EXAMPLE PROBLEM 1

Write down the general forms of the conservation equations in integral and differential form, using vector notation. Do not make any assumptions about the fluid or the flow (i.e., do not assume the fluid is Newtonian, or incompressible, inviscid, etc. and do not assume the flow is steady, fully-developed, etc.). If you know the general forms of the equations, you will be on your way to solving most problems you have seen thus far. These equations should be the *first* thing you write down for each problem.

Now, assume only that the fluid is Newtonian so that the shear stress tensor is given by the relation

$$\vec{\vec{\tau}} = \mu \nabla \vec{u}. \tag{1}$$

Rewrite the Navier-Stokes equation under this assumption.

SOLUTION 1

Let's write the integral form of the conservation laws, which can be applied to a control volume. The continuity or mass conservation equation is

$$\text{conservation of mass}: \quad \frac{\partial}{\partial t}\int_{cv} \rho dV + \oint_{cs} \rho \vec{u}\cdot\hat{n} dA = 0,$$

where the first term describes the rate of change of mass within the control volume (cv) and the second term describes the mass transport across the boundaries of the control volume, i.e., the control surface (cs). The normal to the control surface, $\hat{n}$, is always pointing outward.

The force balance on the control volume gives the momentum conservation equation and is no different than Newton's first law, $\vec{F} = m\vec{a} = m(d\vec{u}/dt)$. If we consider this applied to a control volume, the result is

$$\text{conservation of momentum}: \quad \sum \vec{F} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{u} dV + \oint_{cs} \rho \vec{u}(\vec{u}\cdot\hat{n}) dA,$$

where $\sum \vec{F}$ represents the sum of the forces acting on the control volume or control surface (which means they act on the fluid). In general, we may have something like

$$\sum \vec{F} = -\oint_{cs} p\hat{n} dA + \vec{F}_{body} + \vec{F}_{\tau},$$

where the first term are the pressure forces acting on the control surface, the second includes body forces like gravity, and the third includes shear forces acting parallel or tangent to the control surface.

The differential forms of the equations follow as

$$\text{continuity 1}: \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0,$$

or noting that $\nabla\cdot(\rho\vec{u}) = (\vec{u}\cdot\nabla)\rho + \rho\nabla\cdot\vec{u}$, an equivalent expression is

$$\text{continuity 2}: \quad \frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0,$$

where $D/Dt = \partial/\partial t + (\vec{u} \cdot \nabla)$ is the Lagrangian or material operator. The important thing to remember is that D/Dt is conserved along particle paths. This operator also appears in the Navier-Stokes equation of motion

$$\text{equation of motion :} \quad \rho \frac{D\vec{u}}{Dt} = -\nabla p + \vec{F}_{body} + \nabla \cdot \vec{\vec{\tau}}.$$

Using the Newtonian-fluid stress relation with constant viscosity, this can be written as

$$\text{equation of motion (Newtonian fluid) :} \quad \rho \frac{D\vec{u}}{Dt} = -\nabla p + \vec{F}_{body} + \mu \nabla^2 \vec{u}.$$

Remember that if the forces are conservative (like gravity), we can write $\vec{F}_{body} = \nabla \phi$, where ϕ is a potential.

EXAMPLE PROBLEM 2

A rocket is moving vertically with speed $u_r(t)$. The mass of the rocket system at time t can be denoted by $M(t)$ and includes the mass of the rocket itself, the unspent fuel, and the exhaust, prior to it exiting. At the nozzle exit, the area is A_e and the pressure and density are p_e and ρ_e, respectively. From stationary tests of the rocket nozzle, the exit velocity of the exhaust is u_e.

Find an expression for the rate of change of mass for the system, and an expression for the acceleration of the rocket. The acceleration can be written in terms of the given parameters, atmospheric pressure p_a, and the drag D.

SOLUTION 2

Choose a control volume around the rocket, moving with velocity $u_r(t)$. Remember that all velocities must be measured relative to the control volume in applying the conservation equations from Problem 1. From the point-of-view of the stationary control volume, the exit velocity is u_e.

Conservation of mass gives

$$\frac{dM}{dt} = -\rho_e u_e A_e,$$

where it is understood that

$$M = \int_{cv} \rho dV.$$

Conservation of momentum in the direction of travel of the rocket gives

$$(p_e - p_a)A_e - Mg - D = \frac{\partial}{\partial t} \int_{cv} \rho u_r(t) dV - \rho_e u_e^2 A_e$$

Since $u_r(t)$ is not spatially dependent, the integral on the right can be written as

$$\frac{\partial}{\partial t} \int_{cv} \rho u_r(t) dV = \frac{\partial u_r(t)}{\partial t} \int_{cv} \rho dV + u_r(t) \frac{\partial}{\partial t} \int_{cv} \rho dV = M \frac{du_r}{dt} + u_r \frac{dM}{dt},$$

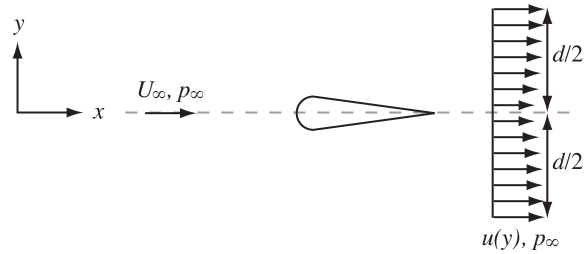
and the momentum equation simplifies to

$$(p_e - p_a)A_e - Mg - D = M \frac{du_r}{dt} + u_r \frac{dM}{dt} + u_e \frac{dM}{dt},$$

where the result from the continuity equation has been substituted in the last term. Solving for the acceleration of the rocket

$$\frac{du_r}{dt} = \frac{1}{M} \left[(p_e - p_a)A_e - Mg - D - (u_e + u_r) \frac{dM}{dt} \right].$$

EXAMPLE PROBLEM 3



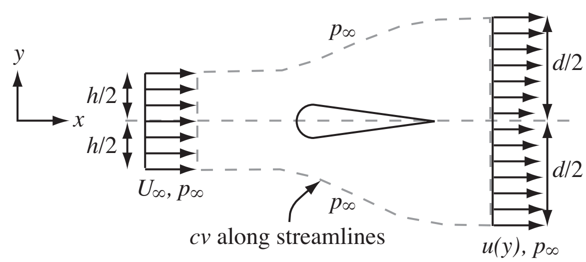
A symmetrical wing is being tested at zero incidence in a wind tunnel. The upstream speed is U_∞ and is uniform across the tunnel, and the pressure is p_∞ (also uniform). At a station far downstream of the wing, the pressure is again uniform at p_∞ and the velocity in the stream-wise direction is

$$u(y) = U_\infty, \quad |y| > d/2; \qquad u(y) = U_\infty - q(1 + \cos(2\pi y/d)), \quad -d/2 \le y \le d/2, \tag{2}$$

where y is the distance from the wing centerline (which lies on the x-axis in Cartesian coordinates) and d and q are the wake-defect length and velocity-scale parameters, respectively. Find the drag on the wing per unit meter of span. You can assume that all along streamlines that pass through $y = \pm d/2$ at the downstream station, the pressure is p_∞ and the shear stresses are negligible.

SOLUTION 3

This problem is essentially identical to Problem B6 on hw2, but we wanted to present a "better" way to approach and solve it. Since we know there is no flow perpendicular to streamlines, it makes more sense to choose a control volume with boundaries parallel to the streamlines around the outside of the wing.



We do not know the height h, but we can solve for it since we know the mass flow between $-h/2$ and $h/2$ must equal the mass flow out between $-d/2$ and $d/2$.

$$-\rho U_\infty h + \rho \int_{-d/2}^{d/2} [U_\infty - q(1 + \cos(2\pi y/d))]\, dy = 0,$$

which gives

$$-\rho U_\infty h + \rho(U_\infty - q)d = 0,$$

and consequently

$$h = \left(\frac{U_\infty - q}{U_\infty}\right) d.$$

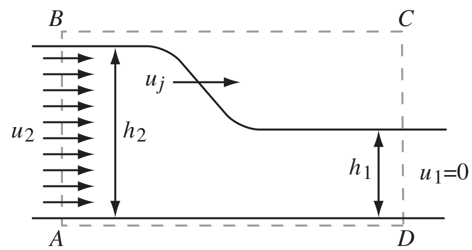
Using the integral momentum equation, the fact that no flow crosses the upper and lower boundaries, and given the pressure is the same around the entire control volume, we arrive at

$$-D = -\rho U_\infty^2 h + \rho \int_{-d/2}^{d/2} \left[U_\infty - q(1 + \cos(2\pi y/d))\right]^2 dy.$$

So the drag per unit meter of span is found by performing the integration, substituting for h, and moving the negative to the right-hand side of the equation

$$D = \rho \left(qU_\infty - \frac{3}{2}q^2 \right) d.$$

EXAMPLE PROBLEM 4



On a horizontal slope, a large-amplitude surface wave (hydraulic bore or tsunami) is propagating to the right at a constant speed u_j. Far upstream of the bore the (undisturbed) fluid is at rest and is of height h_1. The pressure distribution in the fluid can be considered hydrostatic. Far downstream of the bore (to the left) the fluid velocity distribution can be well approximated by uniform speed u_2 and the pressure under the surface is again hydrostatic. The flow in the region of sudden height change is extremely turbulent. Assume however, that the fluid is incompressible and neglect viscosity.

(a) Using the stationary control volume $ABCD$ and the conservation laws of mass and momentum in integral form for this *unsteady flow*, show that the height ratio h_2/h_1 is given by

$$\frac{h_2}{h_1} = \frac{1}{2}\left(\sqrt{1 + 8F_1^2} - 1\right), \tag{3}$$

where $F_1 = u_j/\sqrt{gh_1}$ is the Froude number of the bore and g is the acceleration due to gravity. F_1 is a dimensionless number measuring the ratio of the speed of the bore to the speed of propagation of a long wavelength, small amplitude wave on the surface of the fluid. [Hint: Consider the flow for two different times separated by time Δt.]

(b) Show that the stream-wise pressure-momentum balance for the control voume shown can be written in the form $f(h_1) = f(h_2)$, where

$$f(h) = \frac{gh^2}{2} + \frac{q^2}{h}, \tag{4}$$

and $q = h_1 u_j = h_2(u_j - u_2)$ is the volume flux.

SOLUTION 4

(a) Conservation of mass tells us that

$$\frac{\partial}{\partial t}\int_{cv}\rho dV+\oint_{cs}\rho\vec{u}\cdot\hat{n}dA=0.$$

Let $M(t)=\int_{cv}\rho dV$ be the mass in the control volume $ABCD$ at an arbitrary time t. The mass in $ABCD$ a time Δt later is equal to $M(t)$ plus the amount of mass carried in by the bore, so $M(t+\Delta t)=M(t)+\rho u_j(h_2-h_1)\Delta t$. Therefore

$$\frac{\partial}{\partial t}\int_{cv}\rho dV=\frac{dM}{dt}=\frac{M(t+\Delta t)-M(t)}{\Delta t}=\rho u_j(h_2-h_1).$$

Using this in the conservation of mass relation, and replacing the surface integral with the amount of mass carried in the left boundary gives

$$\rho u_j(h_2-h_1)-\rho u_2 h_2=0,$$

and

$$u_2=\left(\frac{h_2-h_1}{h_2}\right)u_j.$$

Conservation of momentum results in

$$\rho g\left(\int_0^{h_2}ydy-\int_0^{h_1}ydy\right)=u_j\frac{dM}{dt}-\rho u_2^2 h_2,$$

where after integration and substitution for dM/dt,

$$\frac{1}{2}\rho g(h_2^2-h_1^2)=\rho u_j^2(h_2-h_1)-\rho u_2^2 h_2.$$

Substituting for u_2 and solving for h_2/h_1 follows:

$$\frac{1}{2}\rho g(h_2^2-h_1^2)=\rho u_j^2(h_2-h_1)-\rho u_j^2\frac{(h_2-h_1)^2}{h_2},$$

$$\frac{1}{2}g(h_2+h_1)=u_j^2-u_j^2\frac{h_2-h_1}{h_2},$$

$$h_2+h_1=2\frac{u_j^2}{g}\frac{h_1}{h_2},$$

$$\frac{h_2}{h_1}+1=2\frac{u_j^2}{gh_1}\frac{h_1}{h_2},$$

and now use $F_1^2=u_j^2/(gh_1)$

$$\frac{h_2}{h_1}+1=2F_1^2\frac{h_1}{h_2},$$

$$\left(\frac{h_2}{h_1}\right)^2+\frac{h_2}{h_1}-2F_1^2=0,$$

so that solving the quadratic for the positive solution gives

$$\frac{h_2}{h_1}=\frac{1}{2}\left(\sqrt{1+8F_1^2}-1\right).$$

(b) The stream-wise momentum balance was obtained in the previous part

$$\frac{1}{2}\rho g(h_2^2 - h_1^2) = \rho u_j^2(h_2 - h_1) - \rho u_2^2 h_2,$$

and we know the volume flux is defined as $q = h_1 u_j = h_2(u_j - u_2)$. Rearranging the momentum balance to group all h_1 terms on the left and all h_2 terms on the right gives

$$u_j^2 h_1 - \frac{1}{2} g h_1^2 = (u_j^2 - u_2^2)h_2 - \frac{1}{2} g h_2^2,$$

$$q u_j - \frac{1}{2} g h_1^2 = q(u_j + u_2) - \frac{1}{2} g h_2^2,$$

and after more algebra, you will find

$$\frac{g h_2^2}{2} + \frac{q^2}{h_2} = \frac{g h_2^2}{2} + \frac{q^2}{h_2},$$

or $f(h_2) = f(h_1)$, where

$$f(h) = \frac{g h^2}{2} + \frac{q^2}{h}.$$

EXAMPLE PROBLEM 5

An incompressible Newtonian fluid is flowing in a long circular pipe of radius R. Write the simplified forms of the continuity and Navier-Stokes equations for this flow assuming steady, fully-developed flow. Derive an expression for the velocity distribution u and evaluate this to determine u_{max} and the average velocity $\overline{u}$.

The flow then encounters a 90° reducing elbow. At the inlet to the elbow, the pressure is p_{in}, the radius is still given by R, and the velocity follows from above. At the outlet, the radius is $R_{out} < R$ and the velocity is assumed to be uniform. The elbow discharges to atmospheric pressure. Determine the force required to hold the elbow in place. You may neglect the weight of the elbow and the fluid.

SOLUTION 5

Using conventional cylindrical coordinates, our assumptions are:

(1) incompressible flow $\rightarrow$ $\nabla\rho = 0$, i.e., there are no spatial gradients in fluid density;

(2) Newtonian fluid $\rightarrow$ $\vec{\vec{\tau}} = \mu\nabla\vec{u}$, where μ is constant;

(3) steady $\rightarrow$ $\partial/\partial t = 0$;

(4) fully-developed $\rightarrow$ $\partial/\partial x = 0$, assuming the coordinate x is aligned with the axis of the pipe;

(5) due to symmetry, we expect no swirling and no variations with the coordinate θ, so $u_\theta = 0$ and $\partial/\partial\theta = 0$;

(6) neglect gravity and all other body forces;

The assumptions lead to the following simplified continuity equation,

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0,$$

which from a single integration produces, $r u_r = const$ and we know from assumptions (4) and (5) $u_r = u_r(r)$ only. Our one boundary condition tells us that there cannot be any radial flow through the wall of the pipe, or

$u_r = 0$ at $r = R$. This gives $const = 0$ resulting in $u_r(r) = 0$ for all r. We can use this result when writing the simplified N-S equations.

$$r \text{ direction}: \quad 0 = -\frac{\partial p}{\partial r},$$

$$\theta \text{ direction}: \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta},$$

$$x \text{ direction}: \quad 0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r}\right).$$

The r and θ equations tell us $p = p(x)$. In addition, assumptions (4) and (5) tell us that $u_x = u_x(r)$ only. So the x direction equation becomes

$$\frac{dp}{dx} = \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_x}{dr}\right)\right].$$

Two integrations later and you will arrive at

$$u_x(r) = \frac{1}{4\mu}\frac{dp}{dx}r^2 + c_1 \ln r + c_2,$$

where c_1 and c_2 are constants. They are found through the boundary conditions $u_x(R) = 0$ (no slip), and $du_x/dr|_{r=0} = 0$ (symmetry). These give $c_1 = 0$ and $c_2 = [1/(4\mu)](-dp/dx)R^2$. The resulting velocity distribution is

$$u_x(r) = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)(R^2 - r^2),$$

with $u_r = 0$ and $u_\theta = 0$.

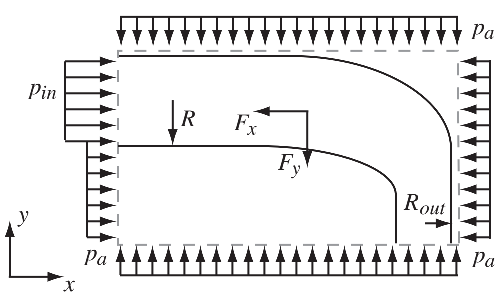
The maximum velocity will occur where $du_x/dr = 0$ which gives $r = 0$. Therefore

$$u_{max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right),$$

where it is understood that $-dp/dx$ is a positive quantity in order to drive the flow in the positive x direction. The average velocity is

$$\overline{u} = \frac{1}{\pi R^2}\int_0^R u_x 2\pi r dr = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) = \frac{1}{2}u_{max}.$$

Now the flow enters the 90° reducing elbow and we are asked to find the force required to hold the elbow in place. Assume that the elbow turns the flow downward, in the $-y$ direction. The area at the inlet is πR^2, the pressure is given as p_{in}, and we just calculated the average velocity (so the volume flow rate is $Q = \overline{u}\pi R^2$). At the outlet, we know the area, πR_{out}^2, and the pressure is atmospheric, p_a. The force can be found using a control volume around the elbow to give a force balance in the x and y directions



$$x \text{ direction}: \quad (p_{in} - p_a)\pi R^2 - F_x = -\rho \int_0^R u_x^2 2\pi r dr,$$

$$F_x = \rho \frac{\pi R^6}{48\mu^2}\left(-\frac{dp}{dx}\right)^2 + (p_{in} - p_a)\pi R^2,$$

where F_x is the force required to hold the elbow in place in the x direction and acts in the direction opposite the incoming stream (assuming the flow is moving to the right, F_x acts to the left).

$$y \text{ direction}: \quad -F_y = -\rho u_{out}^2 \pi R_{out}^2,$$

where the negative on the right comes from the fact that $\vec{u} = -u_{out}\hat{j}$ (note that $\vec{u} \cdot \hat{n}$ is positive at this boundary because $\vec{u}$ and $\hat{n}$ point in the same direction), and u_{out} is assumed uniform and can be found from mass conservation

$$\text{mass conservation}: \quad \overline{u}\pi R^2 = u_{out}\pi R_{out}^2,$$

$$u_{out} = \overline{u}\frac{R^2}{R_{out}^2} = \frac{R^4}{8\mu R_{out}^2}\left(-\frac{dp}{dx}\right),$$

resulting in

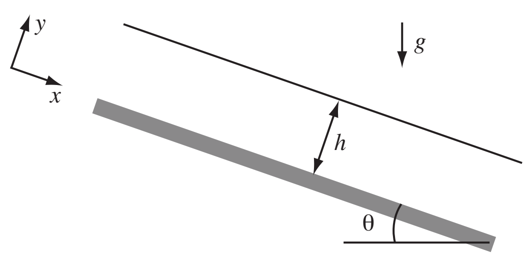
$$F_y = \rho \frac{\pi R^8}{64\mu^2 R_{out}^2}\left(-\frac{dp}{dx}\right)^2,$$

where F_y is the force required to hold the elbow in place in the y direction and acts in the direction of the bend of the elbow, or downward as was assumed. So the force acting on the elbow is

$$\vec{F} = F_x(-\hat{\imath}) + F_y(-\hat{\jmath}) = -\left[\rho \frac{\pi R^6}{48\mu^2}\left(-\frac{dp}{dx}\right)^2 + (p_{in} - p_a)\pi R^2\right]\hat{\imath} - \rho \frac{\pi R^8}{64\mu^2 R_{out}^2}\left(-\frac{dp}{dx}\right)^2 \hat{\jmath}.$$

EXAMPLE PROBLEM 6

A liquid flows down an inclined plane surface (at angle θ) due to gravity in a steady, fully-developed laminar film of thickness h. Simplify the continuity and Navier-Stokes equations to model this flow assuming the fluid is Newtonian. Obtain expressions for the velocity profile in the liquid and the shear stress distribution. Relate the film thickness to the volume flow rate per unit depth of surface normal to the flow.



SOLUTION 6

Consider the directions defined in the figure above. We will use a rotated coordinate system aligned with the plane surface so that x is parallel to the surface and y is normal to it. Our assumptions are:

(1) incompressible flow $\rightarrow$ $\nabla\rho = 0$, i.e., there are no spatial gradients in fluid density;

(2) Newtonian fluid $\rightarrow$ $\vec{\vec{\tau}} = \mu\nabla\vec{u}$, where μ is constant;

(3) steady $\rightarrow$ $\partial/\partial t = 0$;

(4) fully-developed $\rightarrow$ $\partial/\partial x = 0$;

(5) planar flow, so $w = 0$ and $\partial/\partial z = 0$;

The assumptions lead to the following simplified continuity equation,

$$\frac{\partial v}{\partial y} = 0,$$

and using the boundary condition that $v = 0$ at $y = 0$ requires $v(y) = 0$ for all y. The equations of motion are:

$$0 = \rho g \sin\theta + \mu \frac{\partial^2 u}{\partial y^2},$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\theta,$$

because we know $v = w = 0$ and $u = u(y)$ only. The second equation can be solved for the pressure distribution, which we are not interested in. Noting that $\partial u/\partial x = \partial u/\partial z = 0$, the first equation can be written as an ordinary derivative and integrated to give

$$\frac{du}{dy} = -\frac{\rho g y}{\mu}\sin\theta + c_1,$$

and again

$$u(y) = -\frac{\rho g y^2}{2\mu}\sin\theta + c_1 y + c_2.$$

The no-slip boundary condition requires $u(0) = 0$ and therefore $c_2 = 0$. The second boundary condition is zero shear at the free surface. This means $du/dy = 0$ at $y = h$ and results in

$$c_1 = \frac{\rho g h}{\mu}\sin\theta.$$

So the velocity distribution is

$$u(y) = \frac{\rho g}{\mu}\sin\theta\left(hy - \frac{y^2}{2}\right).$$

The shear follows as

$$\tau = \mu\frac{du}{dy} = \rho g \sin\theta(h - y).$$

The volume flow per unit depth is

$$\begin{aligned} Q = \int_0^h u dy &= \frac{\rho g}{\mu}\sin\theta\int_0^h\left(hy - \frac{y^2}{2}\right)dy \\ &= \frac{\rho g}{\mu}\sin\theta\left(\frac{h^3}{2} - \frac{h^3}{6}\right) \\ &= \frac{\rho g}{\mu}\sin\theta\frac{h^3}{3}. \end{aligned}$$

Rearranging to solve for the film thickness

$$h = \left(\frac{3\mu Q}{\rho g \sin\theta}\right)^{1/3}$$

EXAMPLE PROBLEM 7

Consider a spherical cloud of gas of radius $R(t)$ and total mass M. The cloud is expanding into a vacuum in such a fashion that the mass density ρ remains spatially uniform, i.e., $\rho = \rho(t)$ only. Neglect the influence of gravity.

(a) Compute the divergence $\nabla \cdot \vec{u}$ of the velocity field.

(b) Use this result to show that the fluid velocity within the cloud can be represented by

$$u_r(r,t) = r\frac{\dot{R}}{R} \quad \text{where} \quad \dot{R} = \frac{dR}{dt}. \tag{5}$$

Consider that the cloud remains spherically symmetric and the density is spatially uniform at all times.

(c) Find the location r at time t of a particle that was located at r_0 at time t_0 as a function of $R(t)$ and $R(t_0)$.

(d) The pressure goes to zero at the outer edge of the cloud. Find the pressure distribution within the cloud and the relationship between the pressure at the center of the cloud to the rate of expansion $\dot{R}$.

SOLUTION 7

This problem will be solved using spherical coordinates. Our assumptions are:

(1) no spatial gradients in density $\rightarrow$ $\nabla\rho = 0$, $\rho = \rho(t)$ only;

(2) Newtonian fluid $\rightarrow$ $\vec{\vec{\tau}} = \mu\nabla\vec{u}$, where μ is constant;

(3) due to symmetry, we expect $u_\theta = u_\phi = 0$ and $\partial/\partial\theta = \partial/\partial\phi = 0$;

(4) neglect gravity and all other body forces;

(a) The reduced continuity equation is

$$\frac{d\rho}{dt} + \rho\nabla\cdot\vec{u} = 0,$$

$$\nabla\cdot\vec{u} = -\frac{1}{\rho}\frac{d\rho}{dt},$$

and now we can write the density as $\rho = M/V$ where $V = 4\pi[R(t)]^3/3$. Substituting

$$\nabla\cdot\vec{u} = -\frac{4\pi}{3M}R^3\frac{d}{dt}\left(\frac{3M}{4\pi}\frac{1}{R^3}\right) = -R^3\frac{d}{dt}\left(\frac{1}{R^3}\right) = 3\frac{\dot{R}}{R}.$$

(b) Note that the divergence of the velocity in spherical coordinates simplifies to

$$\nabla\cdot\vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r),$$

and equating this with the result in (a),

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 3\frac{\dot{R}}{R}.$$

Solving for the velocity $u_r(r,t)$ gives

$$u_r(r,t) = \frac{\dot{R}}{R}r + \frac{c_1(t)}{r^2},$$

and from symmetry we know $u_r(0,t) = 0$, giving $c_1(t) = 0$ to keep the solution bounded. Thus,

$$u_r(r,t) = \frac{\dot{R}}{R}r.$$

(c) The tricky thing here is to realize that $u_r = dr/dt$, so

$$\frac{dr}{dt} = \frac{\dot{R}}{R}r,$$

or

$$\frac{1}{r}\frac{dr}{dt} = \frac{1}{R}\frac{dR}{dt}.$$

Integrating from t_0 to t,

$$\int_{t_0}^{t} \frac{1}{r}\frac{dr}{dt}dt = \int_{t_0}^{t} \frac{1}{R}\frac{dR}{dt}dt.$$

$$\ln\frac{r(t)}{r_0} = \ln\frac{R(t)}{R(t_0)},$$

so that solving for $r(t)$ gives

$$r(t) = \frac{R(t)}{R(t_0)}r_0.$$

(d) The simplified Navier-Stokes equations are:

$$r \text{ direction}: \quad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) - \frac{2}{r^2}u_r\right],$$

$$\theta \text{ direction}: \quad 0 = -\frac{\partial p}{\partial \theta},$$

$$\phi \text{ direction}: \quad 0 = -\frac{\partial p}{\partial \phi}.$$

The θ and ϕ equations tell us $p = p(r)$. Since we found u_r in part (b), the first equation reduces to a first-order ODE for the pressure for which the boundary condition is $p(R) = 0$. Using the following equations

$$u_r(r,t) = \frac{\dot{R}}{R}r,$$

$$\frac{\partial u_r}{\partial t} = \left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)r,$$

$$\frac{\partial u_r}{\partial r} = \frac{\dot{R}}{R},$$

we make substitutions in the r direction equation and simplify to give

$$\rho\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} + \frac{\dot{R}}{R}\frac{\dot{R}}{R}\right)r = -\frac{dp}{dr} + \mu\left[\frac{2}{r}\frac{\dot{R}}{R} - \frac{2}{r^2}\frac{\dot{R}}{R}r\right],$$

$$\rho\left(\frac{\ddot{R}}{R}\right)r=-\frac{dp}{dr}.$$

Integration over r gives

$$p(r)=-\rho\left(\frac{\ddot{R}}{R}\right)\frac{r^2}{2}+c_1,$$

and using the boundary condition $p(R)=0$, the pressure distribution is

$$p(r)=-\rho\left(\frac{\ddot{R}}{R}\right)\frac{r^2}{2}+\rho\left(\frac{\ddot{R}}{R}\right)\frac{R^2}{2}=\rho\left(\frac{\ddot{R}}{R}\right)\left(\frac{R^2}{2}-\frac{r^2}{2}\right).$$

At the center of the cloud $r=0$ and

$$p(0)=\frac{1}{2}\rho R\ddot{R}=\frac{1}{2}\rho R\frac{d\dot{R}}{dt}.$$