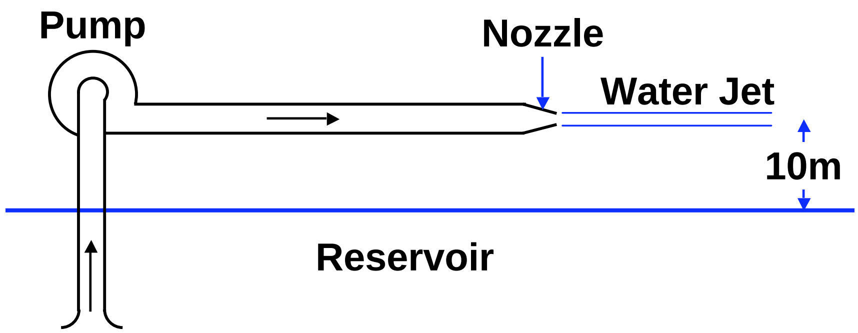
PROBLEM B5

A fire nozzle is to be used at an elevation of 10 m above the level of a reservoir. The velocity of the jet is to be 20 m/s and the flow is provided by a pump: The density of the water may be taken to be 1000 kg/m^3, the



acceleration due to gravity may be taken to be 9.8 m/s^2 and the cross-sectional area of the pipes is 70 cm^2.

The loss coefficient between the reservoir and the inlet to the pump is 4 and the loss coefficient between the discharge from the pump and the end of the nozzle is 3. The ratio of the cross-sectional area of the jet to that of the pipes is 0.1. The inlet and discharge pipes leading to and from the pump have the same cross-sectional area.

- Find the head rise (in m) that the pump must provide.
- If the pump is 75% efficient find the power required to drive the pump in $kg \cdot m^2/s^3$.
- If the pump, pipes and nozzle are mounted on a vehicle find the horizontal force in $kg \cdot m/s^2$ required to hold the vehicle in place.

SOLUTION B5

A modified Bernoulli equation applied between a point on the external free surface outside the submerged pipe (point 1) and the jet after it emerges from the nozzle (point 2) yields

$$\frac{p_A}{\rho g} + H - k_1 \frac{u_1^2}{2g} - k_2 \frac{u_1^2}{2g} = \frac{p_A}{\rho g} + \frac{u_2^2}{2g} + y_2$$

where the elevation of the external free surface is taken to be $y = 0$. In the above p_A is atmospheric pressure, the pressure (and total pressure) at the point 1 is p_A and it is assumed that the pressure in the emerging jet is equal to p_A. Also u_1 is the fluid velocity in the pipes leading to and from the pump, u_2 is the fluid velocity in the jet, y_2 is the elevation of the emerging jet, ρ is the density of the liquid, H is the total head rise across the pump, and k_1 and k_2 are the loss coefficients for the pipes leading to and from the pump.

It also follows from continuity that

$$A_1 u_1 = A_2 u_2$$

where A_1 and A_2 are the cross-sectional areas of the pipes and the jet respectively. Since $u_2 = 20\ m/s$ it follows that $u_1 = 2\ m/s$ and that

$$H = \frac{u_2^2}{2g}\left[1 + \left(\frac{A_2^2}{A_1^2}\right)(k_1 + k_2))\right] + y_2$$

Substituting $u_2 = 20\ m/s$, $y_2 = 10\ m$, $A_2/A_1 = 0.1$, $g = 9.8\ m/s^2$, $k_1 + k_2 = 7$ it follows that

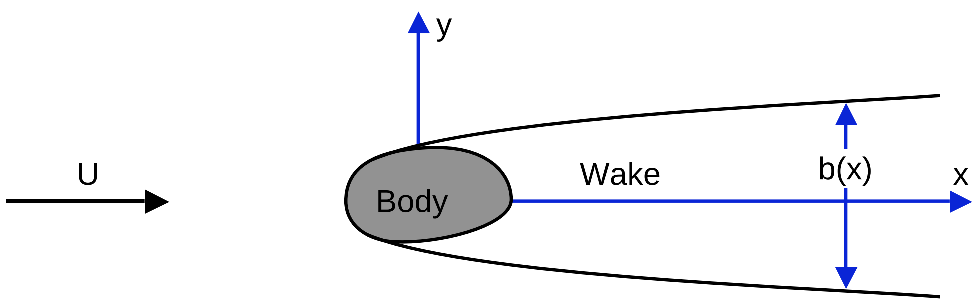
$$H = \frac{400}{19.6}\left[1 + (0.01)\,(7))\right] + 10 = 31.8\ m$$

and this is the total head the pump must provide. Moreover since the volume flow rate is $Q = A_1 u_1 = 0.007 \times 2 = 0.014\ m^3/s$ it follows that the power that must be supplied to the fluid in the pump is $Q\rho g H = 0.014 \times 1000 \times 9.8 \times 31.8 = 4363\ kg \cdot m^2/s^3$. Therefore the power that must be supplied to the pump is $4363/0.75 = 5817\ kg \cdot m^2/s^3$.

If we take a control volume surrounding the pump, the discharge pipe and the nozzle but cutting through the emerging jet and the pipe leading to the pump, the horizontal momentum flux emerging from this control volume (in the direction of the jet) is $\rho A_2 u_2^2$ which equals 280 $kg \cdot m/s^2$ or N. By the momentum thereom, since the pressure on all horizontal sides of the control volume is atmospheric and therefore the net pressure force in the horizontal direction is zero, the force that must be applied to the vehicle to hold it in place is also 280 N in the direction of the jet.

PROBLEM B6

Wake surveys are made in the two-dimensional wake behind a body (cylindrical) which is externally supported in a uniform stream of incompressible fluid approaching the cylinder with velocity, U:



The surveys are made at x locations sufficiently far downstream of the body so that the pressure across the wake is the same as the ambient pressure in the fluid far from the body. They indicate that, to a first approximation, the velocity defect in the wake (the amount by which the velocity, u, is less than U) varies with lateral position, y, according to

$$u = U - A(x)\cos\left[\pi y/b(x)\right] \quad \text{for} \quad -\frac{b}{2} < y < +\frac{b}{2}$$

where $A(x)$ and $b(x)$ are the centerline velocity defect and wake width respectively both of which vary with position, x. If the drag on the body per unit distance normal to the plane of the sketch is denoted by D and the density of the fluid by ρ find the relation for $b(x)$ in terms of $A(x)$, U, ρ and D.

SOLUTION B6

We choose to utilize the control volume $efgh$ shown below and to evaluate fluxes per unit depth normal to the sketch. Then evaluating the mass flow rates through each of the boundaries of this control volume (CV), the equation of continuity requires that

$$\int_{\mathrm{e}}^{\mathrm{f}} \rho U dy = \int_{\mathrm{h}}^{\mathrm{g}} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx$$

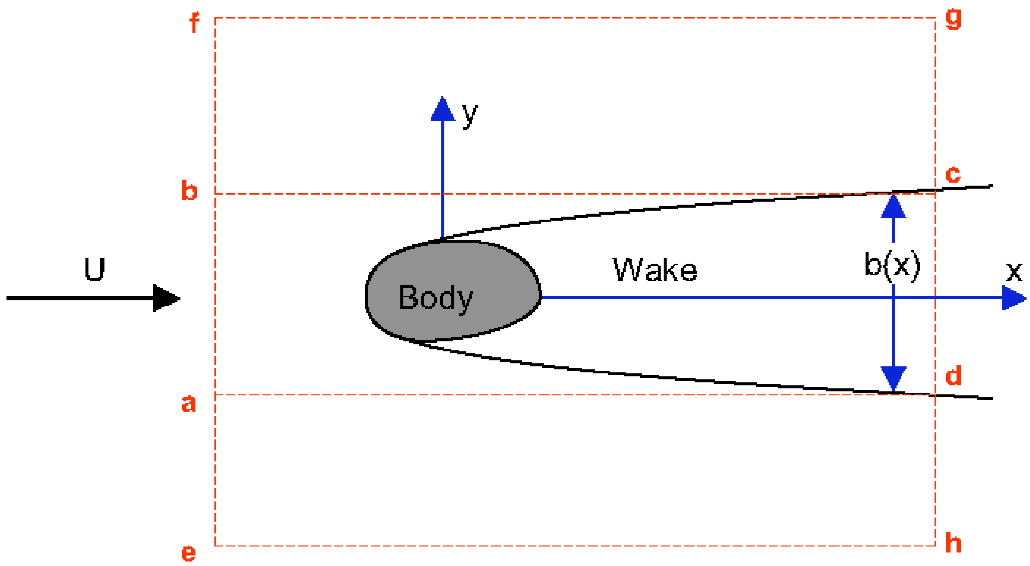


Figure 1: Control volume *abcd* shown by red dashed line.

where ρ is the fluid density, u and v are the velocity components in the x and y directions. The boundaries ef, fg and eh are assumed to be very far away and the boundary hg sufficently far away so that the pressure on all these boundaries is atmospheric pressure and the u velocity on ef, fg and eh is U, the free stream velocity far upstream. It is also assumed that the u velocity on the boundary hg *outside of the wake* in regions cg and hd is equal to U. It follows that

$$\int_{\mathrm{e}}^{\mathrm{a}} \rho U dy = \int_{\mathrm{h}}^{\mathrm{d}} \rho U dy$$

and

$$\int_{\mathrm{b}}^{\mathrm{f}} \rho U dy = \int_{\mathrm{c}}^{\mathrm{g}} \rho U dy$$

and consequently these parts of the integrals in the first equation cancel so that the first equation becomes

$$\int_{\mathrm{a}}^{\mathrm{b}} \rho U dy = \int_{\mathrm{d}}^{\mathrm{c}} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx.$$

Then, assuming the flow is symmetric about the x axis:

$$\int_{-b(x)/2}^{b(x)/2} \rho U dy = \int_{-b(x)/2}^{b(x)/2} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx.$$

Rearranging

$$\int_{\mathrm{f}}^{\mathrm{g}} v dx - \int_{\mathrm{e}}^{\mathrm{h}} v dx = \int_{-b(x)/2}^{b(x)/2} (U - u) dy = \int_{-b(x)/2}^{b(x)/2} A(x) \cos\left[\frac{\pi y}{b(x)}\right] dy$$

$$= A(x)\frac{b(x)}{\pi} \sin\left[\frac{\pi y}{b(x)}\right]\Bigg|_{y=-b(x)/2}^{y=b(x)/2}$$

$$= A(x)\frac{b(x)}{\pi}\left(\sin\left[\frac{\pi}{2}\right] - \sin\left[-\frac{\pi}{2}\right]\right)$$

and therefore

$$\int_{\mathrm{f}}^{\mathrm{g}} v dx - \int_{\mathrm{e}}^{\mathrm{h}} v dx = 2A(x)\frac{b(x)}{\pi}$$

We now apply the momentum thereom in the x direction to obtain the force F_x on the control volume $efgh$ per unit depth normal to the sketch as:

$$F_x = \int_{\mathrm{e}}^{\mathrm{f}} \rho(U)(-U)dy + \int_{\mathrm{h}}^{\mathrm{d}} \rho(U)(U)dy + \int_{\mathrm{d}}^{\mathrm{c}} \rho(u)(u)dy + \int_{\mathrm{c}}^{\mathrm{g}} \rho(U)(U)dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho(U)(v)dx + \int_{\mathrm{e}}^{\mathrm{h}} \rho(U)(-v)dx$$

As with continuity equation the integrals over bf and cg cancel as do the integrals over ea and hd and this cancellation leads to

$$F_x = -\rho U^2 b(x) + \rho \int_{-b(x)/2}^{b(x)/2} \left\{ U - A(x)\cos\left[\frac{\pi y}{b(x)}\right]\right\}^2 dy + \rho U \left[\int_{\mathrm{f}}^{\mathrm{g}} v dx - \int_{\mathrm{e}}^{\mathrm{h}} v dx\right]$$

and substituting for the terms in the square brackets from the result obtained from the continuity equation we obtain:

$$F_x = -\rho U^2 b(x) + \rho \int_{-b(x)/2}^{b(x)/2} \left\{ U^2 - 2UA(x)\cos\left[\frac{\pi y}{b(x)}\right] + A(x)^2\cos^2\left[\frac{\pi y}{b(x)}\right]\right\} dy + \rho U \left[2A(x)\frac{b(x)}{\pi}\right]$$

$$= -\rho U^2 b(x) + \rho U^2 b(x) - 2\rho U A(x)\frac{b(x)}{\pi} \sin\left[\frac{\pi y}{b(x)}\right]\Bigg|_{y=-b(x)/2}^{y=b(x)/2} + \rho A(x)^2 \left\{ \frac{y}{2} + \frac{\sin\left[\frac{2\pi y}{b(x)}\right]}{4\frac{\pi}{b(x)}} \right\}_{y=-b(x)/2}^{y=b(x)/2} + 2\rho U A(x)\frac{b(x)}{\pi}$$

$$= -2\rho U A(x)\frac{b(x)}{\pi} + \frac{1}{2}\rho A(x)^2 b(x)$$

$$F_x = \rho A(x) b(x) \left[\frac{A(x)}{2} - 2\frac{U}{\pi}\right]$$

Because the pressure on all sides of the CV $efgh$ is atmospheric pressure it follows that the contribution of the pressures on the external boundaries to F_x is zero. As a result, the only force on the control volume is the force imposed to keep the body in place. This force is equal and opposite to the drag force on the body, D, or $D = -F_x$ and therefore

$$b(x) = \frac{2\pi D}{\rho A(x)\left[4U - \pi A(x)\right]}$$

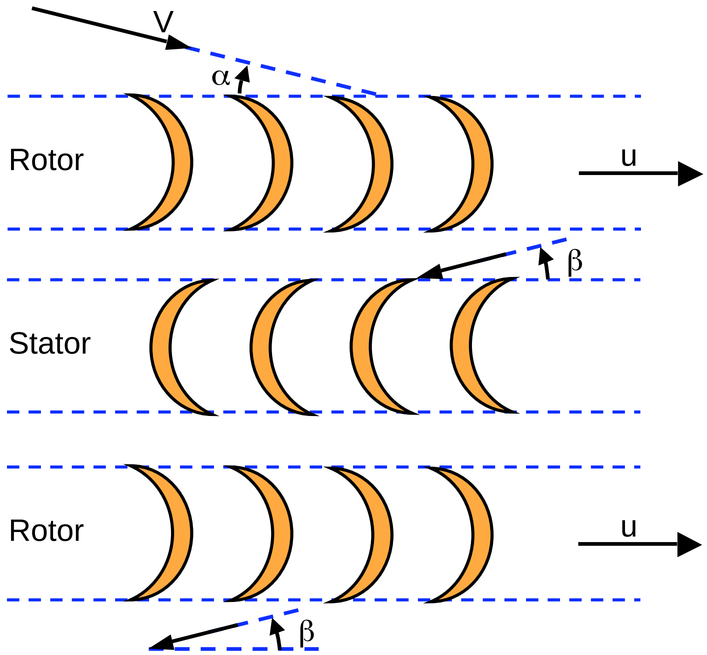
PROBLEM B7

A two-stage turbine (of a type known as an "impulse turbine") consists of a rotor followed by a stator followed by a second rotor:

Visualize that this view is looking down on the ends of the blade rows which are mounted on wheels that have a common axis running longitudinally up and down the page.

It will be assumed that all the angles α and β are sufficiently small so that $\cos\alpha$ and $\cos\beta$ can be approximated by unity. It is also assumed that frictional effects in both the rotors and the stator can be included using the same constant, C, for all three rows of blades where C is defined as follows: relative velocity leaving blades $= -C \times$ the relative velocity entering the blades. Also assume that the pressure is the same before, after and between the blade rows.

The "blade efficiency", η_{rotor}, of the turbine is defined as the ratio of the power transmitted to the rotor (force on the rotor multiplied by the velocity of the rotor) to the available energy in the impinging stream (one half of the mass flow rate of the impinging jet times the square of the velocity of the impinging jet). Evaluate the blade efficiency of the two-stage impulse turbine depicted above as a function of u/V where u is the blade velocity of



both rotors and V is the velocity of the incoming jet. At what value of u/V will this 2-stage turbine have its maximum blade efficiency if $C = 0.9$?

SOLUTION B7

This problem requires analysis of the two-stage turbine consisting of a rotor followed by a stator followed by a second rotor. It is to be assumed that all the angles α and β are sufficiently small so that $\cos\alpha$ and $\cos\beta$ can be approximated by unity. It is also assumed that frictional effects in both the rotors and the stator can be included using the same constant, C, for each of the three rows of blades where C is defined as follows: relative velocity leaving blades $= -C \times$ relative velocity entering blades. The mass flow rate through all stages in a direction perpendicular to u is $\dot{M}$.

First Rotor: In order to use the steady flow version of the momentum theorem we must utilize a control volume around the first rotor which is moving with the first rotor at the velocity u. The *relative velocities* in the u direction are given by (recall $V\cos\alpha \approx V$):

$$u_{R1,in} = V - u \qquad \text{and} \qquad u_{R1,out} = -Cu_{R1,in} = -C(V-u)$$

The momentum theorem tells us that the forces on the fluid in the u direction are equal to

$$F_{u1} = \dot{M}(u_{R1,out} - u_{R1,in}) = -\dot{M}(V-u)(C+1)$$

Consequently the force on the first rotor in the u direction is equal and opposite, $F_{R1} = -F_{u1}$, so

$$F_{R1} = \dot{M}(V-u)(C+1)$$

Note, the "blade efficiency", η_{R1}, for a single rotor impulse turbine, defined as the ratio of the power transmitted to the rotor, $F_{R1}u$, to the available energy in the incoming flow, $\dot{M}V^2/2$, would be

$$\eta_{R1} = \frac{2F_{R1}u}{\dot{M}V^2} = 2(1+C)\frac{u}{V}\left\{1 - \frac{u}{V}\right\}$$

For a value of C equal to 0.9 this becomes

$$\eta_{R1} = \frac{u}{V}\left\{3.8 - 3.8\frac{u}{V}\right\}$$

and we will use this later.

Stator: It follows that the velocities *relative* to the stator at the inlet and outlet in the u direction are approximately:

$$u_{S,in} = u_{R1,out} + u = u(C+1) - CV \qquad \text{and} \qquad u_{S,out} = -Cu_{S,in} = -uC(C+1) + C^2V$$

Second Rotor: In order to use the steady flow version of the momentum theorem we must utilize a control volume around the second rotor which is moving with the second rotor at the velocity u. Then the velocities *relative* to that control volume in the u direction are approximately:

$$u_{R2,in} = u_{S,out} - u = -u(C^2 + C + 1) + C^2V \qquad \text{and} \qquad u_{R2,out} = -Cu_{R2,in} = uC(C^2 + C + 1) - C^3V$$

The force in the u direction is therefore

$$F_{u2} = \dot{M}(u_{R2,out} - u_{R2,in}) = -\dot{M}[-u(C^3 + 2C^2 + 2C + 1) + VC^2(C+1)]$$

and this must be equal and opposite to the force on the second rotor in the u direction

$$F_{R2} = \dot{M}[-u(C^3 + 2C^2 + 2C + 1) + VC^2(C+1)]$$

Adding F_{R1} and F_{R2} the total force, F_R, on the rotors in the u direction is therefore

$$\begin{aligned} F_R &= \dot{M}[(V-u)(C+1) - u(C^3 + 2C^2 + 2C + 1) + VC^2(C+1)] \\ &= \dot{M}[V(C^3 + C^2 + C + 1) - u(C^3 + 2C^2 + 3C + 2)] \end{aligned}$$

Consequently the power transmitted to the rotor is $F_R u$. The "blade efficiency", η_{rotor}, is defined as the ratio of this power to the available energy in the incoming flow prior to the first stage namely $\dot{M}V^2/2$ so that the blade efficiency in this case is given by

$$\eta_{rotor} = \frac{2F_R u}{\dot{M}V^2} = 2\frac{u}{V}\left[(C^3 + C^2 + C + 1) - \frac{u}{V}(C^3 + 2C^2 + 3C + 2)\right]$$

For a value of $C = 0.9$ this becomes

$$\eta_{rotor} = \frac{u}{V}\left(6.878 - 14.098\frac{u}{V}\right)$$

This is a maximum when $u/V = 6.878/(2 \times 14.098) = 0.244$.

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Extra Footnotes:

Consider the results if we add a third stage, namely another stator and rotor. To evaluate this third stage we first need the velocities relative to a new, second stator which become:

$$u_{S2,in} = -C^3V + u(1 + C + C^2 + C^3) \qquad \text{and} \qquad u_{S2,out} = C^4V - uC(1 + C + C^2 + C^3)$$

Then the velocities relative to the new, third rotor become:

$$u_{R3,in} = C^4 V - u(1 + C + C^2 + C^3 + C^4) \qquad \text{and} \qquad u_{R3,out} = -C^5 V + uC(1 + C + C^2 + C^3 + C^4)$$

so that the net momentum flux in the u direction exiting the third rotor is

$$\dot{M}(u_{R3,out} - u_{R3,in}) = -\dot{M}[VC^2(1 + C) - u(1 + 2C + 2C^2 + C^3)]$$

and this must be equal to the force on the fluid in the u direction within the third rotor. Consequently the force on the third rotor in the u direction, F_{R3}, is

$$F_{R3} = \dot{M}[VC^4(1 + C) - u(1 + 2C + 2C^2 + 2C^3 + 2C^4 + C^5)]$$

and the total force on a three stage rotor becomes

$$F_R = \dot{M}[V(1 + C + C^2 + C^3 + C^4 + C^5) - u(3 + 5C + 4C^2 + 3C^3 + 2C^4 + C^5)]$$

and the blade efficiency for a three stage impulse turbine, η_{3rotor}, becomes

$$\eta_{3rotor} = 2\frac{u}{V}\left[(1 + C + C^2 + C^3 + C^4 + C^5) - (3 + 5C + 4C^2 + 3C^3 + 2C^4 + C^5)\frac{u}{V}\right]$$

which, for a value of $C = 0.9$, becomes

$$\eta_{3rotor} = \frac{u}{V}\left(9.37 - 29.66\frac{u}{V}\right)$$

Comparing the one, two and three rotor turbine blade efficiencies, for example for $u/V = 0.1$ we find $\eta_{R1} = 0.342$, $\eta_{rotor} = 0.547$, and $\eta_{3rotor} = 0.640$ and therefore the blade efficiencies increase as more rotors extract more energy from the flow.

A more appropriate comparison would be to examine the maximum blade efficiencies for each of these turbines. For this purpose we differentiate the η expressions with respect to u/V and then set those expressions to zero to find the values of u/V at which the blade efficiencies are a maxiumum. Then we evaluate the blade efficiencies at those values of u/V. In the case of $C = 0.9$ this leads to the following results:

	$(u/V)_{max}$	η_{max}
One Rotor	0.500	0.95
Two Rotors	0.244	0.839
Three Rotors	0.158	0.740

Consequently the lighter the load on the turbine (the larger the value of u/V) the fewer the number of stages needed and the higher the blade efficiency. On the other hand, for larger loads and lower u/V the greater the number of stages needed to extract the energy from the inlet flow.