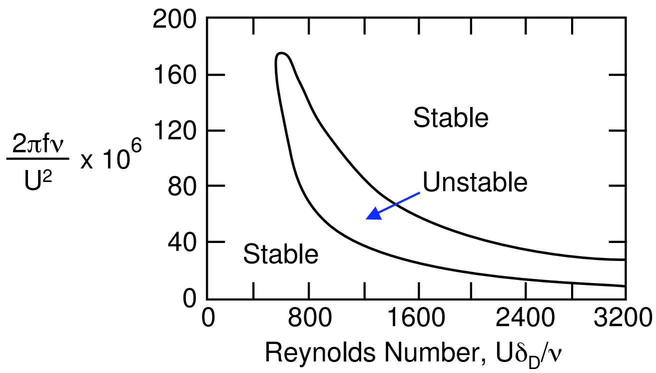
ME19b. **SOLUTIONS.** **Feb. 18, 2010. Due Feb. 25**

PROBLEM B18

The stability diagram for a laminar boundary layer on a flat plate with zero pressure gradient (Blasius problem) is given below:



Using the solid, theoretical curve find the distance from the leading edge of the plate to the point where transition to turbulence begins for a flow of water ($\nu = 10^{-6}\ m^2/s$) when $U = 2\ m/s$. What is the frequency of the most unstable disturbances (in Hz) under these conditions?

SOLUTION B18

To find the distance, x_{crit}, from the leading edge of the plate to the point where transition to turbulence begins, we note from the stability diagram that the critical Reynolds number, $Re_{\delta^*_{\text{crit}}}$, on the left-most edge of the curve is

$$Re_{\delta^*_{\text{crit}}} = \frac{U\delta^*_{\text{crit}}}{\nu} \approx 550$$

Using the Blasius laminar boundary layer solution we also know the expression for the displacement thickness as a function of x:

$$\delta_D = 1.72 \left(\frac{\nu x}{U}\right)^{1/2}$$

and so at x_{crit} we denote $\delta_D = \delta^*_{\text{crit}}$ and it follows that

$$\begin{aligned} x_{\text{crit}} &= \left(\frac{\delta^*_{\text{crit}}}{1.72}\right)^2 \frac{U}{\nu} \\ &= \frac{\nu}{U}\left(\frac{Re_{\delta^*_{\text{crit}}}}{1.72}\right)^2 \\ &= \frac{10^{-6}}{2}\left(\frac{550}{1.72}\right)^2 = 0.0511 \text{ m} \end{aligned}$$

To find the frequency, f, of the most unstable disturbance we also note from the stability diagram that the frequency which becomes unstable at the critical Reynolds number is

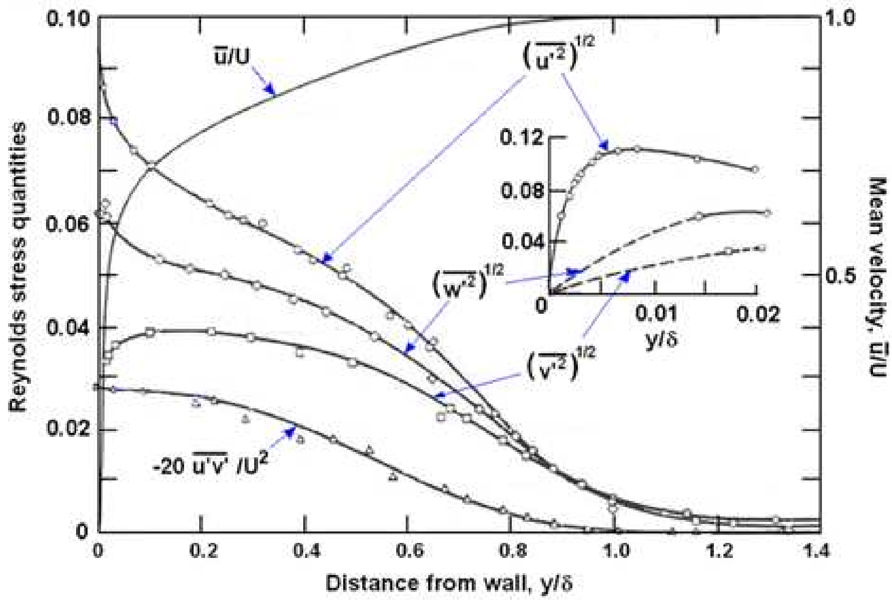
$$\frac{2\pi f\nu}{U^2} = 170 \times 10^{-6}$$

and therefore

$$f = \frac{(170 \times 10^{-6})(2\ m/s)^2}{2\pi(10^{-6}\ m^2/s)} = 108.2 \text{ Hz}$$

PROBLEM B19

Using the data in the graph below estimate the value of the "universal constant", κ, at positions, $y/\delta = 0.1,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ in the turbulent boundary layer.



What is the ratio of the mixing length, ℓ, to the boundary layer thickness, δ, at these distances from the wall?

SOLUTION B19

Consider Prandtl's Mixing Length Model:

$$-\rho\overline{u'v'} = \rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \quad \text{where} \quad l = \kappa y$$

This implies that:

$$\kappa^2 = \frac{-\overline{u'v'}}{y^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2} = \frac{-\frac{\overline{u'v'}}{U^2}}{\left(\frac{y}{\delta}\right)^2\left(\frac{\partial(\bar{u}/U)}{\partial(y/\delta)}\right)^2}$$

and, therefore, in terms of the quantities in the graph provided:

$$\kappa = \frac{\left[\frac{1}{20}\left(-20\frac{\overline{u'v'}}{U^2}\right)\right]^{1/2}}{\left(\frac{y}{\delta}\right)\frac{\partial(\bar{u}/U)}{\partial(y/\delta)}}$$

For various y/δ the quantity $-20\overline{u'v'}/U^2$ can be read from the graph and the quantity $\frac{\partial(\bar{u}/U)}{\partial(y/\delta)}$ can be found by measuring the slope of the graph for $\bar{u}/U$ against y/δ.

Table 1: **Tabulated values of the Karman constant, κ:**

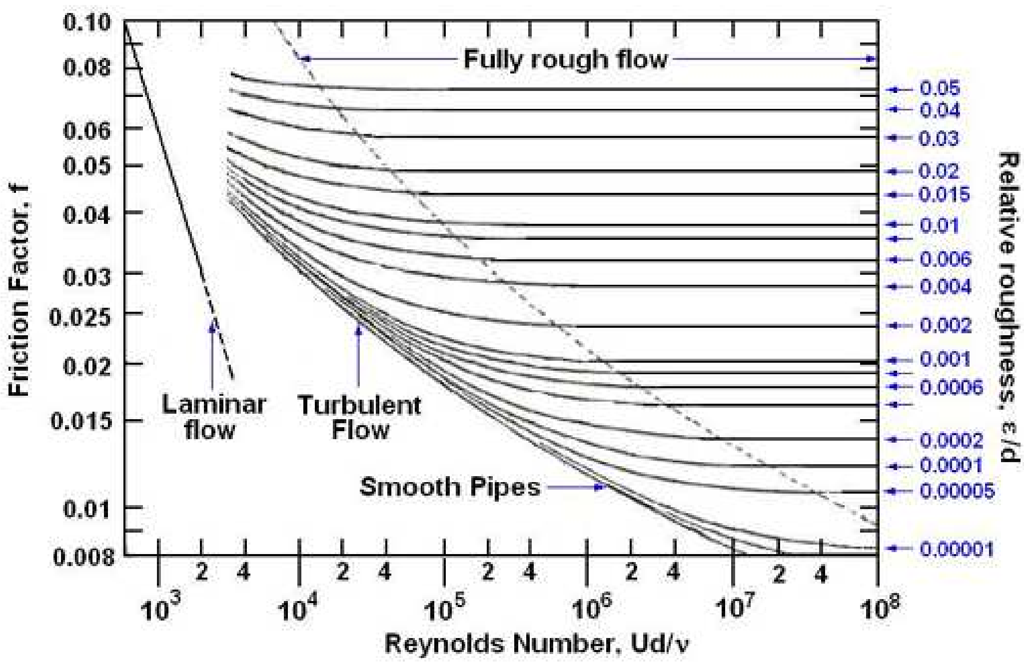
y/δ	$=$	0.1	0.2	0.4	0.6	0.8
$-20\ \overline{u'v'}/U^2$	$\approx$	0.0275	0.025	0.02	0.012	0.0035
$\partial(\bar{u}/U)\ /\ \partial(y/\delta)$	$\approx$	1	0.614	0.392	0.306	0.133
κ	$=$	0.37	0.29	0.20	0.13	0.12
$\ell/\delta\ =\ \kappa y/\delta$	$=$	0.037	0.058	0.080	0.080	0.099

Approximate results are given in the table. Note that κ is only crudely constant. However, the assumption of a constant value yields velocity distributions and wall shear stresses that are reasonable engineering approximations.

PROBLEM B20

A long ventilation duct is used to transport air at normal temperatures (density, $\rho = 1.2\ kg/m^3$, kinematic viscosity, $\nu = 2.3 \times 10^{-6}\ m^2/s$). The duct has a smooth interior surface, a circular cross-section with a diameter of 0.5 m and is 50 m long. A pressure difference of 1 $kg/m \cdot s^2$ is applied between the two ends of the duct. Using the data in the graph below, find (by trial and error or other means) the average velocity of flow through the duct.

[Note that the friction factor, $f = (-dp/dx)d/\frac{1}{2}\rho U^2$, $Re = Ud/\nu$ where d is the diameter and U is the volumetric average velocity of flow.]



SOLUTION B20

There are two analytical tools available to find the average velocity in this pipe flow. First, the friction factor gives

$$\begin{aligned} f &= \frac{\left(-\frac{dp}{dx}\right) d}{\frac{1}{2}\rho U^2} \\ U &= \sqrt{\frac{\left(-\frac{dp}{dx}\right) d}{\frac{1}{2}\rho f}} \\ &= \sqrt{\frac{\left(\frac{1\ kg/m\cdot s^2}{50\ m}\right) 0.5\ m}{\frac{1}{2}(1.2\ kg/m^3) f}} \\ &= \sqrt{\frac{1}{60 f}}\ m/s \end{aligned}$$

Second, the definition of the Reynolds number yields

$$\begin{aligned} Re &= \frac{Ud}{\nu} \\ U &= \frac{Re\ \nu}{d} \\ &= \frac{2.3 \times 10^{-6}\ m^2/s}{0.5\ m} Re \\ &= 4.6 \times 10^{-6}\ Re\ m/s \end{aligned}$$

Thus there are two equations and three unknowns (f, Re, U). To solve the problem, one must guess either the Reynolds number or the friction factor and then use the Moody chart to iterate toward the correct answer. If we start with a guessed value of the Reynolds number of 6×10^4, then the Moody chart yields $f = 0.02$ and the values of U_f (the U value from the friction factor) and U_{Re} (U from the Reynolds number) on the first line follow from the equations above. It also follows that the Reynolds number must actually be greater than 6×10^4 and hence the second iteration on the second line. The other iterations then follow until we find a Reynolds number which yields equal values of U_f and U_{Re} as follows:

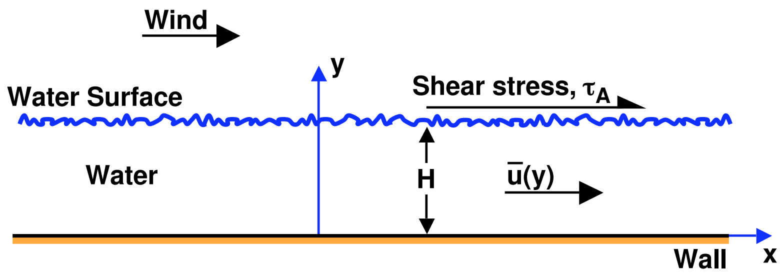
Iteration	Re	f	U_f (m/s)	U_{Re} (m/s)
1	6×10^4	0.02	0.912	0.276
2	2×10^5	0.0155	1.04	0.92
3	3×10^5	0.014	1.09	1.38
4	2.5×10^5	0.015	1.05	1.15
5	2.4×10^5	0.015	1.05	1.104
6	2.3×10^5	0.0151	1.047	1.058

Therefore,

$$U \simeq 1.05\ m/s$$

PROBLEM B21

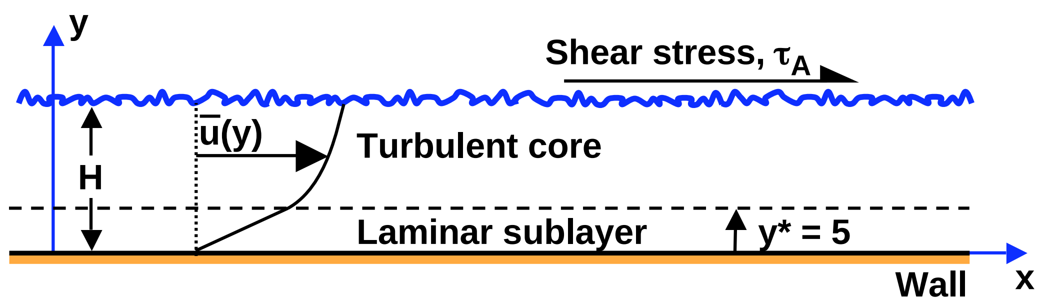
A high wind drives a film of water over a solid surface at such a speed that the flow in the film becomes turbulent. This occurs because the wind applies a shear stress, τ_A, to the surface of the water:



The thickness of the film, H, and the mean water velocity, $\bar{u}(y)$, are constant in time and with position, x. Using the assumptions listed below, find an expression for the mean velocity on the water surface, $\bar{u}(H)$, in terms of τ_A, H, ρ (the water density), ν (the kinematic viscosity of the water), and the Karman universal constant, κ. The assumptions:

- The laminar sublayer next to the solid surface (in which $u^* = y^*$) extends to $y^* = 5$ where the mean velocity is to be matched with that of the turbulent flow in the rest of the water film.
- Outside the laminar sublayer, the Reynolds stresses dominate and the viscous component of the shear stress can be neglected.
- Prandtl's mixing length theory is to be used with a Karman universal constant denoted by κ.

SOLUTION B21



Either by considering the momentum theorem applied to the control volume or by observing the similarity to Couette flow, one can conclude that the shear stress, σ_{xy}, is a constant throughout the flow. Hence $\sigma_{xy} = \tau_A$.

- In the *turbulent core*, since the viscous stresses are negligible

$$\begin{aligned} \sigma_{xy} = \tau_A &= -\rho \overline{u'v'} \\ &= \rho \kappa^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 \end{aligned}$$

by Prandtl's mixing length theory. Hence

$$\kappa y \frac{d\bar{u}}{dy} = \sqrt{\frac{\tau_A}{\rho}} = u_\tau = const.$$

and one integration produces

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} \ln y + C$$

where C is an integration constant.

- Within the *laminar sublayer*, we now know $u_\tau = \sqrt{\frac{\tau_A}{\rho}}$ and $u^* = y^*$ so

$$u^* = \frac{\overline{u}}{u_\tau} = y^* = \frac{u_\tau y}{\nu}$$

Therefore at the edge of the laminar sublayer

$$y^* = 5$$

giving

$$y = \frac{5\nu}{u_\tau}$$

also

$$\frac{\overline{u}}{u_\tau} = 5$$

Using these we can find the constant, C

$$C = 5 - \frac{1}{\kappa}\ln\left(\frac{5\nu}{u_\tau}\right)$$

Therefore in the turbulent core

$$\begin{aligned} \frac{\overline{u}}{u_\tau} &= \frac{1}{\kappa}\ln\left(\frac{u_\tau y}{5\nu}\right) + 5 \\ (\overline{u})_{y=H} &= \sqrt{\frac{\tau_A}{\rho}}\left[\frac{1}{\kappa}\ln\left(\frac{H}{5\nu}\sqrt{\frac{\tau_A}{\rho}}\right) + 5\right] \end{aligned}$$

PROBLEM B22

The velocity profile in a turbulent boundary layer of incompressible fluid on a flat plate (U = constant) is to be approximated by the form:

$$u/U = (y/\delta)^{1/7}$$

[Disregard the fact that this does not exactly satisfy one of the constraints usually imposed on laminar boundary layer profiles namely that du/dy should tend to zero as y tends to δ]. Find the profile parameter α for this profile. If the wall shear stress, τ_w, for this turbulent profile is assumed to be given by the empirical formula

$$\tau_w = 0.023\rho U^2(\nu/\delta U)^{1/4}$$

where ρ and ν are the fluid density and kinematic viscosity, then solve the resulting Karman momentum integral equation to obtain an expression for the thickness of the boundary layer, δ, as a function of distance, x, along the plate. Assume that the layer first becomes turbulent at $x = x_0$ where the thickness is δ_0.

[Do not use $\tau_w = \mu(du/dy)_{y=0}$ which is inappropriate in turbulent boundary layer calculations.]

SOLUTION B22

The velocity profile for a turbulent boundary layer of incompressible fluid on a flat plate (where U = constant) is approximated as

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

The profile parameter, α, follows from

$$\begin{aligned}
\alpha = \frac{\delta_M}{\delta} &= \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)\, d\left(\frac{y}{\delta}\right) \\
&= \int_0^1 \left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right] d\left(\frac{y}{\delta}\right) \\
&= \left.\left[\frac{7}{8}\left(\frac{y}{\delta}\right)^{8/7} - \frac{7}{9}\left(\frac{y}{\delta}\right)^{9/7}\right]\right|_0^1 \\
&= \frac{7}{8} - \frac{7}{9} \\
&= \frac{7}{72} = 0.0972
\end{aligned}$$

From the K.M.I.E.,

$$\begin{aligned}
\tau_W &= \rho\frac{d}{dx}\left(U^2\delta_M\right) + \rho\delta_D U\frac{dU}{dx} \\
&= \rho U^2 \frac{d}{dx}(\alpha\delta) \\
&= \rho U^2 \alpha \frac{d\delta}{dx}
\end{aligned}$$

where α and U are constants. If the wall shear stress for this turbulent profile is assumed to be given by $\tau_W = 0.023\rho U^2 (\nu/\delta U)^{\frac{1}{4}}$,

$$\begin{aligned}
\tau_W = \rho U^2 \alpha \frac{\mathrm{d}\delta}{\mathrm{d}x} &= 0.023\rho U^2 (\nu/\delta U)^{1/4} \\
\delta^{1/4}\mathrm{d}\delta &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{1/4}\mathrm{d}x \\
\frac{4}{5}\delta^{5/4} &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{1/4} x + c
\end{aligned}$$

To evaluate c, we use $x = x_0$, $\delta = \delta_0$:

$$c = \frac{4}{5}\delta_0^{5/4} - \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{1/4} x_0$$

$$\begin{aligned}
\frac{4}{5}\left(\delta^{5/4} - \delta_0^{5/4}\right) &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{1/4}(x - x_0) \\
\delta^{5/4} &= \frac{5}{4}\left(\frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{1/4}(x - x_0) + \frac{4}{5}\delta_0^{5/4}\right) \\
\delta &= \left[\frac{5}{4}\left(\frac{0.023}{\alpha}\right)\left(\frac{\nu}{U}\right)^{1/4}(x - x_0) + \delta_0^{5/4}\right]^{4/5} \\
&= \left[0.296\left(\frac{\nu}{U}\right)^{1/4}(x - x_0) + \delta_0^{5/4}\right]^{4/5}
\end{aligned}$$

This solution will be valid for $x \geq x_0$, i.e., within the turbulent boundary layer.