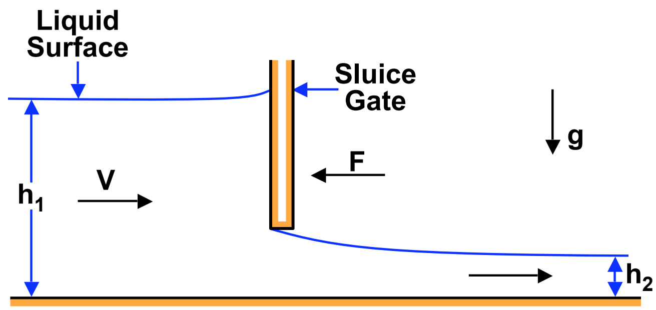
**ME19b.** **SOLUTIONS.** **Jan. 5, 2010. Due Jan. 14**

**PROBLEM B1**

An incompressible, inviscid liquid flow (density, $\rho$) of depth, $h_1$, and velocity, $V$, flows under the action of gravity through a sluice gate:




The depth downstream of the sluice gate is denoted by $h_2$. Determine the force per unit width (normal to sketch), $F$, necessary to hold the plate in place in terms of $\rho$, $g$, $h_1$ and $h_2$.

SOLUTION B1

The continuity equation yields

$$\rho V h_1 = \rho V_2 h_2.$$

or

$$V_2 = V\frac{h_1}{h_2}.$$

The $x$-momentum equation yields

$$\sum F_x = V\rho(-V)h_1 b + V_2\rho(V_2)h_2 b$$

where $b$ is the width of the gate in the direction normal to the sketch and $\sum F_x$ is the total force in the $x$-direction on the gate.

Two forces act on the control volume: (1) the net force due to the pressures acting on the ends of the control volume, $F_p$, and (2) the reaction force necessary to hold the gate in place, $F$. Thus, the $x$-momentum equation becomes

$$F_p - F = \rho V_2^2 h_2 b - \rho V^2 h_1 b$$

where

$$F_p = F_{p1} - F_{p2} = \rho g b \int_0^{h_1} y dy - \rho g b \int_0^{h_2} y dy = \frac{1}{2}\rho g b\left(h_1^2 - h_2^2\right)$$

Substituting

$$\frac{F}{b} = \frac{1}{2}\rho g\left(h_1^2 - h_2^2\right) + \rho V^2 h_1 - \rho V_2^2 h_2$$

where $F/b$ is the reaction force on the gate in the $x$-direction per unit width of the gate.

To eliminate the velocities from this relation, we use Bernoulli's equation

$$\frac{1}{2}\rho V^2 + p_A + \rho g h_1 = \frac{1}{2}\rho V_2^2 + p_A + \rho g h_2,$$

and using the continuity equation

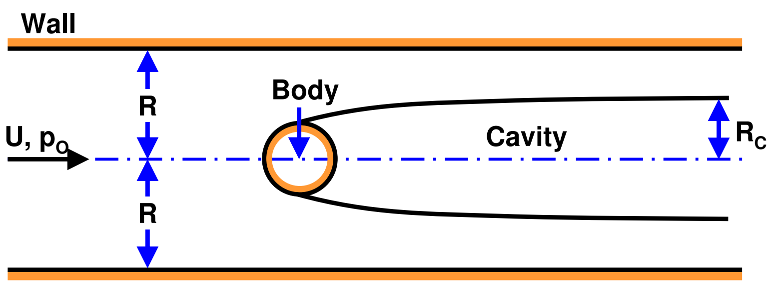
$$V^2 = \frac{2gh_2^2}{h_1 + h_2},$$

Substituting into the expression for $F/b$ leads to

$$\frac{F}{b} = \frac{1}{2}\rho g \frac{(h_1 - h_2)^3}{h_1 + h_2}$$

## PROBLEM B2

An axisymmetric body (a sphere if you wish) is mounted in a water tunnel which has a circular cross-section of radius, $R$. The velocity far upstream, $U$, is fixed. When the pressure far upstream, $p_o$, is lowered sufficiently a large vapor-filled wake or cavity forms behind the body:




The pressure in the cavity is simply given by the vapor pressure, $p_v$, of the water at the operating water temperature ($p_o > p_v$). It is to be assumed that the effect of friction, the effect of gravity, the density of the vapor and the amount of water vaporized at the free surface are all negligible. A parameter called the cavitation number, $\sigma$, is defined as

$$\sigma = \frac{p_o - p_v}{\frac{1}{2}\rho U^2}$$

where $\rho$ is the water density.

(a) Find the relation between $R_c/R$ and $\sigma$ for very long cavities whose asymptotic radius is $R_c$.

(b) Find the drag on the body in terms of $U$, $R$, $\sigma$ and $\rho$.

SOLUTION B2

(a) Applying Bernoulli's equation between a point upstream (point 0) and a point on the surface of the cavity at its maximum radius of $R = R_c$ (point 1) (neglecting gravity)

$$\frac{1}{2}\rho U^2 + p_o = \frac{1}{2}\rho u_1^2 + p_1$$

where $p_1 = p_v$ in order to produce a long, cylindrical cavity. Rearranging

$$p_o - p_v = \frac{1}{2}\rho U^2 \left[\left(\frac{u_1}{U}\right)^2 - 1\right]$$

Substituting into the definition of the cavitation number, $\sigma$:

$$\frac{p_o - p_v}{\frac{1}{2}\rho U^2} = \sigma = \left(\frac{u_1}{U}\right)^2 - 1$$

and therefore

$$u_1 = U\sqrt{1+\sigma}$$

Conservation of mass requires that

$$UA_0 = u_1 A_1 \quad \text{or} \quad U\pi R^2 = u_1 \pi \left(R^2 - R_c^2\right)$$

where it is assumed that a negligible amount of mass has been vaporized. Substituting this expression for $u_1$ leads to

$$U\pi R^2 = \left(U\sqrt{1+\sigma}\right)\pi\left(R^2 - R_c^2\right)$$

or

$$\sqrt{1+\sigma} = \frac{R^2}{R^2 - R_c^2}$$

and solving for $\sigma$ yields

$$\sigma = \left(\frac{R^2}{R^2 - R_c^2}\right)^2 - 1 = \left[\frac{1}{1 - \left(\frac{R_c}{R}\right)^2}\right]^2 - 1$$

or inversely

$$\frac{R_c}{R} = \sqrt{1 - \frac{1}{\sqrt{1+\sigma}}}$$

(b) Using a control volume surrounding all the liquid, the body and the cavity, the momentum theorem in the x-direction yields

$$F_x = \rho u_1 A_1 (u_1) + \rho U A_o(-U) = \rho\pi R^2 U^2 \left(\sqrt{\sigma+1} - 1\right)$$

where $F_x$ is the total force on the control volume in the $x$-direction. Two forces contribute: (1) the net pressure force, $F_p$, on the control volume and (2) the external force, $-F_d$, applied to the body to keep it stationary within the control volume (imposed through an imaginary strut not shown in the sketch). This second contribution is the force imposed on the fluid by the body, which is equal and opposite to the drag, $F_d$. Thus

$$F_x = F_p - F_d$$

Now the pressure force, $F_p$, is given by

$$F_p = \underbrace{p_o \pi R^2}_{\substack{\text{pressure force acting}\\ \text{on left side of CV}}} - \underbrace{p_v \pi R^2}_{\substack{\text{pressure force acting}\\ \text{on right side of CV}}} = \left(p_o - p_v\right)\pi R^2$$

where it is understood that the pressure downstream, $p_v$ is constant across the entire cross section when the cavity has reached an asymptotic radius, $R_c$ (otherwise the shape of the cavity would still be changing). Using this it follows that

$$F_d = \left(p_o - p_v\right)\pi R^2 - \rho\pi R^2 U^2 \left(\sqrt{\sigma+1} - 1\right)$$

Substituting for $p_o - p_v$ yields

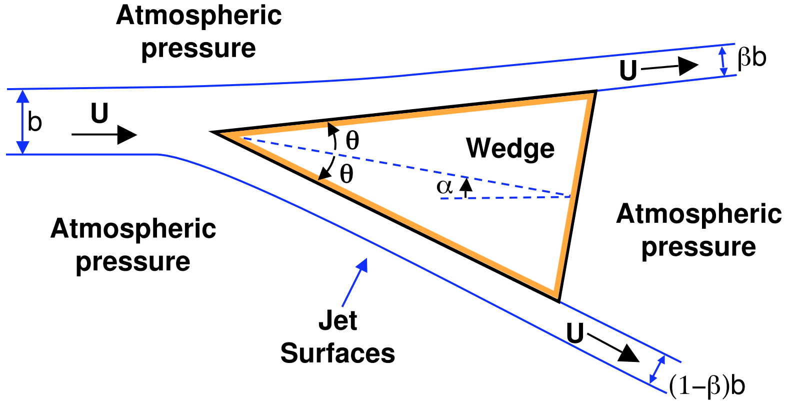
$$F_d = \frac{1}{2}\rho U^2 \sigma \pi R^2 - \rho\pi R^2 U^2 \left(\sqrt{\sigma+1} - 1\right)$$

which upon simplification yields

$$F_d = \pi\rho U^2 R^2 \left(\frac{\sigma}{2} - \sqrt{\sigma+1} + 1\right)$$

**PROBLEM B3**

A wedge with a vertex angle of $2\theta$ is inserted into a jet of water (density, $\rho$) of width, $b$, and velocity, $U$, as shown in the sketch below.




The angle of attack, $\alpha$, of the wedge is also defined in the sketch. The result is that the single incident jet is divided into two jets both of which leave the back edges of the wedge with the velocity, $U$. The widths of the two departing jets are $\beta b$ and $(1-\beta)b$ as indicated in the figure. It is assumed that the flow is planar and that the pressure in surrounding air is everywhere atmospheric.

1. Find the lift and drag on the wedge per unit length normal to the sketch as functions of $\rho$, $U$, $b$, $\beta$, $\alpha$ and $\theta$. Note that drag and lift are defined as the forces on the wedge which are respectively parallel to and perpendicular to the direction of the incident jet.
2. If the angle of attack, $\alpha$, is varied while $\rho$, $U$, $b$, $\beta$ and $\theta$ remain fixed, find the angle of attack at which the lift is zero.
3. If, on the other hand, the wedge is moved in a direction perpendicular to the incident jet while $\rho$, $b$, $U$, $\theta$ and $\alpha$ remain fixed then $\beta$ will change. There is one such position at which the lift is zero; what is the value of $\beta$ at this position in terms of $\theta$ and $\alpha$ ? If the wedge were free to move in such a way would this position represent a position of stable or unstable equilibrium?

SOLUTION B3

1. We define a control volume that includes the wedge and denote the drag and lift forces on the wedge parallel and normal to the $U$ direction by $D$ and $L$ as shown in the sketch:

Applying the momentum theorem in the $x$ or $U$ direction yields

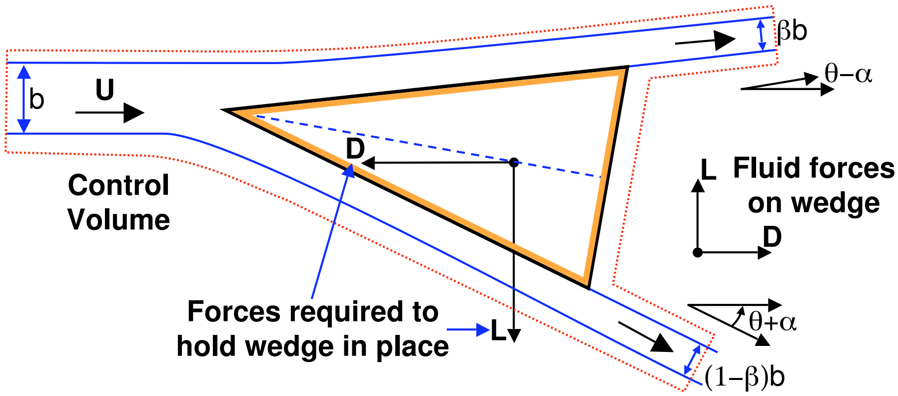
$$\begin{aligned} F_x = -D &= \rho U^2 \beta b \cos(\theta - \alpha) + \rho U^2 (1-\beta) b \cos(\theta + \alpha) - \rho b U^2 \\ D &= \rho U^2 b \left[1 - \beta \cos(\theta - \alpha) - (1-\beta)\cos(\theta + \alpha)\right] \end{aligned}$$

Similarly, using the momentum theorem in the normal direction yields

$$L = \rho U^2 b \left[(1-\beta)\sin(\theta + \alpha) - \beta \sin(\theta - \alpha)\right]$$

2. Therefore the angle of attack for which $L$ is zero is

$$\alpha = \tan^{-1}\left[(2\beta - 1)\tan(\theta)\right]$$




3. Also the $\beta$ for zero lift is

$$\beta = \frac{1}{2}\left[1 + \tan(\alpha)\cot(\theta)\right]$$

Finally to determine whether this position is stable with respect to $\beta$, we require that, for stability, the lift must increase if the wedge is shifted downward. This requires that the lift increase as $\beta$ is increased. But

$$\begin{aligned}\left.\frac{\partial L}{\partial \beta}\right|_{\text{at zero lift}} &= \rho b U^2\left[-\sin(\theta+\alpha) - \sin(\theta-\alpha)\right] \\ &= \text{a } \textbf{negative} \text{ quantity for } (\theta+\alpha) < \pi \text{ and } \theta > \alpha > 0\end{aligned}$$

Thus we have a **unstable equilibrium**. If $\beta$ is increased (body is moved down) the lift becomes negative and further pushes the body down.

**PROBLEM B4**

A turbojet engine in a wind tunnel ingests air at a velocity of 100 $m/s$ and a density of 1 $kg/m^3$. The velocity is uniform and the cross-sectional area of the approaching stream which enters the engine is 0.1 $m^2$. The velocity of the exhaust jet from the engine, however, is not uniform but has a velocity which varies over the cross-section according to

$$u(r) = 2U\left(1 - \frac{r^2}{r_0^2}\right)$$

where the constant $U = 600$ $m/s$ and $r_0$ is the radius of the jet cross-section. The radial position within the axisymmetric jet is denoted by $r$ and the density of the exhaust jet is assumed to be uniform. It is readily shown that the above formula implies that $U$ is the average velocity of the exhaust jet (the volume flow divided by the cross-sectional area of the jet).

(a) Find the thrust of the turbojet engine.

(b) Find what the thrust would be if the exhaust jet had a uniform velocity, $U$.

Assume the pressures in both the inlet and exhaust jets are the same as the surrounding air and that mass is conserved in the flow through the engine (roughly true in practice).

SOLUTION B4

Though you were not asked to do this, I show first that $U$ is indeed the average velocity, $\bar{u}$, of the emerging jet:

$$\bar{u} = \frac{1}{\pi r_0^2}\int_0^{r_0} u(r) 2\pi r dr = \frac{4U}{r_0^2}\left[\frac{r^2}{2} - \frac{r^4}{4r_0^2}\right]_0^{r_0} = U$$

(a) Now consider a control volume beginning with a cut far upstream across the streamtube entering the engine, sides along that streamtube around the outside of the engine and then a cut downstream across the exhaust jet. Since the pressure on all these surfaces is stated to be atmospheric, there is no net force on the control volume due to the external pressures acting on it. Thus the only force is that acting on the control volume transmitted through the structure connecting the engine to the airplane. This force is equal to the thrust, $T$, and is a force acting on the control volume in the direction of the flow.

Before applying the momentum thereom to the problem we consider the areas and velocities entering and exiting the control volume at the upstream and downstream boundaries. Denoting the area, the velocity and the density of the incoming streamtube by $A_i$, $u_i$ and $\rho_i$ it follows from conservation of mass that

$$\rho_i u_i A_i = \rho_e \bar{u} \pi r_0^2$$

where $\rho_e$ is the density of the exhaust jet.

Applying the linear momentum thereom in the axial direction it follows that $T$ must be equal to the net flux of axial momentum out of the control volume and therfore

$$T = \int_0^{r_0} \rho_e (u(r))^2 2\pi r dr - \rho_i u_i^2 A_i$$

$$T = \int_0^{r_0} \rho_e 4U^2 [1 - r^2/r_0^2]^2 2\pi r dr - \rho_i u_i^2 A_i$$

$$T = \frac{4\pi}{3} \rho_e r_0^2 U^2 - \rho_i u_i^2 A_i$$

$$T = \rho_i u_i A_i \left[ \frac{4}{3} U - u_i \right]$$

using the conservation of mass relation.

With $\rho_i = 1\ kg/m^3$, $u_i = 100\ m/s$, $A_i = 0.1\ m^2$, and $U = 600\ m/s$ it follows that the thrust

$$T = 7000\ kg \cdot m/s^2$$

(b) On the other hand with a uniform exit velocity distribution the thrust would be

$$T = \rho_i u_i A_i \left[ U - u_i \right] = 5000\ kg \cdot m/s^2$$