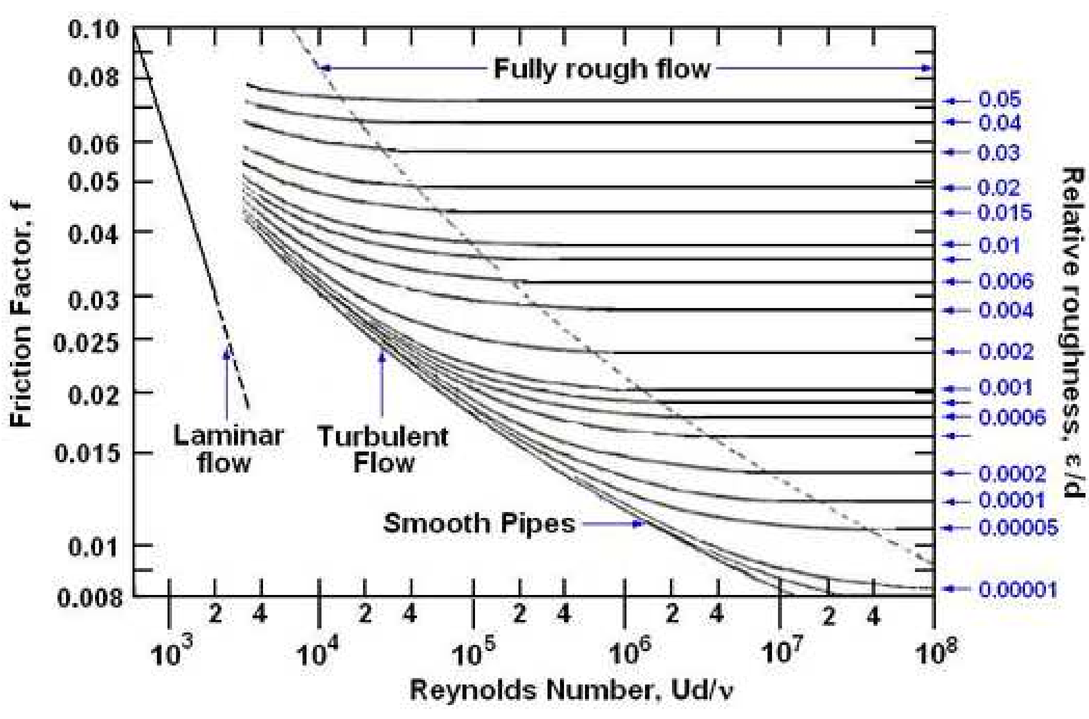
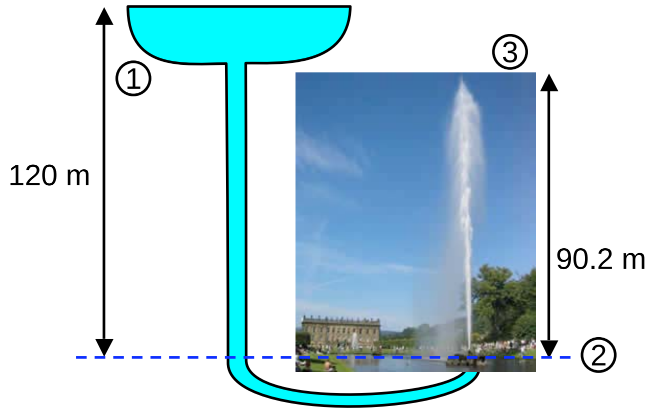
PROBLEM B23

In 1844, to commemorate a visit by the Tsar of Russia, the Duke of Devonshire, the greatest landowner in Britain, wished to construct the tallest fountain ever built in the grounds of his great house at Chatsworth in Derbyshire. He employed the renowned engineer Joseph Paxton to build what was to become known as the Emporer Fountain. That fountain remains the tallest, gravity-fed fountain in the world, with a maximum height of 90.2 m above the pond into which it falls. What Paxton did was to excavate a massive eight acre lake on a nearby hill such that the lake surface was 120 m above the afore-mentioned pond. The pipe to the fountain was 800 m long and had an internal diameter of 0.381 m. (Paxton knew that to maximize the height of the fountain he would have to make the pipe diameter large.) The result was that the maximum flow rate through the pipe (when the control valve was fully opened) was 15000 $liters/min$. Questions:

1. Using the above information find the friction factor for Paxton's pipe.
2. Find the Reynolds number for the flow in Paxton's pipe assuming a water temperature of $15°C$ so that the kinematic viscosity of the water 1.16×10^{-6} m^2/s.
3. What kind of flow is occuring in Paxton's pipe?
4. Using the answers to the first two questions estimate the typical height of the roughnesses in the interior surface of Paxton's pipe.
5. Assuming the same friction factor, what would the maximum height of the fountain have been if Paxton had used a pipe with a half of the above diameter?



SOLUTION B23



1. Taking point 2 as our zero height, Bernoulli's equation shows that the difference in pressure between points 1 and 2 is

$$p_1 + \rho g h_1 = p_2 + \frac{1}{2}V_2^2$$
$$p_1 - p_2 = \frac{1}{2}\rho V_2^2 - \rho g h_1$$

where the velocity at point 2, V_2, is an unknown velocity of the jet (not the velocity in the pipe). Since the pressure at points 2 and 3 are the same (atmospheric) Bernoulli's equation can be used to find the unknown velocity term.

$$p_2 + \frac{1}{2}V_2^2 = p_3 + \rho g h_3$$
$$\frac{1}{2}V_2^2 = \rho g h_3$$

where $p_2 = p_3 = p_a$. Thus the pressure loss in the pipe is given by

$$p_1 - p_2 = \rho g(h_3 - h_1)$$

Assuming all of this loss occurs in the supply pipe it follows that the friction factor, f, in the pipe is

$$\begin{aligned} f &= \frac{\left(-\frac{dp}{dx}\right)d}{\frac{1}{2}\rho V^2} \\ &= \frac{\left(\frac{\rho g(h_1-h_3)}{L}\right)d}{\frac{1}{2}\rho V^2} \\ &= \frac{2g(h_1-h_3)d}{LV^2} \\ &= \frac{2(9.81\ m/s^2)(120\ m - 90.2\ m)(0.381\ m)}{(800\ m)V^2} \end{aligned}$$

where V is the velocity of the flow in the pipe. Since the pipe cross-sectional area is $0.114\ m^2$ and the flow rate is $15000\ L/min$ (or $0.25\ m^3/s$), the velocity, $V = 2.2\ m/s$. Therefore

$$f = 0.058.$$

2. The Reynolds number of the flow in the pipe is

$$\begin{aligned} Re_d &= \frac{Vd}{\nu} \\ &= \frac{(2.2\ m/s)(0.381\ m)}{1.16 \times 10^{-6}\ m^2/s} \\ &= 7.2 \times 10^5 \end{aligned}$$

3. For pipes, flows with a Reynolds number less than about 2000 are laminar and above 4000 are turbulent. With this Reynolds number, the flow is well into the turbulent limit. Moreover, referring to the Moody chart, the pipe flow is also fully rough since, at this Reynolds number, the friction factor is well above that for smooth-walled turbulent flow.

4. Again, referring to the Moody chart it would appear that at this Reynolds number, a friction factor of 0.058 will occur when the roughness has a typical height of $0.03d$ or $1.1\ cm$.

5. With the same friction factor but a pipe diameter of $0.19\ m$ the head loss would be double that of the actual pipe. The maximum height of the fountain would have been $120\ m - 2 \times 29.8\ m$ or $60\ m$—much less impressive.

PROBLEM B24

Consider a turbulent boundary layer on a flat plate (constant and uniform velocity and pressure in the flow outside the boundary layer). The plate is very rough, the size of the roughnesses, ϵ, being very much greater than the laminar sub-layer thickness which would occur in the absence of the roughness. It is anticipated that the velocity distribution within the turbulent part of the boundary layer can be approximated by

$$u^* = C(y/\epsilon)^{\frac{1}{7}}$$

where C is some constant, y is the distance from the wall, $u^* = \bar{u}/u_\tau$, where $\bar{u}$ is the mean velocity and the friction velocity, $u_\tau = (\tau_w/\rho)^{\frac{1}{2}}$, τ_w being the wall shear stress and ρ the fluid density. Using approximate boundary layer methods find an expression for the boundary layer thickness, δ, as a function of x, the distance along the plate from the leading edge. Assume initial conditions $\delta = 0$ at $x = 0$; the result includes ϵ, C and the profile parameter $\alpha = 0.0972$.

SOLUTION B24

Since $u^* = \bar{u}/u_\tau$, the velocity distribution can be written as

$$\bar{u} = C u_\tau \left(\frac{y}{\epsilon}\right)^{1/7}$$

Given a constant and uniform velocity, denoted as U, in the flow outside the boundary layer, we know that $\bar{u} = U$ at $y = \delta$ and it follows that

$$U = C u_\tau \left(\frac{\delta}{\epsilon}\right)^{1/7}$$

$$u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2} = \frac{U}{C}\left(\frac{\epsilon}{\delta}\right)^{1/7}$$

But the Karman Momentum Integral Equation for a case in which U is independent of x (so $dU/dx = 0$) is

$$\frac{\tau_w}{\rho} = \alpha U^2 \frac{d\delta}{dx}$$

and elimnating τ_w/ρ from the last two equations and solving for $d\delta/dx$

$$\frac{d\delta}{dx} = \frac{\epsilon^{2/7}}{\alpha C^2 \delta^{2/7}}$$

Separating variables and integrating

$$\delta^{2/7} d\delta = \frac{\epsilon^{2/7}}{\alpha C^2} dx$$

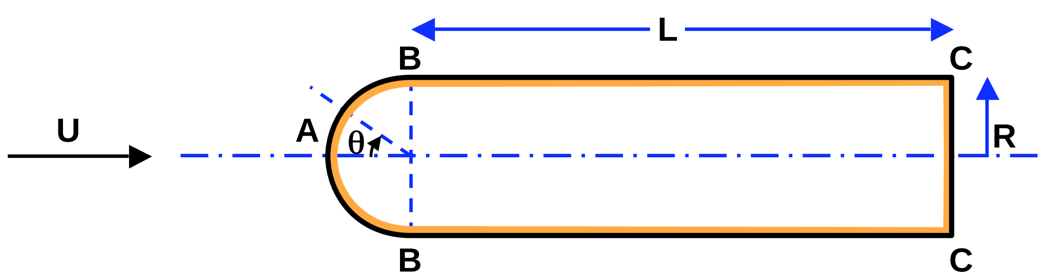
$$\frac{7}{9}\delta^{9/7} = \frac{\epsilon^{2/7}}{\alpha C^2} x + c$$

and applying $\delta = 0$ at $x = 0$ gives $c = 0$ so

$$\delta = \left(\frac{9\epsilon^{2/7}}{7\alpha C^2} x \right)^{7/9}$$

PROBLEM B25

The sketch below defines the geometry of an axisymmetric underwater body that is quite streamlined in the sense that L/R is large. This body travels through the incompressible water at a velocity, U, parallel to the axis.



It is to be assumed:

- that the velocity distribution over the spherical nose, BAB, is the same as in potential flow, that is to say the velocity outside the boundary layer is $\frac{3}{2}U \sin\theta$.
- that the flow separates at the sharp trailing edge, C, so that the pressure coefficient acting on the circular base, CC, is

$$C_p = -0.5$$

  Remember that the pressure coefficient is defined as, $C_p = (p - p_\infty)/\frac{1}{2}\rho U^2$ where p is the pressure, p_∞ is the pressure far upstream and ρ is the fluid density.
- that the skin friction forces on the spherical nose are negligible.

If the drag coefficient is defined as the drag divided by $\frac{1}{2}\rho U^2$ and the frontal projected area (πR^2) find:

1. The contribution of the form drag to the total drag coefficient (denote this by C_{DF}).

2. An estimate of the contribution of the skin friction on the cylindrical surface of the body (between B and C) to the total drag coefficient, assuming the boundary layer remains laminar. This should be in terms of the Reynolds number, $Re = 2UR/\nu$, where ν is the kinematic viscosity of the fluid.

3. For what aspect ratio, L/R, will the drag be comprised of equal parts of form and skin friction drag if $Re = 10000$?

SOLUTION B25

1. Find the form drag contribution, C_{DF}, to the total drag coefficient.

   To find the form drag, we must examine the pressure distribution on the nose and the flat trailing portion of the body. Since we can consider the flow over the nose to be described by potential flow, $u(\theta) = \frac{3}{2}U \sin\theta$, we can use Bernoulli's equation to get the corresponding pressure distribution over the surface.

$$p_\infty + \frac{1}{2}\rho U^2 = p(\theta) + \frac{1}{2}\rho[u(\theta)]^2$$

$$\Rightarrow p(\theta) - p_\infty = \frac{1}{2}\rho U^2 \left[1 - \frac{9}{4}\sin^2\theta\right]$$

   This leads to a pressure coefficient on the nose that varies with θ

$$C_{p,N} = \frac{p(\theta) - p_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{9}{4}\sin^2\theta$$

   Since the flow separates at the sharp trailing edge, we can consider the pressure to be constant on the circular base with a constant pressure coefficient.

$$C_{p,T} = -0.5$$

   Our goal is the drag coefficient due to form effects, C_{DF}.

$$C_{DF} = \frac{D_F}{\frac{1}{2}\rho U^2 \pi R^2}$$

   where D_F is the form drag. We calculate this drag by finding the difference between the pressure integrated over the nose and the base of the streamlined body.

$$D_F = \int p_N dA - \int p_T dA = \int (p_N - p_\infty) dA - \int (p_T - p_\infty) dA$$

   Dividing this equation by $\frac{1}{2}\rho U^2 \pi R^2$ gives us the form drag coefficient in terms of integrals of the pressure coefficients over the nose and tail.

$$C_{DF} = \frac{\int C_{p,N} dA}{\pi R^2} - \frac{\int C_{p,T} dA}{\pi R^2}$$

   The second integral is trivial since the pressure coefficient is constant.

$$\frac{\int C_{p,T} dA}{\pi R^2} = \frac{-0.5\pi R^2}{\pi R^2} = -0.5$$

The integral over the nose is slightly more involved.

$$\begin{aligned}
\frac{\int C_{p,N} dA}{\pi R^2} &= \frac{\int \left(1-\frac{9}{4}\sin^2\theta\right)}{\pi R^2} \\
&= \frac{1}{\pi R^2}\int_0^{\pi/2}\int_0^{2\pi}\left(1-\frac{9}{4}\sin^2\theta\right) R^2 \sin\theta\cos\theta d\phi d\theta \\
&= 2\int_0^{\pi/2}\left(1-\frac{9}{4}\sin^2\theta\right)\sin\theta\cos\theta d\theta \\
&= 2\int_0^{1}\left(\sin\theta-\frac{9}{4}\sin^3\theta\right) d(\sin\theta) \\
&= 2\left(\frac{1}{2}\sin^2\theta-\frac{9}{16}\sin^4\theta\right)\Big|_0^1 = -\frac{1}{8}
\end{aligned}$$

The form drag coefficient is the difference between these integrals.

$$C_{DF} = \frac{\int C_{p,N} dA}{\pi R^2} - \frac{\int C_{p,T} dA}{\pi R^2} = -\frac{1}{8}+\frac{1}{2} = \frac{3}{8}$$

Note: The form drag coefficient could also have been evaluated as a single integral, more akin to what was done in class. As was shown in class, it is sufficient to integrate over the projected area.

2. Find the skin friction (on the cylindrical surface) contribution, C_{DS}, to the total drag coefficient.

Since we assume that the boundary layer remains laminar, we can use the Blasius solution to calculate the skin friction drag, D_S, on the cylindrical surface.

$$\begin{aligned}
D_S &= \int \tau_0 dA \\
&= \int_0^L \frac{1}{2}\rho U^2\left(0.664\sqrt{\frac{\nu}{Ux}}\right) 2\pi R dx \\
&= \frac{1}{2}\rho U^2 2\pi R\int_0^L 0.664\sqrt{\frac{\nu}{Ux}} dx \\
&= \frac{1}{2}\rho U^2 2\pi R L\cdot 1.328\sqrt{\frac{\nu}{UL}}
\end{aligned}$$

This leads to a drag coefficient due to skin drag of:

$$C_{DS} = \frac{D_S}{\frac{1}{2}\rho U^2\pi R^2} = \frac{2L}{R}\cdot 1.328\sqrt{\frac{\nu}{UL}}$$

The skin friction drag coefficient can be rewritten in terms of a radius-based Reynolds number, $Re = 2UR/\nu$ to give:

$$C_{DS} = 1.328\cdot\frac{2L}{R}\sqrt{\frac{2R\nu}{2RUL}} = 1.328\cdot 2\sqrt{2}\sqrt{\frac{L}{R}}\sqrt{\frac{1}{Re}}$$

Note: Here we used the original free stream velocity in the Blasius calculation of the shear stress. One may argue that it may be more appropriate to use the accelerated value, $\frac{3}{2}U$ for the free stream. The flow after the nose can no longer be considered potenial and the higher free stream value will decrease back to U with distance along the body. Thus, there are likely regimes, based on the length of the body, in which one assumption for the velocity is preferable to the other. But, after all, we only desire an estimate of the contribution of the skin friction drag so either choice is fine.

3. Calculate the aspect ratio, L/R, at $Re = 10000$ for which the total drag is composed of equal parts form and skin friction drag.

   Here we equate the two drag coefficients for the given Reynolds number.

$$C_{DF} = C_{DS}$$

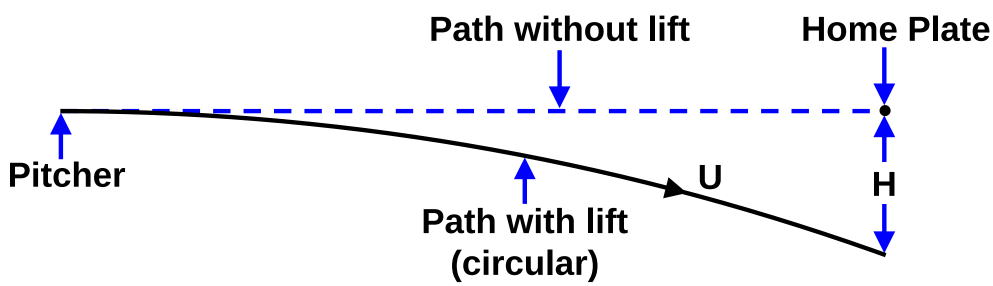
$$\frac{3}{8} = 1.328 \cdot 2\sqrt{2}\sqrt{\frac{L}{R}}\sqrt{\frac{1}{10000}}$$

$$\Rightarrow \sqrt{\frac{L}{R}} = \frac{3}{8}\frac{100}{1.328 \cdot 2\sqrt{2}} = 9.984$$

$$\frac{L}{R} = 99.67$$

PROBLEM B26

Suppose that the lift force experienced by a spinning baseball is to be estimated by $\rho U \Gamma a$ where ρ is the air density ($1\ kg/m^3$), U is its forward velocity (say $40\ m/s$), a is its radius ($0.03\ m$) and Γ is a circulation which is estimated as $2\pi a^2 \omega$ where ω is the velocity of spin (take $\omega = 200\ rad/s$). If the path of the baseball between the pitcher's mound and home plate (distance $\approx 20\ m$) is modeled as part of a circle, estimate the distance (in m) between the home plate arrival locations with and without the spin, in other words estimate the distance, H:



Neglect gravity (what!?). The mass of the baseball is $0.145\ kg$.

SOLUTION B26

If we assume a circular path, as the problem states, the lift force due to spin must be equal to the centripetal force on the baseball.

$$\rho U \Gamma a = \frac{mU^2}{R}$$

With the given relationship for the circulation, $\Gamma = 2\pi a^2 \omega$, the radius of the ball's trajectory is:

$$R = \frac{mU}{2\pi \rho a^3 \omega} = \frac{(0.145\ kg)(40\ m/s)}{(2\pi)(1\ kg/m^3)(0.03\ m)^3(200\ rad/s)} = 170.9\ m$$

From the geometry:

$$(R - H)^2 + L^2 = R^2$$

$$\Rightarrow H = R - \sqrt{R^2 - L^2} = 1.174\ m$$