INTRODUCTION TO FLUID MECHANICS

SIXTH EDITION

ROBERT W. FOX
Purdue University
ALAN T. McDONALD
Purdue University
PHILIP J. PRITCHARD
Manhattan College

JOHN WILEY & SONS, INC.

9-4 MOMENTUM INTEGRAL EQUATION

Blasius' exact solution involved performing a rather subtle mathematical transformation of two differential equations based on the insight that the laminar boundary layer velocity profile is self-similar—only its scale changes as we move along the plate. Even with this transformation, we note that numerical integration was necessary to obtain results for the boundary-layer thickness $\delta(x)$, velocity profile u/U versus y/δ, and wall shear stress $\tau_w(x)$. Furthermore, the analysis is limited to laminar boundary layers only (Eq. 9.4 does not include the turbulent Reynolds stresses discussed in Chapter 8), and for a flat plate only (no pressure variations).

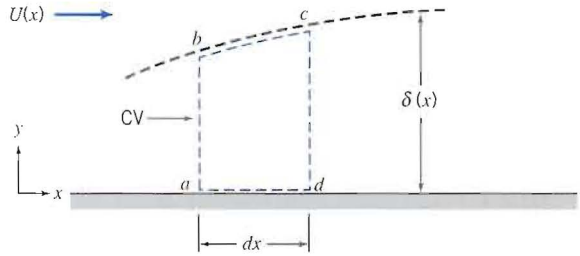

Fig. 9.4 Differential control volume in a boundary layer.

To avoid these difficulties and limitations, we now consider a method for deriving an algebraic equation that can be used to obtain approximate information on boundary-layer growth for the general case (laminar or turbulent boundary layers, with or without a pressure gradient). The approach is one in which we will again apply the basic equations to a control volume. The derivation, from the mass conservation (or continuity) equation and the momentum equation, will take several pages.

Consider incompressible, steady, two-dimensional flow over a solid surface. The boundary-layer thickness, δ, grows in some manner with increasing distance, x. For our analysis we choose a differential control volume, of length dx, width w, and height $\delta(x)$, as shown in Fig. 9.4. The freestream velocity is $U(x)$.

We wish to determine the boundary-layer thickness, δ, as a function of x. There will be mass flow across surfaces ab and cd of differential control volume $abcd$. What about surface bc? Will there be a mass flow across this surface? In Example Problem 9.2, (on the CD), we showed that the edge of the boundary layer is not a streamline. Thus there will be mass flow across surface bc. Since control surface ad is adjacent to a solid boundary, there will not be flow across ad. Before considering the forces acting on the control volume and the momentum fluxes through the control surface, let us apply the continuity equation to determine the mass flux through each portion of the control surface.

a. Continuity Equation

Basic equation:

$$\cancelto{0(1)}{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall} + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \tag{4.12}$$

Assumptions: (1) Steady flow.
(2) Two-dimensional flow.

Then

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd} = 0$$

or

$$\dot{m}_{bc} = -\dot{m}_{ab} - \dot{m}_{cd}$$

Now let us evaluate these terms for the differential control volume of width w:

Surface	Mass Flux
ab	Surface ab is located at x. Since the flow is two-dimensional (no variation with z), the mass flux through ab is $$\dot{m}_{ab} = -\left\{\int_0^\delta \rho u\, dy\right\} w$$
cd	Surface cd is located at $x + dx$. Expanding $\dot{m}$ in a Taylor series about location x, we obtain $$\dot{m}_{x+dx} = \dot{m}_x + \left.\frac{\partial \dot{m}}{\partial x}\right]_x dx$$ and hence $$\dot{m}_{cd} = \left\{\int_0^\delta \rho u\, dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx\right\} w$$
bc	Thus for surface bc we obtain $$\dot{m}_{bc} = -\left\{\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx\right\} w$$

Now let us consider the momentum fluxes and forces associated with control volume *abcd*. These are related by the momentum equation.

b. Momentum Equation

Apply the x component of the momentum equation to control volume *abcd:*

Basic equation:

$$F_{S_x} + \cancel{F_{B_x}}^{\,=0(3)} = \cancel{\frac{\partial}{\partial t}}^{\,=0(1)} \int_{CV} u\, \rho\, d\forall + \int_{CS} u\, \rho \vec{V} \cdot d\vec{A} \tag{4.18a}$$

Assumption: (3) $F_{B_x} = 0$.

Then

$$F_{S_x} = \mathrm{mf}_{ab} + \mathrm{mf}_{bc} + \mathrm{mf}_{cd}$$

where mf represents the x component of momentum flux.

To apply this equation to differential control volume *abcd*, we must obtain expressions for the x momentum flux through the control surface and also the surface forces acting on the control volume in the x direction. Let us consider the momentum flux first and again consider each segment of the control surface.

Surface	Momentum Flux (mf)
ab	Surface *ab* is located at x. Since the flow is two-dimensional, the x momentum flux through *ab* is $$\mathrm{mf}_{ab} = -\left\{\int_0^\delta u\,\rho u\,dy\right\}w$$
cd	Surface *cd* is located at $x + dx$. Expanding the x momentum flux (mf) in a Taylor series about location x, we obtain $$\mathrm{mf}_{x+dx} = \mathrm{mf}_x + \left.\frac{\partial \mathrm{mf}}{\partial x}\right]_x dx$$ or $$\mathrm{mf}_{cd} = \left\{\int_0^\delta u\,\rho u\,dy + \frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\,dy\right]dx\right\}w$$
bc	Since the mass crossing surface *bc* has velocity component U in the x direction, the x momentum flux across *bc* is given by $$\mathrm{mf}_{bc} = U\dot{m}_{bc}$$ $$\mathrm{mf}_{bc} = -U\left\{\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\,dy\right]dx\right\}w$$

From the above we can evaluate the net x momentum flux through the control surface as

$$\int_{\mathrm{CS}} u\,\rho\vec{V}\cdot d\vec{A} = -\left\{\int_0^\delta u\,\rho u\,dy\right\}w + \left\{\int_0^\delta u\,\rho u\,dy\right\}w + \left\{\frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\,dy\right]dx\right\}w - U\left\{\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\,dy\right]dx\right\}w$$

Collecting terms, we find that

$$\int_{\mathrm{CS}} u\,\rho\vec{V}\cdot d\vec{A} = +\left\{\frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\,dy\right]dx - U\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\,dy\right]dx\right\}w$$

Now that we have a suitable expression for the x momentum flux through the control surface, let us consider the surface forces acting on the control volume in the x direction. (For convenience the differential control volume has been redrawn in Fig. 9.5.) We recognize that normal forces having nonzero components in the x direction act on three surfaces of the control surface. In addition, a shear force acts on surface *ad*.

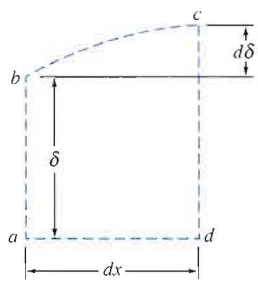

Fig. 9.5 Differential control volume.

Since the velocity gradient goes to zero at the edge of the boundary layer, the shear force acting along surface *bc* is negligible.

Surface	Force
ab	If the pressure at x is p, then the force acting on surface *ab* is given by $$F_{ab} = pw\delta$$ (The boundary layer is very thin; its thickness has been greatly exaggerated in all the sketches we have made. Because it is thin, pressure variations in the y direction may be neglected, and we assume that within the boundary layer, $p = p(x)$.)
cd	Expanding in a Taylor series, the pressure at $x + dx$ is given by $$p_{x+dx} = p + \left.\frac{dp}{dx}\right]_x dx$$ The force on surface *cd* is then given by $$F_{cd} = -\left(p + \left.\frac{dp}{dx}\right]_x dx\right) w(\delta + d\delta)$$
bc	The average pressure acting over surface *bc* is $$p + \frac{1}{2}\left.\frac{dp}{dx}\right]_x dx$$ Then the x component of the normal force acting over *bc* is given by $$F_{bc} = \left(p + \frac{1}{2}\left.\frac{dp}{dx}\right]_x dx\right) w\, d\delta$$
ad	The average shear force acting on *ad* is given by $$F_{ad} = -\left(\tau_w + \tfrac{1}{2}\, d\tau_w\right) w\, dx$$

Summing the x components of all forces acting on the control volume, we obtain

$$F_{S_x} + \left\{ -\frac{dp}{dx}\,\delta\, dx - \frac{1}{2}\frac{dp}{dx}\, \cancelto{\simeq 0}{dx\, d\delta} - \tau_w\, dx - \frac{1}{2}\, \cancelto{\simeq 0}{d\tau_w\, dx} \right\} w$$

where we note that $dx\, d\delta << \delta\, dx$ and $d\tau_w << \tau_w$, and so neglect the second and fourth terms.

Substituting the expressions for $\int_{CS} u\,\rho\vec{V}\cdot d\vec{A}$ and F_{S_x} into the x momentum equation, we obtain

$$\left\{-\frac{dp}{dx}\,\delta\, dx - \tau_w\, dx\right\} w = \left\{\frac{\partial}{\partial x}\left[\int_0^\delta u\,\rho u\, dy\right] dx - U\frac{\partial}{\partial x}\left[\int_0^\delta \rho u\, dy\right] dx\right\} w$$

Dividing this equation by $w\, dx$ gives

$$-\delta\frac{dp}{dx} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta u\,\rho u\, dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\, dy \tag{9.16}$$

Equation 9.16 is a "momentum integral" equation that gives a relation between the x components of the forces acting in a boundary layer and the x momentum flux.

The pressure gradient, dp/dx, can be determined by applying the Bernoulli equation to the inviscid flow outside the boundary layer: $dp/dx = -\rho U\, dU/dx$. If we recognize that $\delta = \int_0^\delta dy$, then Eq. 9.16 can be written as

$$\tau_w = -\frac{\partial}{\partial x}\int_0^\delta u\,\rho u\,dy + U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy + \frac{dU}{dx}\int_0^\delta \rho U\,dy$$

Since

$$U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy = \frac{\partial}{\partial x}\int_0^\delta \rho u U\,dy - \frac{dU}{dx}\int_0^\delta \rho u\,dy$$

we have

$$\tau_w = \frac{\partial}{\partial x}\int_0^\delta \rho u(U-u)\,dy + \frac{dU}{dx}\int_0^\delta \rho(U-u)\,dy$$

and

$$\tau_w = \frac{\partial}{\partial x}U^2\int_0^\delta \rho\frac{u}{U}\left(1-\frac{u}{U}\right)dy + U\frac{dU}{dx}\int_0^\delta \rho\left(1-\frac{u}{U}\right)dy$$

Using the definitions of displacement thickness, δ^* (Eq. 9.1), and momentum thickness, θ (Eq. 9.2), we obtain

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U\frac{dU}{dx} \tag{9.17}$$

Equation 9.17 is the *momentum integral equation.* This equation will yield an ordinary differential equation for boundary-layer thickness, provided that a suitable form is assumed for the velocity profile and that the wall shear stress can be related to other variables. Once the boundary-layer thickness is determined, the momentum thickness, displacement thickness, and wall shear stress can then be calculated.

Equation 9.17 was obtained by applying the basic equations (continuity and x momentum) to a differential control volume. Reviewing the assumptions we made in the derivation, we see that the equation is restricted to steady, incompressible, two-dimensional flow with no body forces parallel to the surface.

We have not made any specific assumption relating the wall shear stress, τ_w, to the velocity field. Thus Eq. 9.17 is valid for either a laminar or turbulent boundary-layer flow. In order to use this equation to estimate the boundary-layer thickness as a function of x, we must:

1. Obtain a first approximation to the freestream velocity distribution, $U(x)$. This is determined from inviscid flow theory (the velocity that would exist in the absence of a boundary layer) and depends on body shape. The pressure in the boundary layer is related to the freestream velocity, $U(x)$, using the Bernoulli equation.
2. Assume a reasonable velocity-profile shape inside the boundary layer.
3. Derive an expression for τ_w using the results obtained from item **2**.

To illustrate the application of Eq. 9.17 to boundary-layer flows, we consider first the case of flow with zero pressure gradient over a flat plate (Section 9-5)—the results we obtain for a laminar boundary layer can then be compared to the exact Blasius results. The effects of pressure gradients in boundary-layer flow are then discussed in Section 9-6.