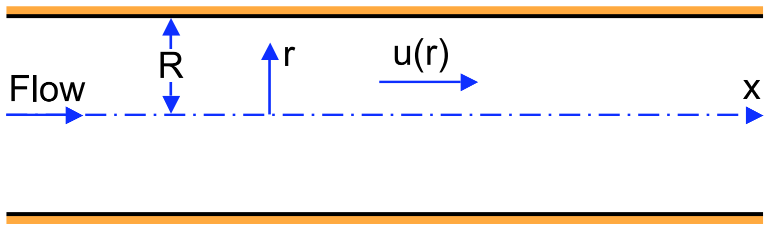
ME19b. SOLUTIONS. Jan. 28, 2010. Due Feb. 4

PROBLEM B12

Consider the fully developed laminar pipe flow of an incompressible, non-Newtonian fluid :



This fluid is such that the normal stress in the x direction is equal to $-p$ where p is the pressure and the shear stress, τ, is related to the velocity gradient by

$$\tau = C\left(-\frac{du}{dr}\right)^2$$

where C is a known constant. Find the friction factor, f , for this pipe flow in terms of C, ρ (the fluid density) and R (the radius of the pipe).

[Note: Remember the definition of the friction factor, f, namely

$$f = \frac{4R}{\rho V^2}\left(-\frac{dp}{dx}\right)$$

where V is the volumetric average velocity of flow in the pipe (the volume flow rate divided by the cross-sectional area).]

SOLUTION B12

In class we derived a relation between the shear stress and the pressure gradient for fully developed pipe flow that was good for any kind of fluid. I repeat that derivation here for the sake of completeness:

Consider a cylindrical control volume within the pipe of axial length dx and extending out to some arbitrary radius r within the pipe. Since the velocities at the two ends of this control volume are identical there is no net momentum flux out of this control volume. Consequently the forces acting on this control volume must balance. If the pressure at the inlet end is denoted by p the force in the x direction due to that pressure is $p\pi r^2$ in the positive direction. Then the pressure on the outlet end will be $(p + (dp/dx)dx)$ and this creates a force $(p + (dp/dx)dx)\pi r^2$ in the negative x direction. Thirdly, if the shear stress at radius r is denoted by τ this will impose a force equal to $\tau 2\pi r dx$ in the negative x direction. Equating these forces leads to the result:

$$\tau(r) = \frac{r}{2}\left(-\frac{dp}{dx}\right)$$

Therefore, in the present problem

$$\tau(r) = \frac{r}{2}\left(-\frac{dp}{dx}\right) = C\left(-\frac{du}{dr}\right)^2$$

where, since $(-dp/dx)$ is independent of r, it can be regarded as a constant. Then

$$\left(-\frac{du}{dr}\right) = \frac{r^{1/2}}{(2C)^{1/2}}\left(-\frac{dp}{dx}\right)^{1/2}$$

and, integrating:

$$u = -\frac{2r^{3/2}}{3(2C)^{1/2}}\left(-\frac{dp}{dx}\right)^{1/2} + constant$$

where the integration constant is determined by the condition that $u = 0$ at $r = R$ so that

$$u = \frac{2}{3(2C)^{1/2}}\left(-\frac{dp}{dx}\right)^{1/2} [R^{3/2} - r^{3/2}]$$

To find the average volumetric velocity, $\bar{u} = V$, we integrate over the cross-section of the pipe and divide by the area, πR^2:

$$\bar{u} = V = \frac{1}{\pi R^2}\int_0^R 2\pi r u dr = \frac{2}{7(2C)^{1/2}}\left(-\frac{dp}{dx}\right)^{1/2} R^{3/2}$$

and it then follows that the friction factor, f, is given by

$$f = \frac{4R}{\rho}\left[\frac{98C}{4R^3(-dp/dx)}\right]\left(-\frac{dp}{dx}\right) = \frac{98C}{\rho R^2}$$

PROBLEM B13

Consider a flow in which the density, ρ, of a particular fluid element remains unchanged as it moves along in the flow but in which the density may vary from one fluid element to another. The fluid will be assumed to be inviscid and the body forces are assumed to be conservative.

(a) What are the forms of the equation of continuity and the equation of motion which are appropriate for such a flow?

(b) Now consider such a flow which is in the process of accelerating from rest at time, $t = 0$. The velocity vector $\vec{u}$ is zero but the acceleration, $\partial\vec{u}/\partial t$, is not zero. Show that the rate of increase of vorticity, $\vec{\omega}$, in the flow at time $t = 0$ is directly related to the density gradient and the acceleration by

$$\left(\frac{\partial\vec{\omega}}{\partial t}\right)_{t=0} = -\frac{1}{\rho}\left[(\nabla\rho)\times\frac{\partial\vec{u}}{\partial t}\right]_{t=0}$$

Note the following vector identities:

$$\nabla\times(\rho\vec{a}) = (\nabla\rho)\times\vec{a} + \rho\nabla\times\vec{a} \qquad \text{where } \rho \text{ is a scalar and } \vec{a} \text{ is a vector.}$$

$$\nabla\times(\nabla y) = 0 \qquad \text{where } y \text{ is a scalar.}$$

SOLUTION B13

(a) In its most general form, the equation of continuity can be written as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

which can be expanded as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho = 0$$

Now express this equation in terms of the Lagrangian derivative, D/Dt,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0$$

In this flow, the density of each fluid element is constant. As a result, $D\rho/Dt = 0$. The continuity equation then becomes

$$\nabla \cdot \vec{u} = 0$$

Because each fluid element is incompressible, the momentum equation can be written as

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \vec{F}$$

which can be expanded into the form

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\, \vec{u} \right] = -\nabla p + \nabla U$$

where it is assumed the body forces are conservative and given by $\vec{F} = \nabla U$.

(b) Taking the curl of the momentum equation,

$$\nabla \times \left[\rho \frac{\partial \vec{u}}{\partial t} + \rho\, (\vec{u} \cdot \nabla) \right] = -\nabla \times (\nabla p) + \nabla \times (\nabla U)$$

$$\nabla \times \left(\rho \frac{\partial \vec{u}}{\partial t} \right) + \nabla \times [\rho\, (\vec{u} \cdot \nabla)\, \vec{u}] = 0$$

$$(\nabla \rho) \times \frac{\partial \vec{u}}{\partial t} + \rho \left(\nabla \times \frac{\partial \vec{u}}{\partial t} \right) + \nabla \times [\rho\, (\vec{u} \cdot \nabla)\, \vec{u}] = 0$$

Evaluating this expression at time $t = 0$ when $u = 0$ but $\partial u/\partial t \neq 0$ it follows that

$$\left[(\nabla \rho) \times \frac{\partial \vec{u}}{\partial t} \right]_{t=0} = \left(-\rho \frac{\partial \vec{\omega}}{\partial t} \right)_{t=0}$$

Thus

$$\left(\frac{\partial \vec{\omega}}{\partial t} \right)_{t=0} = -\frac{1}{\rho} \left[(\nabla \rho) \times \frac{\partial \vec{u}}{\partial t} \right]_{t=0}$$