PROBLEM B14

Consider the long thin racing boats used in competitive rowing events. Assume that the major component of resistance to motion is the skin friction drag on the hull of the boat. Estimate this drag force for any velocity, U, by assuming that

1. the hull/water interface is like a flat plate with a length of 10 m and a width of 1 m.
2. the boundary layer on the hull remains laminar (even though in practice this would not be the case) and unseparated.

The water density, ρ, and kinematic viscosity, $\nu = \mu/\rho$, are respectively 1000 kg/m^3 and 10^{-6} m^2/s.

If the boat is propelled by eight humans *each* capable of a rate of output of work of 0.1 HP (1 $HP = 746\ kg{\cdot}m^2/s^3$) and if half of this energy is uselessly dissipated in the rowing process what would be the top speed of the boat?

SOLUTION B14

From the Blasius solution for a laminar boundary layer on a flat plate with zero pressure gradient, the drag on one side of the flat plate is

$$D = \rho U^2 w \left[(\delta_M)_{\text{trailing edge}} - (\delta_M)_{\text{leading edge}} \right]$$

where w is the width of the plate and δ_M is the momentum thickness of the boundary layer defined as

$$\delta_M = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Using the Blasius boundary layer solution, the momentum thickness evaluates to

$$\delta_M = 0.664 \left(\frac{\nu x}{U}\right)^{1/2}$$

At the leading edge of the plate, $x = 0$ and therefore $(\delta_M)_{\text{leading edge}} = 0$. Substituting $x = L$ at the trailing edge, the first equation yields

$$D = 0.664 \rho w \left(\nu L U^3\right)^{1/2}$$

which, after substituting $\rho = 10^3\ kg/m^3$, $\nu = 10^{-6}\ m^2/s$, $L = 10\ m$, and $w = 1\ m$, becomes

$$D \simeq 2.1 U^{3/2}\ kg \cdot m/s^2$$

The total power generated by the eight rowers at 50% efficiency (because half the power is uselessly dissipated) is

$$P = 0.5\,(8P_i)$$

Each rower can produce $P_i = 0.1\ HP$, so that

$$P = 0.4\ HP = 298.4\ W$$

The power is related to the force necessary to move the boat by

$$P = D\,U = 2.1 U^{5/2}\ W$$

Thus, the boat can reach a top speed of

$$U = 7.26\ m/s$$

PROBLEM B15

Air enters a long horizontal ventilation duct of circular cross-section (radius 0.25 m) with a velocity of 1.0 m/s. At the entrance it is assumed that this velocity is uniform over the entire cross-section. However as the flow proceeds down the duct a thin laminar boundary develops on the inside wall of the duct. If we first assume that this is like the boundary layer on a flat plate and that the velocity away from the boundary layer remains at 1.0 m/s find the displacement thickness, δ_D (in m), at a distance x (in m) from the entrance. Assume the kinematic viscosity of the air is $2.5 \times 10^{-6}\ m^2/s$.

Having calculated this displacement thickness we recognize that the velocity outside the boundary layer cannot remain precisely constant at 1 m/s. Using the above calculated displacement thickness find the uniform velocity outside the boundary layer at a point 200 m from the entrance. What is the pressure difference between the entrance and this point 200 m from the entrance?

SOLUTION B15

If the boundary layer is like that of a flat plate (for which $dp/dx = 0$) then the Blasius solution applies and

$$\begin{aligned}
\delta_D &= 1.72 \left(\frac{\nu x}{U}\right)^{1/2} \\
&= 1.72 \left(\frac{2.5 \times 10^{-6}\ x}{1.0}\right)^{1/2} \quad m \\
&= 2.7 \times 10^{-3}\ x^{1/2}\ m
\end{aligned}$$

The effect of this displacement thickness is to yield a volume flow rate in the tube which is the same as the volume flow rate of a *uniform* stream in a tube of radius $(R - \delta_D)$ where R is the actual radius of the tube. Therefore the velocity of the flow outside the boundary layer is not 1 m/s but U_x where

$$U \pi R^2 = U_x \pi \left(R - \delta_D\right)^2$$

At $x = 200\ m$, where $\delta_D = 0.038\ m$ this yields

$$U_x = \frac{U}{\left(1 - \delta_D/R\right)^2} = 1.39\ m/s$$

Since Bernoulli's equation applies outside the boundary layer the pressure at $x = 200\ m$ is related to the pressure at the inlet $(x = 0\ m)$ by

$$\begin{aligned}
p_{x=200\ \mathrm{m}} - p_{x=0\ \mathrm{m}} &= \frac{1}{2}\rho\left[U^2 - U_x^2\right] \\
&= -0.57\ \mathrm{Pa}
\end{aligned}$$

where $\rho = 1.2\ kg/m^3$.

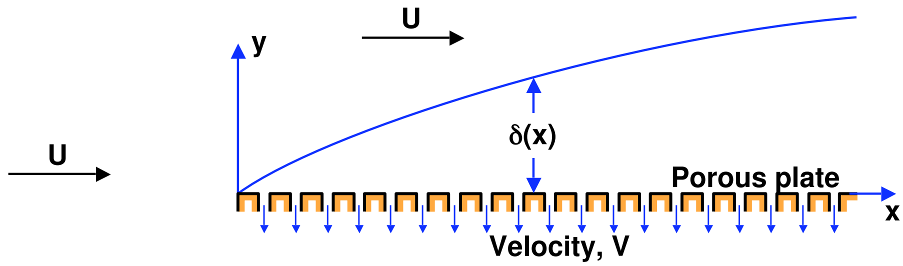
In order to proceed to a more accurate solution that takes this pressure and velocity gradient into account, one might approximately estimate the value of the Falkner-Skan m from

$$m = \frac{x}{U}\frac{dU}{dx} \approx \frac{U_{x=200\ m}}{U} - 1 = 0.39$$

and then use the Falkner-Skan solution for this value of m instead of the Blasius solution to evaluate the displacement thickness.

PROBLEM B16

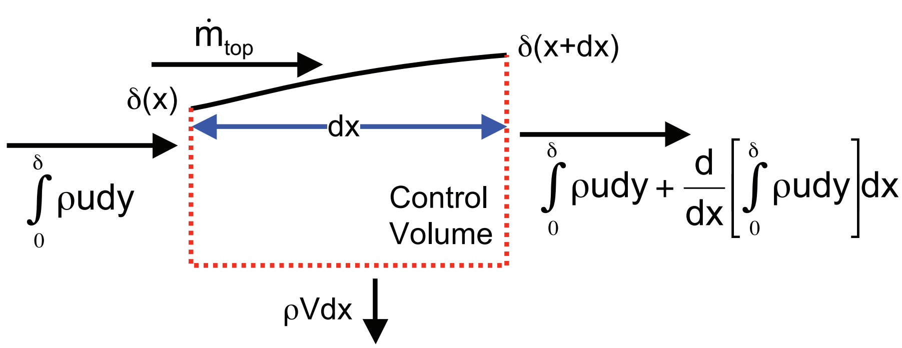
A laminar boundary layer forms on a **porous** flat plate. Fluid is removed through the porous flat plate at a uniform velocity, V.



In other words, the volume of fluid removed through the porous plate per unit plate length, per unit breadth (perpendicular to figure) and per unit time is equal to V. The thickness of the boundary layer is denoted by $\delta(x)$ and the velocity outside the boundary layer is a constant, U. Using approximate boundary layer methods assuming similarity of the velocity profile (in other words that $u/U = F(y/\delta)$ where the function F is not a function of x) find a relation between the coefficient of friction ($= \tau_w/\frac{1}{2}\rho U^2$) and the quantities V, U, $d\delta/dx$ and α where α is the profile parameter

$$\alpha = \int_0^1 F(1-F)d\left(\frac{y}{\delta}\right)$$

SOLUTION B16



We define a control volume between x and $x + dx$, bounded by the surface of the plate and extending to the edge of the boundary layer. Continuity (conservation of mass) tells us that

$$\dot{m}_{\text{top}} + \dot{m}_{\text{left}} - \dot{m}_{\text{right}} - \dot{m}_{\text{bottom}} = 0$$

$$\dot{m}_{\text{top}} + \int_0^\delta \rho u dy - \left(\int_0^\delta \rho u dy + \frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx\right) - \rho V dx = 0$$

This requires that the mass flow rate into the top of the control volume, $\dot{m}_{\text{top}}$, is:

$$\dot{m}_{\text{top}} = \frac{d}{dx}\left[\int_0^{\delta} \rho u dy\right] dx + \rho V dx$$

Now, applying the momentum thereom in the x direction, the sum of the forces acting on the control volume is equal to the the net flux of momentum out of the control volume

$$\begin{aligned} \sum F_x &= -\dot{m}_{top} U - \int_0^{\delta} \rho u^2 dy + \left(\int_0^{\delta} \rho u^2 dy + \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right] dx\right) \\ &= -\dot{m}_{top} U + \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right] dx \end{aligned}$$

[Note that there is no x-direction momentum flux through the bottom plate surface.] After substituting for $\dot{m}_{top}$ from continuity this becomes

$$\begin{aligned} \sum F_x &= -\left(\frac{d}{dx}\left[\int_0^{\delta} \rho u U dy\right] dx + \rho V U dx\right) + \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right] dx \\ &= -\rho U^2 \left(\frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right] dx + \frac{V}{U} dx\right) \\ &= -\rho U^2 \left(\frac{d}{dx}\left[\delta \int_0^1 F(1-F)\, d\eta\right] dx + \frac{V}{U} dx\right) \\ &= -\rho U^2 \left(\alpha \frac{d\delta}{dx} + \frac{V}{U}\right) dx \end{aligned}$$

where the constant α:

$$\alpha = \int_0^1 F(1-F)\, d\eta$$

and $\eta = y/\delta$.

Summing the forces on the control volume in the x-direction gives:

$$\sum F_x = -\tau_w dx - \frac{dp}{dx}\delta(x) dx$$

and since there is no pressure gradient in the exterior flow (because dp/dx is proportional to dU/dx and U is constant in this problem) and therefore no force due to a pressure difference it follows that

$$\tau_w = \rho U^2 \left(\alpha \frac{d\delta}{dx} + \frac{V}{U}\right)$$

The skin friction coefficient C_f is then given by:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2\left(\alpha \frac{d\delta}{dx} + \frac{V}{U}\right)$$

PROBLEM B17

A laminar boundary layer in a planar, incompressible flow experiences a velocity, U, external to the boundary layer which increases with distance, x, measured along the surface as follows:

$$U = Ax^{1/2}$$

where A is a known constant.

Approximate boundary layer methods (the Karman momentum integral equation) are to be used to find the boundary layer thickness, δ. An approximate velocity profile is assumed and the profile parameters, α, β, and γ, are calculated. Assume that this has been done and that α, β, and γ are known.

The answer is that the boundary layer thickness is given by

$$\delta = Cx^m$$

where C and m are constants. Determine the index m. Find the constant C as a function of A, α, β, γ, and the kinematic viscosity, ν, of the fluid.

SOLUTION B17

The Karman Momentum Integral Equation is :

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2\delta_M\right) + \delta_D U \frac{dU}{dx}$$

where α, β and γ are the usual profile parameters, τ_w is the wall shear stress, U is the velocity exterior to the boundary layer, δ is the boundary layer thickness, ν is the kinematic viscosity of the fluid and x is the streamwise distance along the wall surface.

According to the definitions of $\beta = \frac{d(u/U)}{d(y/\delta)}$ and $\tau_w = \mu\frac{du}{dy}$, the left hand side of the Karman Momentum Integral Equation (KMIE) can be expressed as:

$$\frac{\tau_w}{\rho} = \frac{\nu U\beta}{\delta}$$

The definitions of α and γ give:

$$\delta_M = \alpha\delta$$

$$\delta_D = \gamma\delta$$

Substitute into the KMIE:

$$\begin{aligned}
\frac{\nu U\beta}{\delta} &= \frac{d}{dx}\left(U^2\alpha\delta\right) + \gamma\delta U\frac{dU}{dx} \\
&= \alpha U^2\frac{dU}{dx} + 2\alpha\delta U\frac{d\delta}{dx} + \gamma\delta U\frac{dU}{dx} \\
\frac{\nu\beta}{\alpha\delta} &= U\frac{d\delta}{dx} + \left(2 + \frac{\gamma}{\alpha}\right)\delta\frac{dU}{dx}
\end{aligned}$$

Substituting $U = Ax^{1/2}$ and $\delta = Cx^m$ yields:

$$\frac{\nu\beta}{\alpha Cx^m} = mACx^{m-1/2} + \left(1 + \frac{\gamma}{2\alpha}\right)ACx^{m-1/2}$$

By matching powers of x,

$$m = \frac{1}{4}$$

and

$$C = \left[\frac{4\nu\beta}{\alpha A(5 + \frac{2\gamma}{\alpha})}\right]^{1/2}$$