf{a}^* \in \mathbb{R}^n_+$ be a Lindahl outcome and $\mathbf{P}$ the associated price matrix, satisfying the conditions of Definition 5 (recall that this is an equivalent definition of a Lindahl outcome, given in the proof of Theorem 1 above). Then we have

$$\mathbf{a}^* \in \operatorname{argmax} u_i(\mathbf{a}) \text{ s.t. } \mathbf{a} \in \mathbb{R}^n_+ \text{ and } \sum_{j \in N} P_{ij} a_j \leq 0.$$

We will refer to this convex program as the Lindahl problem. We now use these properties of $\mathbf{a}^*$ to show that it is robust to coalitional deviations. Pareto efficiency of $\mathbf{a}^*$, which follows by Proposition 1, ensures the grand coalition doesn't have a profitable deviation. We now rule out all other possible coalitional deviations. Toward a contradiction, suppose $\mathbf{a}^*$ is not robust to coalitional deviations, and therefore that there exists a nonempty proper coalition M and an $\mathbf{a}'$ (with $a'_i = 0$ for $i \notin M$) for which $u_i(\mathbf{a}') \geq u_i(\mathbf{a}^*)$ for each $i \in M$, with strict inequality for some $i \in M$. Since $\mathbf{a}^*$ solves the Lindahl problem, we must have that the action profile $\mathbf{a}'$ is weakly unaffordable to i at prices $\mathbf{P}$: $\sum_{j \in N} P_{ij} a'_j \geq 0$ for each $i \in M$.[52]

There are then two cases to consider. Suppose first that there is some $i \in M$ such that $u_i(\mathbf{a}') > u_i(\mathbf{a}^*)$ and so $\sum_{j \in N} P_{ij} a'_j > 0$. If this is true, then:

$$\sum_{i \in M} \sum_{j \in M} P_{ij} a'_j > 0. \tag{19}$$

[52]Suppose $\sum_{j \in N} P_{ij} a'_j < 0$ for some $i \in M$. It follows that, while satisfying the assumption $\sum_{j \in N} P_{ij} a'_j \leq 0$, every a_j for $j \neq i$ can be increased slightly; by Assumption 3, this makes i better off.

On the other hand,

$$\sum_{i\in M}\sum_{j\in M} P_{ij}a'_j = \sum_{j\in M} a'_j \sum_{i\in M} P_{ij} \leq \sum_{j\in M} a'_j \sum_{i\in N} P_{ij} = 0. \tag{20}$$

The first equality follows by switching the order of summation, the inequality holds because $P_{ij} \geq 0$ for $j \neq i$, and the final equality follows from $P_{ii} = -\sum_{j:j\neq i} P_{ji}$ for all i. Equation (20) contradicts equation (19). ∎

Appendix D. Additional Results

Proposition 8. The following are equivalent:

(i) $r(\mathbf{B}(\mathbf{0})) \leq 1$;
(ii) $\mathbf{0}$ is a Pareto efficient action profile;
(iii) $\mathbf{0}$ is a Lindahl outcome.

Proof. Proposition 1(ii) establishes the equivalence between (i) and (ii).

(ii) ⇒ (iii): The construction of prices is exactly analogous to the proof of Theorem 1; the only difference is that rather than the Pareto weights, we use Pareto weights adjusted by the Lagrange multipliers on the binding constraints $a_i \geq 0$.

(iii) ⇒ (ii): The standard proof of the First Welfare Theorem goes through without modification; see, e.g., Foley (1970).

□

ONLINE APPENDIX: A NETWORK APPROACH TO PUBLIC GOODS

Throughout the online appendix, we refer often to sections, results, and equations in the main text and its appendix using the numbering established there (e.g., Section 2.2, Appendix A, equation (4)). The numbers of sections, results, and equations in this online appendix are all prefixed by OA to distinguish them, and we always use this prefix in referring to them.

OA1. Multiple Actions

This section extends our environment to permit each agent to take actions in multiple dimensions, and then proves analogues of our main results. We focus on what we consider to be the essence of our analysis—namely the equivalence of certain eigenvalue properties, and certain matrix equations, to efficient and Lindahl outcomes. Other important matters—existence of efficient and Lindahl points, as well as their strategic microfoundations—are not treated here, but we believe that the techniques introduced in the main text would establish analogous results.

OA1.1. **Environment.** We adjust the environment only by permitting players to take multi-dimensional actions $\mathbf{a}_i \in \mathbb{R}_+^k$, with entry d of player i's action vector being denoted by a_i^d. Each player then has a utility function $u_i : \mathbb{R}_+^{nk} \to \mathbb{R}$. When we need to think of $\mathbf{a}$ as a vector—i.e., when we need an explicit order for its coordinates—we will use the following one. First we list all actions on the first dimension, then all actions on the second dimension, etc.:

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_\cdot^{[1]} \\ \mathbf{a}_\cdot^{[2]} \\ \vdots \\ \mathbf{a}_\cdot^{[k]} \end{bmatrix}.$$

For each $d \in \{1, 2, \ldots, k\}$, we construct the n-by-n Jacobian $\mathbf{J}^{[d]}(\mathbf{a})$ by setting $J_{ij}^{[d]}(\mathbf{a}) = \partial u_i(\mathbf{a})/\partial a_j^d$. We define the benefits matrix:

$$B_{ij}^{[d]}(\mathbf{a}; \mathbf{u}) = \begin{cases} \frac{J_{ij}^{[d]}(\mathbf{a};\mathbf{u})}{-J_{ii}^{[d]}(\mathbf{a};\mathbf{u})} & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

The following assumptions are made on these new primitives. First, utility functions are concave and continuously differentiable. Second, all actions are costly.[1] Third, there are weakly positive externalities from all actions.[2] Fourth, benefit flows are connected, so that each matrix $\mathbf{B}^{[d]}(\mathbf{a})$ is irreducible, for all $\mathbf{a}$. These assumptions are very similar to those we required in the one-dimensional case.

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[1] $\partial u_i(\mathbf{a})/\partial a_i^k < 0$ for all i and all k.

[2] $\partial u_i(\mathbf{a})/\partial a_j^k \geq 0$ for all $j \neq i$ and all k.

OA1.2. **Efficiency.** The generalization of our efficiency result is as follows. Recall that, by the Perron–Frobenius theorem, any nonnegative, irreducible square matrix $\mathbf{M}$ has a left eigenvector $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}\mathbf{M} = r(\mathbf{M})\boldsymbol{\theta}$, where $r(\mathbf{M})$ is the spectral radius. This eigenvector is determined uniquely up to scale, and imposing the normalization that $\boldsymbol{\theta} \in \Delta_n$ (the simplex in $\mathbb{R}^n_+$) we call it *the* Perron vector of $\mathbf{M}$.

PROPOSITION **OA1.** Consider an interior action profile $\mathbf{a} \in \mathbb{R}^{nk}_{++}$. Then the following are equivalent:

(i) $\mathbf{a}$ is Pareto efficient;
(ii) every matrix in the set $\{\mathbf{B}^{[d]}(\mathbf{a}) : d = 1, \ldots, k\}$ has spectral radius 1, and they all have the same left Perron vector.

Proof. For any nonzero $\boldsymbol{\theta} \in \Delta_n$ define $\mathcal{P}(\boldsymbol{\theta})$, the Pareto problem with weights $\boldsymbol{\theta}$ as:

$$\text{maximize } \sum_{i \in N} \theta_i u_i(\mathbf{a}) \text{ subject to } \mathbf{a} \in \mathbb{R}^{nk}_+.$$

By a standard fact, an action profile $\mathbf{a}$ is Pareto efficient if and only if it solves $\mathcal{P}(\boldsymbol{\theta})$ for some $\boldsymbol{\theta} \in \Delta_n$. The first order conditions for this problem consist of the equations $\sum_i \theta_i \partial u_i(\mathbf{a})/\partial a_j^d = 0$ for all j and all d. Rearranging, and recalling the assumption that $\partial u_j(\mathbf{a})/\partial a_j^d < 0$ for every i and d, we have:

$$\theta_j = \sum_{i \neq j} \theta_i \frac{\partial u_i(\mathbf{a})/\partial a_j^d}{-\partial u_j(\mathbf{a})/\partial a_j^d}. \tag{OA-1}$$

Given the concavity of $\mathbf{u}$, these conditions are necessary and sufficient for an interior optimum. We can summarize these conditions as the system of (matrix) equations:

$$\boldsymbol{\theta} = \boldsymbol{\theta}\mathbf{B}^{[d]}(\mathbf{a}) \quad d = 1, 2, \ldots, k. \tag{OA-2}$$

In summary, (i) is equivalent to the statement "system (OA-2) holds for some nonzero $\boldsymbol{\theta} \in \Delta_n$," and so we will treat the two interchangeably.

We can now show (i) and (ii) are equivalent. System (OA-2) holding for a nonzero $\boldsymbol{\theta} \in \Delta_n$ entails that the spectral radius of each $\mathbf{B}^{[d]}(\mathbf{a})$ is 1, because (by the Perron–Frobenius Theorem) a nonnegative eigenvector can correspond only to a largest eigenvalue. And the same system says a single $\boldsymbol{\theta}$ is a left Perron vector for each $\mathbf{B}^{[d]}(\mathbf{a})$. So (ii) holds. Conversely, if (ii) holds, then there is some left Perron vector $\boldsymbol{\theta} \in \Delta_n$ so that the system in (OA-2) holds, which (as we have observed) is equivalent to (i). □

OA1.3. **Characterizing Lindahl Outcomes.** Our characterization of Lindahl outcomes will rely on some "stacked" versions of matrices we have encountered before. We define a stacked n-by-nk Jacobian as follows:

$$\underline{\mathbf{J}}(\mathbf{a}) = \begin{bmatrix} \mathbf{J}^{[1]}(\mathbf{a}) & \mathbf{J}^{[2]}(\mathbf{a}) & \cdots & \mathbf{J}^{[k]}(\mathbf{a}) \end{bmatrix}.$$

For defining a Lindahl outcome, we will need to think of a larger price matrix. In particular, we will introduce an n-by-nk matrix

$$\underline{\mathbf{P}} = \begin{bmatrix} \mathbf{P}^{[1]} & \mathbf{P}^{[2]} & \cdots & \mathbf{P}^{[k]} \end{bmatrix},$$

where $P_{ij}^{[d]}$ (with $i \neq j$) is interpreted as the price i pays for the effort of agent j on dimension d.

To generalize our main theorem on the characterization of Lindahl outcomes, we now define a Lindahl outcome in the multi-dimensional setting. (Recall Definition 1, and from Section 4.1 that the budget balance condition can be restated as $\mathbf{Pa}^* \leq \mathbf{0}$.)

Definition **OA1.** An action profile $\mathbf{a}^*$ is a *Lindahl outcome* for a preference profile u_i if there is an n-by-nk price matrix $\underline{\mathbf{P}}$, with each column summing to zero, so that the following conditions hold for every i:

(i) The inequality

$$\underline{\mathbf{P}}\mathbf{a} \leq \mathbf{0} \tag{$\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$}$$

is satisfied when $\mathbf{a} = \mathbf{a}^*$;

(ii) for any $\mathbf{a}$ such that $\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$ is satisfied, we have $\mathbf{a}^* \succeq_{u_i} \mathbf{a}$.

Definition **OA2.** The action vector $\mathbf{a} \in \mathbb{R}^{nk}_+$ is defined to be *scaling-indifferent* if $\mathbf{a} \neq \mathbf{0}$ and $\underline{\mathbf{J}}(\mathbf{a})\mathbf{a} = \mathbf{0}$.

We will establish that Lindahl outcomes are characterized by being scaling-indifferent and Pareto efficient.

Theorem **OA1.** Under the maintained assumptions, an interior action profile is a Lindahl outcome if and only if it is scaling-indifferent and Pareto efficient.

Proof. First, we show Lindahl outcomes are scaling-indifferent and Pareto efficient. Suppose $\mathbf{a}^* \in \mathbb{R}^{nk}_{++}$ is a nonzero Lindahl outcome. Its Pareto efficiency follows by the standard proof of the first welfare theorem. Let $\underline{\mathbf{P}}$ be the price matrix in Definition OA1. Consider the following program for each agent i, denoted by $\Pi_i(\underline{\mathbf{P}})$:

$$\text{maximize } u_i(\mathbf{a}) \text{ subject to } \mathbf{a} \in \mathbb{R}^{nk}_+ \text{ and } \widehat{\mathrm{BB}}_i(\underline{\mathbf{P}}).$$

By definition of a Lindahl outcome, $\mathbf{a}^*$ solves $\Pi_i(\underline{\mathbf{P}})$. By the assumption of connected benefit flows, there is always some other agent j and some dimension d so that i is better off when a^d_j increases. So the budget balance constraint $\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$ is satisfied with equality. Note that this is equivalent to the statement $\underline{\mathbf{P}}\mathbf{a}^* = \mathbf{0}$.

Because $\mathbf{a}^*$ is interior, the gradient of the maximand u_i (viewed as a function of $\mathbf{a}$) must be orthogonal to the budget constraint $\underline{\mathbf{P}}\mathbf{a} \leq \mathbf{0}$. In other words, row i of $\underline{\mathbf{J}}(\mathbf{a}^*)$ is parallel to row i of $\underline{\mathbf{P}}$. This combined our earlier deduction that $\underline{\mathbf{P}}\mathbf{a}^* = \mathbf{0}$ implies $\underline{\mathbf{J}}(\mathbf{a}^*)\mathbf{a}^* = \mathbf{0}$.

We now prove the converse implication of the theorem. Take any scaling-indifferent and Pareto efficient outcome $\mathbf{a}^* \in \mathbb{R}^{nk}_+$. Because $\mathbf{a}^*$ is Pareto efficient, by Proposition OA1 there is a nonzero vector $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*) = \mathbf{0}$ for each d. We need to find prices supporting $\mathbf{a}^*$ as a Lindahl outcome. Define the matrix $\mathbf{P}^{[d]}$ by $P^{[d]}_{ij} = \theta_i J^{[d]}_{ij}(\mathbf{a}^*)$ and note that for all $j \in N$ we have

$$\sum_{i \in N} P^{[d]}_{ij} = \sum_{i \in N} \theta_i J^{[d]}_{ij}(\mathbf{a}^*) = \left[\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*)\right]_j = 0, \tag{OA-3}$$

where $\left[\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*)\right]_j$ refers to entry j of the vector $\boldsymbol{\theta}\mathbf{J}^{[d]}(\mathbf{a}^*)$.

Now, recalling the definition of the n-by-nk matrix $\underline{\mathbf{P}}$, we see that each column of $\underline{\mathbf{P}}$ sums to zero. Further, each row of $\underline{\mathbf{P}}$ is just a scaling of the corresponding row of $\underline{\mathbf{J}}(\mathbf{a}^*)$. We therefore have:

$$\underline{\mathbf{P}}\mathbf{a}^* = \mathbf{0}, \tag{OA-4}$$

and these prices satisfy budget balance.

Finally, we claim that, for each i, the vector $\mathbf{a}^*$ solves $\Pi_i(\underline{\mathbf{P}})$. This is because the gradient of u_i at $\mathbf{a}^*$, which is row i of $\underline{\mathbf{J}}(\mathbf{a}^*)$, is normal to the constraint set by construction of $\mathbf{P}$ and, by (OA-4) above, $\mathbf{a}^*$ satisfies the constraint $\widehat{\mathrm{BB}}_i(\underline{\mathbf{P}})$. The claim then follows by the concavity of u_i. $\square$

OA2. Transfers of a Numeraire Good

It is natural to ask what happens in our model when transfers are possible. If utility is transferable—that is, if a "money" term enters additively into all payoffs, but utility functions are otherwise the same—then Coasian reasoning implies that the only Pareto-efficient solutions involve action profiles that maximize $\sum_i u_i(a_{\mathrm{X}}, a_{\mathrm{Y}}, a_{\mathrm{Z}})$. But in general, agents' preferences over environmental or other public goods need not be quasilinear in any numeraire—especially when the changes being contemplated are large. It is in this case that our analysis extends in an interesting way, and that is what we explore in this section, via two different modeling approaches.

OA2.1. **The Multiple Actions Approach.** We can use the extension to multiple actions to consider what will happen if we permit transfers of a numeraire good. We extend the environment in the main part of the paper by letting each agent choose, in addition to an action level, how much of a numeraire good to transfer to each other agent. We model this by assuming that each agent has $k = n$ dimensions of action. For agent i, action a_i^i corresponds to the actions we consider in the one-dimensional model of the paper and action a_i^j for $j \neq i$ corresponds to a transfer of the numeraire good from agent i to agent j. We assume agents' utility functions are concave, and that all of them always have strictly positive marginal value from consuming the numeraire good. For agent i, the transfer action a_i^j (for $j \neq i$) is then individually costly (as i can then consume less of the numeraire good) but provides weak benefits to all others. Moreover, we assume for this section that $\partial u_i / \partial a_j^j > 0$ for every i and j—meaning that the original problem has strictly positive externalities.[3] As a consequence, each $\mathbf{B}^{[d]}$ is irreducible. This extension of the single action model to accommodate transfers fits the multiple actions framework above and we can then simply apply Proposition OA1 and Theorem OA1 to show how our results change once transfers are possible.

Proposition OA1 and Theorem OA1 show that the main results of our paper extend in a natural way to environments with transfers. However, it is important to note that although we are assuming transfers are possible, we are *not* assuming that agents' preferences are quasi-linear in any numeraire. Under the (strong) additional assumption of transferable utility, the problem becomes much simpler, as mentioned above.

OA2.2. **An Inverse Marginal Utility of Money Characterization of Lindahl Outcomes with a Transferable Numeraire.** It is possible to extend Theorem 1 in a different way to a setting with a transferable, valuable numeraire. Consider the basic setting of the paper in which each player can put forth externality-generating effort on one dimension, and suppose that each agent's utility function is $u_i : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}$. We write a typical payoff as $u_i(\mathbf{a}; m_i)$, where $\mathbf{a} \in \mathbb{R}_+^n$ is an action profile as in the

[3] We suspect this assumption can be relaxed substantially without affecting the conclusions.

main text, and m_i is a net transfer of "money"—a numeraire—to agent i. We assume preferences are concave and continuously differentiable on the domain $\mathbb{R}^n_+ \times \mathbb{R}$. We also assume that for all fixed vectors $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, the utility functions satisfy the maintained assumptions of Section 2.2 in the main text. We assume that the numeraire is valuable: $\frac{\partial u_i}{\partial m_i} > 0$ on the whole domain. Finally, to streamline things, we assume that $\frac{\partial u_i}{\partial a_i}(\mathbf{a}; m_i) = -1$ for all values of $(\mathbf{a}; m_i)$. The benefits matrix is defined as in Section 2.3.

Now we can define a Lindahl outcome in this setting, taking all prices to be in terms of the numeraire.

Definition **OA3.** An outcome $(\mathbf{a}^*; \mathbf{m}^*)$ is a *Lindahl outcome* for a preference profile $\mathbf{u}$ if $\sum_{i \in N} m_i = 0$ and there is an n-by-n matrix (of prices) $\mathbf{P}$ so that the following conditions hold for every i:

(i) The inequality

$$\sum_{j:j \neq i} P_{ij} a_j + m_i \leq a_i \sum_{j:j \neq i} P_{ji} \tag{BB$_i(\mathbf{P})$}$$

is satisfied when $(\mathbf{a}; \mathbf{m}) = (\mathbf{a}^*; \mathbf{m}^*)$;

(ii) for any $(\mathbf{a}; m_i)$ such that the inequality $\text{BB}_i(\mathbf{P})$ is satisfied, we have

$$(\mathbf{a}^*; m_i^*) \succeq_{u_i} (\mathbf{a}; m_i).$$

We can now characterize the Lindahl outcomes in this setting in a way that is reminiscent of both Proposition 1 in Section 3 and of Theorem 1. To do this, we make one final definition.

Define

$$\mu_i(\mathbf{a}, m_i) = \left[\frac{\partial u_i}{\partial m_i}(\mathbf{a}, m_i)\right]^{-1}.$$

This is the reciprocal of i's marginal utility of the numeraire at a given outcome. We will write $\boldsymbol{\mu}(\mathbf{a}, \mathbf{m})$ for the vector of all these inverse marginal utilities.

Proposition **OA2.** An interior outcome $(\mathbf{a}; \mathbf{m})$ is a Lindahl outcome if and only if

$$\boldsymbol{\theta} = \boldsymbol{\theta} \mathbf{B}(\mathbf{a}; \mathbf{m}) \tag{OA-5}$$

where $\boldsymbol{\theta} = \boldsymbol{\mu}(\mathbf{a}, \mathbf{m})$ and

$$m_i = \theta_i \cdot \left(a_i - \sum_j B_{ij} a_j\right) \tag{OA-6}$$

for each i.

Without going through the proof, which is analogous to that of Theorem 1, we discuss the key parts of the reasoning. Given a pair $(\mathbf{a}; \mathbf{m})$ such that (OA-5) and (OA-6) hold, we will construct prices supporting $(\mathbf{a}; \mathbf{m})$ as a Lindahl outcome. For $i \neq j$, we set

$$P_{ij} = \theta_i B_{ij}(\mathbf{a}, \mathbf{m}).$$

The prices agent i faces are proportional to his marginal utilities for various other agents' contributions, so i is making optimal tradeoffs in setting the a_j for $j \neq i$. Now we turn to the "labor supply decision" of agent i, i.e., what a_i should be. The wage that i makes from working is θ_i per unit of effort, because (by equation OA-5) we can write $\sum_{j:j \neq i} P_{ji} = \sum_{j:j \neq i} \theta_j B_{ji} = \theta_i$. Thus, recalling that the price of the numeraire is 1 by definition, we have

$$\frac{\text{price of numeraire}}{i\text{'s wage}} = \frac{1}{\theta_i} = \frac{1}{[\partial u_i/\partial m_i]^{-1}} = \frac{\partial u_i/\partial m_i}{1}.$$

Recalling that 1 is the marginal disutility of effort (by assumption), this shows that the price ratio above is equal to the corresponding ratio of i's marginal utilities. Finally, the condition $m_i = \theta_i \cdot \left(a_i - \sum_j B_{ij}a_j\right)$ can be written, in terms of our prices, as

$$m_i = a_i \sum_{j:j\neq i} P_{ji} - \sum_{j:j\neq i} P_{ij}a_j.$$

In rewriting the first term, we have again used (OA-5). This equation just says that i's budget balance condition holds: The net transfer of the numeraire he obtains is the difference between the wages paid to him and what he owes others for their contributions.

This shows that the conditions of Proposition OA2 are sufficient for a Lindahl outcome. The omitted argument for the converse is simpler; the proof essentially involves tracing backward through the reasoning we have just given.

The important thing to note about the conditions of Proposition OA2 is that, like the characterization of Theorem 1, there are no prices explicitly involved. The content of the Lindahl solution can be summarized succinctly in an eigenvector equation. Here the equation says that an agent's θ_i, his inverse marginal utility of income (so a higher θ_i corresponds to more wealth), satisfies the eigenvector centrality equation $\theta_i = \sum_j B_{ji}\theta_j$. Equivalently, the θ_i's are proportional to agents' eigenvector centralities in the network $\mathbf{B}(\mathbf{a})^\mathsf{T}$. Using the walks interpretation discussed in Section 5, we can say the following: In the presence of transfers, wealthier (higher θ_i) agents are the ones who sit at the *origin* of large flows in the benefits matrix: They are the ones capable of conferring large direct and indirect benefits on others.

OA3. Formalizing the Implementation-Theoretic Approach to Negotiations

Section 6.2 discussed the unique robustness of Lindahl outcomes from the perspective of a mechanism design problem. In this section, we present the notation and results to make that discussion fully precise.

Let $\mathcal{U}_A$ be the set of all functions $u : \mathbb{R}^n_+ \to \mathbb{R}$. We denote by $\succeq_u$ and $\succ_u$ the weak and strict preference orderings, respectively, induced by $u \in \mathcal{U}_A$. The domain of possible preference profiles[4] is a set $\mathcal{U} \subseteq \mathcal{U}_A^n$; we will state specific assumptions on it in our results.

A *game form* is a tuple $H = (\Sigma_1, \ldots, \Sigma_n, g)$ where:

- Σ_i is a set of *strategies* that agent i can play; we write $\Sigma = \prod_{i\in N} \Sigma_i$;
- $g : \Sigma \to \mathbb{R}^n_+$ is the *outcome function* that maps strategy profiles to action profiles.

[4]The standard approach (e.g., Maskin, 1999) is to work with preference relations. We use sets of utility functions to avoid carrying around two parallel notations.

Definition **OA4.** In a game form $H = (\Sigma_1, \ldots, \Sigma_n, g)$, a strategy profile $\boldsymbol{\sigma} \in \Sigma$ is a *Nash equilibrium* for preference profile $\mathbf{u} \in \mathcal{U}$ if for any $i \in N$ and any $\widetilde{\sigma}_i \in \Sigma_i$, it holds that $g(\boldsymbol{\sigma}) \succeq_{u_i} g(\widetilde{\sigma}_i, \boldsymbol{\sigma}_{-i})$. We define $\Sigma^*(H, \mathbf{u})$ to be the set of all such $\boldsymbol{\sigma}$.

A *social choice correspondence* $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ maps each preference profile to a nonempty set of outcomes. Any game form for which equilibrium existence is guaranteed[5] naturally induces a social choice correspondence: its Nash equilibrium outcome correspondence $F_H(\mathbf{u}) = g(\Sigma^*(H, \mathbf{u}))$. The set $F_H(\mathbf{u})$ describes all the outcomes the participants with preferences $\mathbf{u}$ can end up with if they are left with a game form H and they play some Nash equilibrium. We say that F_H is the social choice correspondence that the game form H *implements*[6]. A social choice correspondence is said to be *implementable* if there is some game form H that implements it.

There are two basic normative criteria we impose on such correspondences. A social choice correspondence F is *Pareto efficient* if, for any $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a} \in F(\mathbf{u})$, the profile $\mathbf{a}$ is Pareto efficient under $\mathbf{u}$. A social choice correspondence F is *individually rational* if, for any $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a} \in F(\mathbf{u})$, it holds that $\mathbf{a} \succeq_{u_i} \mathbf{0}$ for all i. An individually rational social choice correspondence is one that leaves every player no worse off than the status quo.

We will also refer to a technical condition—upper hemicontinuity. A social choice correspondence F is *upper hemicontinuous* if: For every sequence of preference profiles $(\mathbf{u}^{(k)})$ converging compactly[7] to $\mathbf{u}$, and every sequence of outcomes $\left(\mathbf{a}^{(k)}\right)$ with $\mathbf{a}^{(k)} \in F(\mathbf{u}^{(k)})$, if $\mathbf{a}^{(k)} \to \mathbf{a}$, then $\mathbf{a} \in F(\mathbf{u})$. This condition has some normative appeal in that a social choice correspondence not satisfying upper hemicontinuity is sensitive to arbitrarily small changes in preferences that may be difficult for the agents themselves to detect.[8]

Definition **OA5.** The *Lindahl correspondence* $L : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ is defined by

$$L(\mathbf{u}) = \{\mathbf{a} \in \mathbb{R}^n_+ : \mathbf{a} \text{ is a Lindahl outcome for } \mathbf{u}\}.$$

Fix $\mathcal{U}$. Let $\mathcal{F}$ be the set of implementable social choice correspondences $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ that are Pareto efficient, individually rational, and upper hemicontinuous. For any $\mathbf{u} \in \mathcal{U}$, define the set of outcomes prescribed at $\mathbf{u}$ by *every* such correspondence:

$$R(\mathbf{u}) = \bigcap_{F \in \mathcal{F}} F(\mathbf{u}). \tag{OA-7}$$

This defines a correspondence $R : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$. We call this the *robustly attainable* correspondence.

If the set of possible preferences is rich enough, then the robustly attainable correspondence is precisely the Lindahl correspondence. We can now state Proposition 5 more formally.

Proposition **OA3.** Suppose $\mathcal{U}$ is the set of all preference profiles satisfying the assumptions of Section 2.2, and the number of players n is at least 3. Then the robustly attainable correspondence is equal to the Lindahl correspondence: $R = L$.

[5]Otherwise, we can still talk about the correspondence, but it will not be a social choice correspondence, which is required to be nonempty-valued.

[6]To be more precise, this is the definition of full Nash implementation. Since we consider only this kind of implementation, we drop the adjectives.

[7]That is, the sequence $(\mathbf{u}^{(k)})$ converges uniformly on every compact set.

[8]The other way for upper hemicontinuity to fail is for the values of F not to be closed sets.

From this proposition, we can deduce that the Lindahl correspondence is the *minimum* solution in $\mathcal{F}$—it is the unique one that is a subcorrespondence of every other. For details on this, see Section OA4 below.

OA3.1. **Proof.** We begin by recalling Maskin's Theorem. Assuming that the number of agents n is at least 3 and that a social choice correspondence F satisfies *no veto power*[9] (a condition that is vacuously satisfied in our setting), then F is implementable if and only if it satisfies *Maskin monotonicity.*

DEFINITION **OA6.** A social choice correspondence $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ satisfies *Maskin monotonicity* if: Whenever $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$ and for some $\mathbf{u} \in \mathcal{U}$ it holds that

$$\forall i \in N, \forall \mathbf{a} \in \mathbb{R}^n_+, \quad \mathbf{a}^* \succeq_{\widehat{u}_i} \mathbf{a} \ \Rightarrow \ \mathbf{a}^* \succeq_{u_i} \mathbf{a}, \tag{OA-8}$$

then $\mathbf{a}^* \in F(\mathbf{u})$.[10]

We now show that[11] $R \subseteq L$. By the definition that $R(\mathbf{u}) = \bigcap_{F \in \mathcal{F}} F(\mathbf{u})$, it suffices to show that $L \in \mathcal{F}$, i.e., that L is an implementable, individually rational, Pareto efficient, and upper hemicontinuous social choice correspondence. First, a social choice correspondence must be nonempty-valued; Proposition 2 in Section 6.1 guarantees that L complies. By Assumption 3, the no veto power condition is vacuous in our setting. It is verified immediately from Definition 1 that L satisfies Maskin monotonicity.[12] Thus, L is implementable by Maskin's Theorem. Also, L is individually rational since, by definition of a Lindahl outcome, each agent prefers a Lindahl outcome to $\mathbf{0}$, which is always feasible. By the standard proof of the First Welfare Theorem, L is Pareto efficient (see, e.g., Foley, 1970). Similarly, the standard argument for the upper hemicontinuity of equilibria in preferences transfers to our setting.

Now assume F is implementable, Pareto efficient, individually rational, and upper hemicontinuous. Fix $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a}^* \in L(\mathbf{u})$. We will show $\mathbf{a}^* \in F(\mathbf{u})$. Define

$$\widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})\mathbf{a}.$$

Lemma OA1, proved later in this section, states that since F is individually rational, Pareto efficient, and upper hemicontinuous, it follows that $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$.[13] Note that for all $\mathbf{a} \in \mathbb{R}^n_+$, we have

$$\widehat{\mathbf{u}}(\mathbf{a}^*) - \widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})(\mathbf{a}^* - \mathbf{a}) \leq \mathbf{u}(\mathbf{a}^*) - \mathbf{u}(\mathbf{a})$$

by concavity of $\mathbf{u}$, so (OA-8) holds. Since F is implementable, it satisfies Maskin monotonicity, so we conclude that $\mathbf{a}^* \in F(\mathbf{u})$. ∎

[9] A social choice correspondence $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ satisfies no veto power if, for every $\mathbf{u} \in \mathcal{U}$, whenever there is an $\mathbf{a} \in \mathbb{R}^n_+$ and an agent i' such that $\mathbf{a} \succeq_{u_i} \mathbf{a}'$ for all $i \neq i'$ and all $\mathbf{a}' \in \mathbb{R}^n_+$, then $\mathbf{a} \in F(\mathbf{u})$.

[10] In words: If an alternative $\mathbf{a}^*$ was selected by F under $\widehat{\mathbf{u}}$ and then we change those preferences to a profile $\mathbf{u}$ so that (under each agent's preference) the outcome $\mathbf{a}^*$ defeats all the same alternatives that it defeated under $\widehat{\mathbf{u}}$ and perhaps some others, then $\mathbf{a}^*$ is still selected under $\mathbf{u}$.

[11] For two correspondences $F, F^{\ddagger} : \mathcal{U} \to \mathbb{R}^n_+$, we write $F \subseteq F^{\ddagger}$ if for every $\mathbf{u} \in \mathcal{U}$, it holds that $F(\mathbf{u}) \subseteq F^{\ddagger}(\mathbf{u})$. In this case, we say that F is a *sub-correspondence* of $F^{\ddagger}$.

[12] If $\widehat{\mathbf{u}}$ and $\mathbf{u}$ are as in the above definition of Maskin monotonicity and $\mathbf{a}$ is a Lindahl outcome under preferences $\widehat{\mathbf{u}}$, then using the same price matrix $\mathbf{P}$, the outcome $\mathbf{a}$ still satisfies condition (ii) in Definition 1.

[13] The proof of that lemma constructs a sequence of preference profiles $(\widehat{\mathbf{u}}^{(k)})$ converging to $\widehat{\mathbf{u}}$ such that individual rationality and Pareto efficiency alone force the set $F(\widehat{\mathbf{u}}^{(k)})$ to converge to $\mathbf{a}^*$. Then by upper hemicontinuity of F, it follows that $F(\widehat{\mathbf{u}})$ contains $\mathbf{a}^*$.

The Hurwicz rationale for the Lindahl outcomes is actually more general than we have so far stated. We will now formalize and prove this, as well as tying up the proof of Proposition 5.

Let $\mathcal{A}$ be the set of preference profiles $\mathbf{u}$ satisfying the assumptions of Section 2.2. Endow this space with the compact-open topology.[14]

DEFINITION **OA7.** A set of preferences $\mathcal{U} \subseteq \mathcal{A}$ is called *rich* if, for every $\mathbf{u} \in \mathcal{U}$ and $\mathbf{a}^* \in \mathbb{R}^n_+$, there is a (linear) preference profile $\widehat{\mathbf{u}} \in \mathcal{U}$ defined by

$$\widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})\mathbf{a} \tag{OA-9}$$

and a neighborhood of $\widehat{\mathbf{u}}$ relative to $\mathcal{A}$ is contained in $\mathcal{U}$.

Richness of $\mathcal{U}$ requires that for every preference profile $\mathbf{u} \in \mathcal{U}$ and every $\mathbf{a}^* \in \mathbb{R}^n_+$, there are preferences in $\mathcal{U}$ that are linear over outcomes and have the same *marginal* tradeoffs that $\mathbf{u}$ does at $\mathbf{a}^*$, as well as a neighborhood of these preferences. To take a simple example, $\mathcal{A}$ itself is rich.

PROPOSITION **OA4.** Suppose $\mathcal{U}$ is rich and the number of players, n, is at least 3. Then the robustly attainable correspondence is equal to the Lindahl correspondence: $R = L$.

The proof is exactly as in Section OA3. The only thing that remains to do is to establish the following lemma used in that proof under the hypothesis that $\mathcal{U}$ is rich (the result needed in Section OA3 is then a special case).

LEMMA **OA1.** Fix $\mathbf{u}$ satisfying the assumptions of Section 2.2 and an $\mathbf{a}^* \in L(\mathbf{u})$. Define $\widehat{\mathbf{u}}$ as in (OA-9), i.e.,

$$\widehat{\mathbf{u}}(\mathbf{a}) = \mathbf{J}(\mathbf{a}^*; \mathbf{u})\mathbf{a}.$$

Suppose $F : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ is a Pareto efficient, individually rational, and upper hemicontinuous social choice correspondence. If $\mathcal{U}$ is rich, then $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$.

Proof of Lemma OA1: First assume $\mathbf{a}^* \neq \mathbf{0}$. (We will handle the other case at the end of the proof.) By Lemma 1 in Section C, $\mathbf{a}^*$ is interior—all its entries are positive. Write $\mathbf{J}^*$ for $\mathbf{J}(\mathbf{a}^*; \mathbf{u})$ and $\mathbf{B}^*$ for $\mathbf{B}(\mathbf{a}^*; \mathbf{u})$.

For $\gamma > 0$, and $i \in N$, define $\widehat{u}_i^{[\gamma]} : \mathbb{R}^n_+ \to \mathbb{R}$ by

$$\widehat{u}_i^{[\gamma]}(\mathbf{a}) = J^*_{ii} \left(\gamma + a_i\right)^{1+\gamma} + \sum_{j \neq i} J^*_{ij} a_j.$$

This is just an adjustment obtained from $\widehat{\mathbf{u}} = \widehat{\mathbf{u}}^{[0]}$ by building some convexity into the costs. Note that for all γ close enough to 0, the profile $\widehat{\mathbf{u}}^{[\gamma]}$ is in $\mathcal{U}$ by the richness assumption.[15]

Choose $\mathbf{a}^{[k]} \in F(\widehat{\mathbf{u}}^{[1/k]})$; this is legitimate since F is a social choice correspondence, and hence nonempty-valued. We will show that by the properties of F, a subsequence of the sequence $(\mathbf{a}^{[k]})$ converges to $\mathbf{a}^*$. Then by upper hemicontinuity of F, it will

[14]For any compact set $K \subseteq \mathbb{R}^n_+$ and open set $V \subseteq \mathbb{R}^n$, let $U(K, V)$ be the set of all preference profiles $\mathbf{u} \in \mathcal{A}$ so that $\mathbf{u}(K) \subseteq V$. The compact-open topology is the smallest one containing all such $U(K, V)$.

[15]The key fact here is that the topology of compact convergence is the same as the compact-open topology (Bourbaki, 1989, Chapter X, §3.4). As $\gamma \downarrow 0$, the functions $\mathbf{u}^{[\gamma]}$ converge compactly to $\widehat{\mathbf{u}}$, and thus any neighborhood of $\widehat{\mathbf{u}}$ under the compact-open topology contains $\mathbf{u}^{[\gamma]}$ for sufficiently small $\gamma > 0$. Therefore $\mathcal{U}$ contains these functions as well (recall the definition of richness).

follow that $\mathbf{a}^* \in F(\widehat{\mathbf{u}}^{[0]})$, as desired. The trickiest part of the argument is showing that the $\mathbf{a}^{[k]}$ lie in some compact set, so we can extract a convergent subsequence; it will then be fairly easy to show that the limit point of that subsequence is $\mathbf{a}^*$.

Let $\mathrm{IR}^{[\gamma]}$ be the set of individually rational points under $\widehat{\mathbf{u}}^{[\gamma]}$, and let $\mathrm{PE}^{[\gamma]}$ be the set of Pareto efficient points under $\widehat{\mathbf{u}}^{[\gamma]}$. Let $a^*_{\max} = \max_i a^*_i$, and define the box $K = [0, 2a^*_{\max}]^n$.

Claim **OA1.** For all k, the point $\mathbf{a}^{[k]}$ is either in K or on the ray

$$Z = \{\mathbf{a} \in \mathbb{R}^n_+ : \mathbf{J}^*\mathbf{a} = \mathbf{0}\}.$$

To show the claim, we first establish that

$$\mathrm{IR}^{[0]} = Z.$$

The proof is as follows: First note that $\widehat{\mathbf{u}}^{[0]}(\mathbf{a}) = \mathbf{J}^*\mathbf{a}$. There cannot be an $\mathbf{a}$ such that $\mathbf{J}^*\mathbf{a}$ is nonnegative in all entries and positive in some entries.[16] Thus, if $\mathbf{J}^*\mathbf{a}$ is nonzero, it must have some negative entries, i.e., $\widehat{u}^{[0]}_i(\mathbf{a}) < 0$ for some i, and then $\mathbf{a} \notin \mathrm{IR}^{[0]}$, contradicting the fact that F is individually rational.

Next, it can be seen that for $\mathbf{a}$ outside the box K, we have for small enough γ

$$\widehat{\mathbf{u}}^{[\gamma]}(\mathbf{a}) \leq \widehat{\mathbf{u}}^{[0]}(\mathbf{a}).$$

From this and the fact that $\widehat{\mathbf{u}}^{[\gamma]}(\mathbf{0}) = \mathbf{0}$ for all γ, we have the relation

$$\mathrm{IR}^{[\gamma]} \cap K^c \subseteq \mathrm{IR}^{[0]} \cap K^c.$$

Since we have established that $\mathrm{IR}^{[0]} = Z$, the claim follows.

We now deduce that, in fact, $\mathbf{a}^{[k]} \in K$ for all k. It is easily checked[17] that if $\mathbf{a} \in Z$ and $\mathbf{a} > \mathbf{a}^*$, then for $\gamma > 0$ we have $r(\mathbf{B}(\mathbf{a}; \widehat{\mathbf{u}}^{[\gamma]})) < r(\mathbf{B}^*) = 1$, where the latter equality holds by the efficiency of eigenvector centrality action profiles. Therefore, by Proposition 1, no point on the ray Z outside K is Pareto efficient for $\gamma > 0$. This combined with Claim OA1 shows that $\mathrm{IR}^{[\gamma]} \cap \mathrm{PE}^{[\gamma]} \subseteq K$, and therefore (since F is Pareto efficient and individually rational) it follows that $\mathbf{a}^{[k]} \in K$ for all k.

As a result we can find a sequence $(j(k))_k$ such that the sequence $\left(\mathbf{a}^{(j(k))}\right)_k$ converges to some $\overline{\mathbf{a}} \in \mathbb{R}^n_+$. Define $\mathbf{a}^{(k)} = \mathbf{a}^{[j(k)]}$ and set $\widehat{\mathbf{u}}^{(k)} = \widehat{\mathbf{u}}^{[1/j(k)]}$. Note that the $\widehat{\mathbf{u}}^{(k)}$ converge uniformly to $\widehat{\mathbf{u}}^{[0]}$ on K and, indeed, on any compact set (thus, they converge compactly to $\widehat{\mathbf{u}}^{[0]}$). By upper hemicontinuity of F, it follows that $\overline{\mathbf{a}} \in F(\widehat{\mathbf{u}}^{[0]})$. It remains only to show that $\overline{\mathbf{a}} = \mathbf{a}^*$, which we now do.

If $\overline{\mathbf{a}} \notin Z$, then it is easy to see that for large enough k, we would have $\widehat{u}^{(k)}_i(\mathbf{a}^{(k)}) < 0$ for some i. This would contradict the hypothesis that F is individually rational. Thus, $\mathbf{a}^{(k)} \to \zeta\mathbf{a}^*$ for some $\zeta \geq 0$. If $\zeta = 0$, then eventually $\mathbf{a}^{(k)}$ is not Pareto efficient for preferences $\mathbf{u}^{(k)}$, because $(\gamma + a_i)^{1+\gamma}$ with $a_i = 0$ tends to zero as $\gamma \downarrow 0$, making increases in action arbitrarily cheap (while marginal benefits remain constant). But

[16] Otherwise, $\mathbf{a}^*$ would not have been Pareto efficient under $\mathbf{u}$: moving in the direction $\mathbf{a}$ would have yielded a Pareto improvement. But $\mathbf{a}^*$ *is* Pareto efficient—see Section 4.

[17] We do a very similar calculation below in this proof.

that contradicts the Pareto efficiency of F. So assume $\zeta > 0$. In that case we would have:

$$J_{ij}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)}) \to \begin{cases} \zeta J^*_{ij} & \text{if } j = i \\ J^*_{ij} & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{B}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)}) \to \zeta \mathbf{B}^*.$$

Recall from Section 4 that $r(\mathbf{B}^*) = 1$. Since the spectral radius is linear in scaling the matrix and continuous in matrix entries, it follows that

$$r(\mathbf{B}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)})) \to \zeta,$$

By the Pareto efficiency of F, we know that $r(\mathbf{B}(\mathbf{a}^{(k)}; \mathbf{u}^{(k)})) = 1$ whenever $\mathbf{a}^{(k)}$ is interior, which holds for all large enough k since $\zeta \neq 0$. Thus $\zeta = 1$. It follows that $\overline{\mathbf{a}} = \mathbf{a}^*$ and the argument is complete.

It remains to discuss the case that $\mathbf{a}^* = \mathbf{0}$ is a Lindahl outcome. In that case, by Proposition 8 in Section D (or simply the First Welfare Theorem), the outcome $\mathbf{0}$ is Pareto efficient. It follows that there cannot be any $\mathbf{a} \in \mathbb{R}^n_+$ such that $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}$ is nonzero and nonnegative; for if there were, we would be able to find a (nearby) Pareto improvement on $\mathbf{0}$ under $\mathbf{u}$. There are thus two cases: (i) $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}$ has at least one negative entry for every nonzero $\mathbf{a} \in \mathbb{R}^n_+$; or (ii) there is some nonzero $\mathbf{a}^{**} \in \mathbb{R}^n_+$ such that $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}^{**} = \mathbf{0}$.

In case (i), it follows by concavity of $\mathbf{u}$ that $\mathbf{0}$ is the only individually rational and Pareto efficient outcome under $\widehat{\mathbf{u}}$. So $\mathbf{a}^* \in F(\widehat{\mathbf{u}})$.

In case (ii), $\mathbf{J}(\mathbf{0}; \mathbf{u})\mathbf{a}^{**} = \mathbf{0}$ can be rewritten as $\mathbf{B}(\mathbf{0}; \mathbf{u})\mathbf{a}^{**} = \mathbf{a}^{**}$. The Perron-Frobenius Theorem implies that $\mathbf{a}^{**}$ has only positive entries (because it is a right eigenvector of $\mathbf{B}(\mathbf{0}; \mathbf{u})$, which is nonnegative and irreducible by our maintained assumptions). Now, recall the argument we carried through above in the case $\mathbf{a}^* \neq \mathbf{0}$, involving a sequence of utility functions converging to $\widehat{\mathbf{u}}$. This argument goes through without change if we replace all instances of $\mathbf{J}^*$ by $\mathbf{J}(\mathbf{0}; \mathbf{u})$; all instances of $\mathbf{B}^*$ by $\mathbf{B}(\mathbf{0}; \mathbf{u})$; and if we redefine[18] $\mathbf{a}^* = \beta \mathbf{a}^{**}$ for any $\beta > 0$. That shows that $\beta \mathbf{a}^{**} \in F(\widehat{\mathbf{u}})$ for every $\beta > 0$. Now, since F is an upper hemi-continuous correspondence, its values are closed: in particular, the set $F(\widehat{\mathbf{u}})$ is closed. So $\mathbf{0} \in F(\widehat{\mathbf{u}})$ as well, completing the proof.

OA4. The Lindahl Correspondence as the Smallest Solution Satisfying the Desiderata

In Section OA3, we defined a set $\mathcal{F}$ of solutions having some desirable properties (those that are Pareto efficient, individually rational, and upper hemicontinuous) and showed that the Lindahl correspondence satisfies $L(\mathbf{u}) = \bigcap_{F \in \mathcal{F}} F(\mathbf{u})$. After stating that result in Proposition OA3, we claimed that this implies that L is the unique minimum correspondence in $\mathcal{F}$. In this section, we supply the details to make that statement precise, and contrast the notion of a minimum solution with the weaker notion of a minimal one.

Let $\mathcal{U}$ be a set of problems or environments (in our case, preference profiles) and let X be a set of available allocations (in our case, action profiles in $\mathbb{R}^n_+$). Fix a particular

[18] Except as the argument in the definitions of $\mathbf{J}^*$ or $\mathbf{B}^*$.

set $\mathcal{F}$ of nonempty-valued correspondences $F : \mathcal{U} \rightrightarrows X$.[19] Given $F, G \in \mathcal{F}$, recall that we say $F = G$ if $F(\mathbf{u}) = G(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$.

Definition **OA8.** An $F \in \mathcal{F}$ is a *minimum in* $\mathcal{F}$ if: for every $G \in \mathcal{F}$ and every $\mathbf{u} \in \mathcal{U}$, we have $F(\mathbf{u}) \subseteq G(\mathbf{u})$.

This differs from the definition of a *minimal* social choice correspondence:

Definition **OA9.** An $F \in \mathcal{F}$ is *minimal in* $\mathcal{F}$ if: there is no $G \in \mathcal{F}$ satisfying $G(\mathbf{u}) \subseteq F(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$, with strict containment for some $\mathbf{u} \in \mathcal{U}$.

Minimal correspondences exist under fairly general conditions (of the Zorn's Lemma type); the existence of a minimum is a more stringent condition.[20] However, what the minimum lacks in general existence results it makes up for in uniqueness in the cases where it does exist. When a minimum exists, it is uniquely determined. (In contrast, there may in general be multiple correspondences that are minimal in $\mathcal{F}$.)

Proposition **OA5.** If each of F and G is a minimum in $\mathcal{F}$, then $F = G$.

Proof of Proposition OA5: Take any $\mathbf{u} \in \mathcal{U}$. By definition of F being a minimum in $\mathcal{F}$, we have $F(\mathbf{u}) \subseteq G(\mathbf{u})$. By definition of G being a minimum in $\mathcal{F}$, we have $G(\mathbf{u}) \subseteq F(\mathbf{u})$. Thus $F(\mathbf{u}) = G(\mathbf{u})$. Since $\mathbf{u}$ was arbitrary, this establishes the equality. ∎

We can give a more "constructive" characterization of the minimum that connects it with our discussion in Section OA3.

Proposition **OA6.** If F is a minimum in $\mathcal{F}$, then $F(\mathbf{u}) = \bigcap_{G \in \mathcal{F}} G(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$.

Proof of Proposition OA6: Define the correspondence $H : \mathcal{U} \rightrightarrows X$ by $H(\mathbf{u}) = \bigcap_{G \in \mathcal{F}} G(\mathbf{u})$. (At this point nothing is claimed about whether H is in $\mathcal{F}$.) Now take any $\mathbf{u} \in \mathcal{U}$. By definition of F being a minimum in $\mathcal{F}$, for every $G \in \mathcal{F}$ we have $F(\mathbf{u}) \subseteq G(\mathbf{u})$. Thus, $F(\mathbf{u})$ lies in the intersection of all the sets $G(\mathbf{u})$: that is, $F(\mathbf{u}) \subseteq H(\mathbf{u})$. On the other hand, since $F \in \mathcal{F}$ is one of the correspondences over which the intersection $\bigcap_{G \in \mathcal{F}} G(\mathbf{u})$ is taken, we have the reverse inclusion $H(\mathbf{u}) \subseteq F(\mathbf{u})$. Since $\mathbf{u}$ was arbitrary, we have shown[21] $F = H$. ∎

OA5. Irreducibility of the Benefits Matrix

In Assumption 3, we posited that $\mathbf{B}(\mathbf{a})$ is irreducible—i.e., that it is not possible to find an outcome and a partition of society into two nonempty groups such that, at that outcome, one group does not care about the effort of the other at the margin.

How restrictive is this assumption? We now discuss how our analysis extends beyond it. Suppose that whether $B_{ij}(\mathbf{a})$ is positive or 0 does not depend on $\mathbf{a}$, so that

[19] In our case, these are the Nash-implementable, upper hemi-continuous correspondences F so that, for each $\mathbf{u} \in \mathcal{U}$, the set $F(\mathbf{u})$ contains only Pareto efficient outcomes that leave nobody worse off than the endowment. But nothing in the present section relies on this structure.

[20] Suppose F is a minimum in $\mathcal{F}$. We will show it is minimal in $\mathcal{F}$. Suppose we have $G \in \mathcal{F}$ such that $G(\mathbf{u}) \subseteq F(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{U}$. By definition of F being a minimum it is also the case that $F(\mathbf{u}) \subseteq G(\mathbf{u})$ for every $\mathbf{u} \in \mathcal{U}$. Thus $G = F$ and it is impossible for $G(\mathbf{u})$ to be strictly smaller than $F(\mathbf{u})$, for any $\mathbf{u}$. So F is, indeed, minimal. In particular, existence of a minimum in $\mathcal{F}$ implies existence of a minimal correspondence in $\mathcal{F}$. The converse does not hold.

[21] In particular, we see $H \in \mathcal{F}$.

the directed graph describing whose effort matters to whom is constant, though the nonzero marginal benefits may change as we vary $\mathbf{a}$. Let $\mathbf{G}$ be a matrix defined by

$$G_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } B_{ij}(\mathbf{a}) > 0 \text{ for all } \mathbf{a} \\ 0 & \text{otherwise.} \end{cases}$$

We say a subset $S \subseteq N$ is *closed* if $G_{ij} = 0$ for every $i \in S$ and $j \notin S$. We say S is *irreducible* if $\mathbf{G}$ is irreducible when restricted to S.

We can always partition N into some closed, irreducible subsets

$$S^{(1)}, S^{(2)}, \ldots, S^{(m)}$$

and a remaining class T of agents who are in no closed, irreducible subset. The utility of any agent in a set $S^{(k)}$ is independent of the choices of anyone outside the set (and these are the minimal sets with that property). So it seems reasonable to consider negotiations restricted to each such set; that is, to take the set of players to be $S^{(k)}$. All our analysis then goes through without modification on each such subset.

When entries $B_{ij}(\mathbf{a})$ change from positive to zero depending on $\mathbf{a}$, then the analysis becomes substantially more complicated, and we leave it for future work.

OA6. Explicit Formulas for Lindahl Outcomes

OA6.1. **A Parametric Family of Preferences and a Formula for Eigenvector Centrality Actions.** Here we provide more interpretations regarding explicit formulas for Lindahl outcomes, following up on the discussion of Section 5.2. In that section, we defined:[22]

$$u_i(\mathbf{a}) = -a_i + \sum_j \left[G_{ij} a_j + H_{ij} \log a_j\right].$$

for non-negative matrices $\mathbf{G}$ and $\mathbf{H}$ with zeros on the diagonal, assuming $r(\mathbf{G}) < 1$. Letting $h_i = \sum_j H_{ij}$, the eigenvector centrality actions $\mathbf{a}$ are then characterized by $\mathbf{a} = \mathbf{h} + \mathbf{G}\mathbf{a}$ or

$$\mathbf{a} = (\mathbf{I} - \mathbf{G})^{-1}\mathbf{h}. \tag{OA-10}$$

Note that the vector $\mathbf{a}$ is well-defined and nonnegative[23] by the assumption that $r(\mathbf{G}) < 1$. These eigenvector centrality outcomes correspond to agents' degree centralities, Bonacich centralities, or eigenvector centralities on some network $\mathbf{M}$, for specific parametrizations of the above utility functions.

OA6.2. **Degree Centrality.** To obtain agents' degree centralities as their eigenvector centrality actions, we set $\mathbf{H} = \mathbf{M}$ and let $\mathbf{G} = \mathbf{0}$. Then equation (OA-10) says that $\mathbf{a} = \mathbf{h}$. When costs are linear in one's own action and benefits are logarithmic in others' actions, then an agent i's contribution is determined by how much he benefits from everyone else's effort at the margin: the sum of coefficients H_{ij} as j ranges across the other agents. The agents who are particularly dependent on the rest are the ones who are contributing the most.

[22] These should be viewed as functions $u_i : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$, with $0 \cdot \log 0$ understood as 0. In other words, preferences should be completed by continuity to the extended range. No result in the paper is affected by this slight departure from the framework of Section 2.

[23] See Ballester, Calvó-Armengol, and Zenou (2006, Section 3).

OA6.3. **Bonacich Centrality.** To obtain agents' Bonacich centralities as their eigenvector centrality actions, we set $\mathbf{G} = \alpha \mathbf{M}$ for $\alpha < 1/r(\mathbf{M})$, and let each row of $\mathbf{H}$ sum to 1. Dropping the arguments, the defining equation for Bonacich centrality[24] says that for every i, we have:

$$\beta_i = 1 + \alpha \sum_j M_{ij} \beta_j.$$

Thus, every node gets a baseline level of centrality (one unit) and then additional centrality in proportion to the centrality of those it is linked to. To shed further light on this result, recall the definitions and notation related to walks from Section 5, and let

$$V_i(\ell; \mathbf{M}) = \sum_{w \in \mathcal{W}_i^{\downarrow}(\ell; \mathbf{M})} v(w; \mathbf{M}).$$

This is the sum of the values of all walks of length ℓ in $\mathbf{M}$ ending at i. Then we have:

Fact **OA1.** $\beta_i(\mathbf{M}, \alpha) = 1 + \sum_{\ell=1}^{\infty} \alpha^{\ell} V_i(\ell; \mathbf{M}^{\mathsf{T}})$.

Fact OA1 is established, e.g., in Ballester, Calvó-Armengol, and Zenou (2006, Section 3). Thus, the Bonacich centrality is equal to 1 plus a weighted sum of values of all walks in $\mathbf{M}^{\mathsf{T}}$ terminating at i, with longer walks downweighted exponentially.

In contrast to the case of degree centrality treated in the previous section, it is not only how much i benefits from his immediate neighborhood that matters in determining his contribution, but also how much i's neighbors benefit from *their* neighbors, etc.

OA6.4. **Eigenvector Centrality.** Eigenvector centrality is a key notion throughout the paper. Theorem 1 establishes a general connection between eigenvector centrality and Lindahl outcomes. However, this theorem characterizes $\mathbf{a}$ through an *endogenous* eigenvector centrality condition—a condition that depends on $\mathbf{B}(\mathbf{a})$. In this section, we study the special case in which action levels approximate eigenvector centralities defined according to an *exogenous* network.

We continue with the specification from Section OA6.3, with one exception: We consider networks $\mathbf{M}$ such that $r(\mathbf{M}) = 1$.[25] Thus,

$$\mathbf{a} = \boldsymbol{\beta}(\mathbf{M}, \alpha).$$

By the Perron–Frobenius Theorem, $\mathbf{M}$ has a unique right-hand Perron eigenvector $\mathbf{e}$ (satisfying $\mathbf{e} = \mathbf{M}\mathbf{e}$) with entries summing to 1. As we take the limit $\alpha \to 1$, agents' Bonacich centralities become large but $a_i/a_j \to e_i/e_j$, for every i, j. That is, each agent's share of the total of all actions converges to his eigenvector centrality according to $\mathbf{M}$.[26] The reason for this convergence is presented in the proof of Theorem 3 of Golub and Lever (2010); see also Bonacich (1991).

[24] An important antecedent was discussed by Katz (1953).

[25] This is just a normalization here: For any $\mathbf{M}$, we can work with the matrix $(1/r(\mathbf{M}))\mathbf{M}$, as this has spectral radius 1.

[26] To loosely gain some intuition for this, note that $\mathbf{a} = \alpha \mathbf{M}\mathbf{a} + \mathbf{1}$; as $\alpha \to 1$, actions grow large and we can think of this equation as saying $\mathbf{a} \approx \mathbf{M}\mathbf{a}$.

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