**PROBLEM B8**

Consider the laminar, viscous, planar flow of an incompressible fluid contained between two parallel plates a distance $H$ apart. The coordinates $x$ and $y$ are measured parallel to and perpendicular to these plates, respectively. We shall take $y = 0$ at the static plate and $y = H$ at the moving plate for convenience. The plate at $y = H$ moves with a steady velocity, $U$, in the $x$ direction. However, unlike simple Couette flow, a pressure gradient, $dp/dx$, is also applied to the fluid. Find:

[1] The velocity distribution, $u(y)$, in the flow as a function of $y$, $U$, $H$, $dp/dx$ and the viscosity of the fluid, $\mu$.

[2] The magnitude and direction of the particular pressure gradient for which there would be zero net volume flow in the $x$ direction.

SOLUTION B8

[1] Since the flow is steady, planar, and incompressible the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The velocity in the vertical direction, $v$, is zero at both boundaries and thus everywhere in the flow, so the continuity equation dictates that:

$$\frac{\partial u}{\partial x} = 0$$

so $u$ is only a function of $y$, $u = u(y)$.

The Navier-Stokes equation in the **y-direction** reduces to

$$\frac{\partial p}{\partial y} = 0$$

and therefore the pressure can only be a function of $x$.

The Navier-Stokes equation in the **x-direction** is:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{dp}{dx} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Since the flow is steady, planar, $v = 0$, and $u = u(y)$, this becomes:

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Integrating twice with respect to $y$ and noting that $dp/dx$ is a simple constant for this operation (because $p$ does not depend on the integration variable $y$):

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + c_1 y + c_2$$

We now use the boundary conditions to evaluate the constants $c_1, c_2$ :

$$u(0) = c_2 = 0$$

$$u(H) = \frac{1}{2\mu}\frac{dp}{dx}H^2 + c_1 H = U$$

Therefore

$$c_1 = \frac{U}{H} - \frac{H}{2\mu}\frac{dp}{dx}$$

Inserting these values for the constants, the velocity distribution is:

$$\frac{u(y)}{U} = \frac{y}{H} - \frac{H^2}{2\mu U}\frac{dp}{dx}\frac{y}{H}\left(1 - \frac{y}{H}\right)$$

[2] Find the magnitude and direction of the particular pressure gradient for which there would be zero net volume flow in the $x$ direction. Evaluating the volume flow rate, $Q$, per unit depth normal to the sketch:

$$\begin{aligned} Q &= \int_0^H u(y)dy \\ &= \int_0^H \left\{ U\frac{y}{H} + \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - Hy\right)\right\} dy \\ &= \frac{1}{2}UH - \frac{1}{12\mu}\frac{dp}{dx}H^3 \end{aligned}$$

Therefore the particular pressure gradient, $\frac{\hat{dp}}{dx}$, for which there will be no net volume flow ($Q = 0$) will be:

$$\frac{\hat{dp}}{dx} = \frac{6\mu U}{H^2}$$

The pressure gradient is positive, so the pressure will need to increase in the positive x-direction to offset the effect of the moving upper plate.

**PROBLEM B9**

In cylindrical coordinates, $(r, \theta, z)$, the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity, $\mu$, and density, $\rho$, are

$$\rho\left[\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right] = -\frac{\partial p}{\partial r} + f_r + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

$$\rho\left[\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta + \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]$$

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + f_z + \mu\nabla^2 u_z$$

where $u_r, u_\theta, u_z$ are the velocities in the $r, \theta, z$ cylindrical coordinate directions, $p$ is the pressure, $f_r, f_\theta, f_z$ are the body force components in the $r, \theta, z$ directions and the operators $D/Dt$ and $\nabla^2$ are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

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Now consider the steady, planar, incompressible, viscous flow between two concentric cylinders. The inner cylinder has radius, $a$, and is rotating with angular velocity, $\Omega$ (radians/second). The outer cylinder has radius, $b$, and is static. There is no flow in the direction parallel to the axis of the cylinders so only the velocity, $u_\theta$, is non-zero. Body forces are to be neglected. The density of the fluid is denoted by $\rho$. Find:

(a) The velocity distribution, $u_\theta(r)$, in the gap between the two cylinders.

(b) The difference between the pressure on the outer surface of the inner cylinder and the pressure on the inner surface of the outer cylinder.

Note: The solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0 \qquad \text{is} \qquad y = A/x + Bx$$

where $A$ and $B$ are constants.

SOLUTION B9

(a) With the prescription of the flow in this problem, the Navier-Stokes equations become

$$-\rho\frac{u_\theta^2}{r} = -\frac{dp}{dr}$$

$$0 = \mu\left(\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2}\right)$$

The note at the end of the problem provides the solution to the differential equation,

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} - \frac{X}{r^2} = 0$$

namely

$$X = Ar + \frac{B}{r}$$

where $A$ and $B$ are integration constants. In the present problem this yields

$$u_\theta = Ar + \frac{B}{r}$$

We now apply the boundary conditions to determine the values of $A$ and $B$. At $r = b$ (the surface of the outer, stationary cylinder) $u_\theta = 0$ by the no-slip condition, so that

$$0 = Ab + \frac{B}{b} \quad\Longrightarrow\quad B = -Ab^2$$

Also at $r = a$ (the surface of the inner, rotating cylinder) $u_\theta = \Omega a$, where $\Omega$ is the angular velocity of the inner cylinder, so that

$$\Omega a = Aa + \frac{B}{a} = A\left(a - \frac{b^2}{a}\right) \quad\Longrightarrow\quad A = -\frac{\Omega a^2}{b^2 - a^2}$$

Substituting these expressions for $A$ and $B$ into the flow solution yields

$$u_\theta(r) = \frac{\Omega a^2}{b^2 - a^2}\left(\frac{b^2}{r} - r\right)$$

(b) Using this solution, the first equation yields

$$\frac{dp}{dr} = \rho\frac{u_\theta^2}{r} = \rho\frac{\Omega^2 a^4}{(b^2-a^2)^2}\left(\frac{b^4}{r^3} - 2\frac{b^2}{r} + r\right)$$

and integrating this gives

$$p(r) = \rho\frac{\Omega^2 a^4}{(b^2-a^2)^2}\left(-\frac{b^4}{2r^2} - 2b^2\ln r + \frac{r^2}{2}\right) + C$$

where $C$ is an integration constant. This can be used to find the pressure difference between the surfaces of the two cylinders, namely

$$p(b)-p(a) = \left[\rho\frac{\Omega^2 a^4}{(b^2-a^2)^2}\left(-\frac{b^2}{2} - 2b^2\ln b + \frac{b^2}{2}\right)\right] - \left[\rho\frac{\Omega^2 a^4}{(b^2-a^2)^2}\left(-\frac{b^4}{2a^2} - 2b^2\ln a + \frac{a^2}{2}\right)\right]$$

which simplifies to

$$p(b)-p(a) = \rho\frac{\Omega^2 a^4}{(b^2-a^2)^2}\left[\frac{b^4}{2a^2} - 2b^2\left(\ln b - \ln a\right) - \frac{a^2}{2}\right]$$

and further

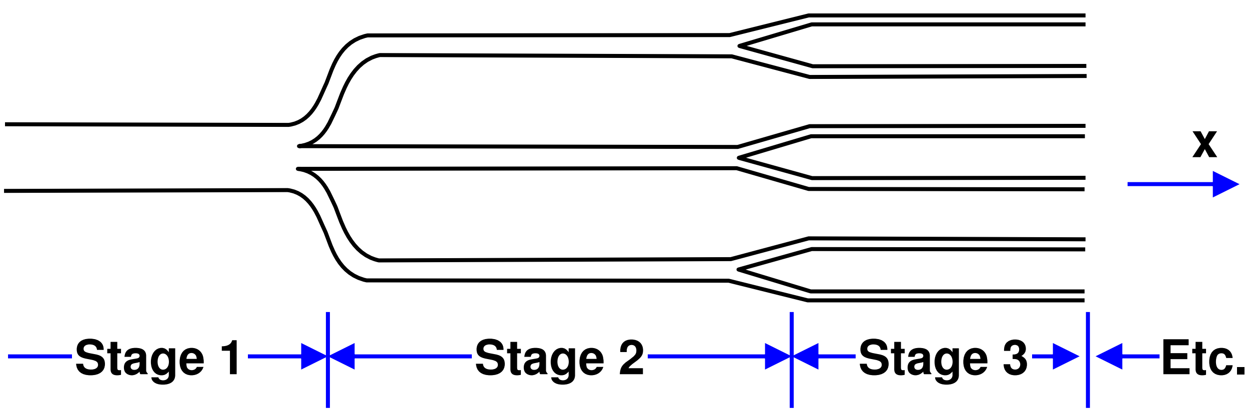
$$p(b)-p(a) = \rho\frac{\Omega^2 a^2}{(b^2-a^2)^2}\left[\frac{1}{2}\left(b^4-a^4\right) - 2a^2b^2\ln\frac{b}{a}\right]$$

or

$$p(b)-p(a) = \rho\frac{\Omega^2 a^2}{(b^2-a^2)}\left[\frac{1}{2}\left(b^2+a^2\right) - \frac{2a^2b^2}{(b^2-a^2)}\ln\frac{b}{a}\right]$$

**PROBLEM B10**

Both the mammalian respiration system and the mammalian blood circulation system are networks of tubes in which the flow from one large tube (respectively the trachea and the aorta) branches into parallel flows in tubes of smaller size. This branching continues through a number of stages:




If, for each stage, the number of tubes is denoted by $n$ and the cross-sectional area for each and every tube in that stage is denoted by $A_n$, find the relation between $A_n$ and $n$ such that the pressure gradient, $dp/dx$, is the same for each stage. How does the average velocity depend on $n$? Assume steady, fully-developed Poiseuille flow in all tubes even though this may not be the case in the actual systems.

If the diameter of the aorta is 3 $cm$ and the diameter of the microcirculation (the smallest tubes) is $8 \times 10^{-6}$ $m$, calculate the number of tubes at the microcirculation stage which would be present if the above property were to exist. The actual number is much smaller than this. Where, then, does most of the pressure drop occur in the blood circulation system?

SOLUTION B10

Consider the volume flow rate for a stage with $n$ tubes:

$$Q = nA_n\overline{u}_n$$

From the solution for laminar Poiseuille flow it follows that:

$$\overline{u} = \frac{R^2}{8\mu}\left(\frac{dp}{dx}\right)$$

and therefore,

$$A_1\frac{R_1^2}{8\mu}\left(\frac{dp}{dx}\right) = nA_n\frac{R_n^2}{8\mu}\left(\frac{dp}{dx}\right)$$

Since,

$$A_n = \pi R_n^2$$

it follows that

$$A_1^2 = nA_n^2$$

and therefore the desired relation between $A_n$ and $n$ is

$$A_n = \frac{A_1}{\sqrt{n}}$$

From the continuity relation

$$A_1\overline{u}_1 = nA_n\overline{u}_n$$

and therefore the desired relation between the velocity and $n$ is

$$\overline{u}_n = \frac{\overline{u}_1}{\sqrt{n}}$$

Using the numerical values given

$$\pi(0.015)^2 = \frac{\pi(4\times 10^{-6})^2}{\sqrt{n}}$$

and hence

$$n = 1.98\times 10^{14}$$

The actual number is much smaller than this, which implies that the velocity (and therefore, the pressure drop) is greater in the microcirculation stages.

**PROBLEM B11**

A semi-infinite domain of fluid is bounded only by a single infinite flat plate. The fluid is incompressible with a constant and uniform viscosity, $\mu$, and density, $\rho$. The plate is then set in accelerating motion, moving in its own plane with an accelerating velocity, $Ue^{kt}$, where $U$ and $k$ are constants and $t$ is time. If the fluid only reacts by moving parallel with the plate with a velocity, $u(y,t)$, where $y$ is the distance from the plate and if the velocities in the other directions are zero, write down the simplified form of the Navier-Stokes equation that govern this flow and must be solved to find $u(y,t)$. Note that $p$ is uniform; that the velocity far from the plate is zero; and neglect gravitational effects. The result is a partial differential equation for $u(y,t)$ that only includes $u$, $y$, $t$ and $\mu/\rho$.

Using separation of variables (or otherwise) solve this equation to find $u(y,t)$ and the vorticity, $\omega(y,t)$, in terms of $y$, $t$, $U$, $k$, and the fluid properties. If we define a boundary layer next to the plate as the region within which the velocity is at least 10% of the plate velocity, derive an expression for the thickness of the boundary layer as a function of time.

SOLUTION B11

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Since the flow is planar and incompressible this simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since the velocity, $v$, normal to the plate is zero everywhere in the flow it follows from continuity that

$$\frac{\partial u}{\partial x} = 0$$

so $u$ is only a function of $y$ and $t$, $u = u(y,t)$.

Navier-Stokes:

**x-direction:**

$$\rho \left[ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Since the flow is planar, since $v = 0$ and $u = u(y,t)$, and since the pressure is constant, this becomes:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$

We use separation of variables to solve this partial differential equation. Assume

$$u(y,t) = Y(y)T(t)$$

Substituting this into the partial differential equation and rearranging, the result can be written as a term which is a function only of $y$ equal to a term which is a function only of $t$. It follows that both must be equal to a simple constant, $\lambda$:

$$\frac{1}{T}\frac{dT}{dt} = \frac{\mu}{\rho}\frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda$$

The equation for $t$ is then:

$$\frac{dT}{dt} = \lambda T$$

and the solution to this is:

$$T(t) = c_1 e^{\lambda t}$$

The equation for $y$ is:

$$\frac{d^2 Y}{dy^2} - \frac{\rho\lambda}{\mu} Y = 0$$

and the solution to this is:

$$Y(y) = c_2 e^{\sqrt{\rho\lambda/\mu}\; y} + c_3 e^{-\sqrt{\rho\lambda/\mu}\; y}$$

The boundary condition at the plate gives

$$u(0,t) = U(t) = Ue^{kt}$$

and the condition as $y \to \infty$ gives

$$u(y \to \infty, t) = 0$$

The second condition yields $c_2 = 0$. It follows that the solution for $u(y,t)$ is:

$$u(y,t) = c_4 e^{-\sqrt{\rho\lambda/\mu}\ y}\ e^{\lambda t}$$

where $c_4 = c_1 c_3$. Applying the no-slip boundary condition at the surface of the plate:

$$u(0,t) = c_4 e^{\lambda t} = U e^{kt}$$

so the values of the unknown constant $c_4 = U$ and $\lambda = k$ are now determined. This yields a velocity profile:

$$u(y,t) = U e^{kt} e^{-\sqrt{k/\nu}\ y}$$

where $\nu$ is the kinematic viscosity $\nu = \mu/\rho$. The vorticity, $\vec{\omega}(y,t)$, is given by (note that $\vec{u} = u(y,t)\hat{\mathrm{i}}$)

$$\vec{\omega}(y,t) = \nabla \times \vec{u} = -\frac{\partial u}{\partial y}\hat{\mathrm{k}}$$

$$\vec{\omega}(y,t) = U\sqrt{\frac{k}{\nu}}\ e^{kt} e^{-\sqrt{k/\nu}\ y}\hat{\mathrm{k}}$$

The boundary layer thickness, $\delta$, is defined as that distance from the plate where the velocity is 10% of the plate velocity:

$$\begin{aligned} 0.1\ U e^{kt} &= U e^{kt} e^{-\sqrt{k/\nu}\ \delta} \\ 0.1 &= e^{-\sqrt{k/\nu}\ \delta} \\ \delta &= \ln(10)\sqrt{\frac{\nu}{k}} \end{aligned}$$