EXAMPLE PROBLEM 1

A laboratory wind tunnel has a square test section with side length L. Boundary-layer velocity profiles are measured at two cross-sections and displacement thicknesses are evaluated from the measured profiles. At section 1, where the free-stream speed is U_1, the displacement thickness is δ_{D1}. At section 2, located downstream from 1, the displacement thickness is δ_{D2}.

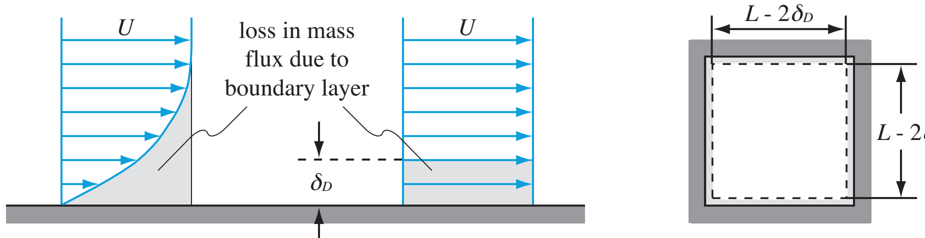
(a) Calculate the free-stream velocity ratio U_2/U_1 as a function of the known parameters.

(b) Calculate the change in static pressure between sections 1 and 2. Express the result as a fraction of the free-stream dynamic pressure at section 1, i.e., find $(p_1 - p_2)/\frac{1}{2}\rho U_1^2$.

(c) ...and just for fun, write down the displacement thickness, δ_D, and the momentum thickness, δ_M in integral form. What are the results for these from the laminar Blasius solution? What is the physical significance of the disturbance thickness (or boundary layer height), δ, and the other two thicknesses δ_D and δ_M?

SOLUTION 1

[Note: This problem illustrates a basic application of the displacement-thickness concept. It is somewhat unusual in that, because the flow is confined, the reduction in flow area caused by the boundary layer leads to the result that the pressure in the inviscid flow region drops (only slightly). In most applications the pressure distribution is determined from the inviscid flow and *then* applied to the boundary layer. We saw a similar phenomenon in Problem B15, where we discovered that the centerline velocity at the entrance of the circular duct increases due to the boundary layer "squeezing" the effective flow area.]



(*a*) Actual velocity profile (*b*) Hypothetical velocity profile (*c*) Cross-section of wind tunnel

(a) The idea here is that at each location, the boundary layer displacement thickness effectively reduces the area of uniform flow. Location 2 has a smaller effective flow area than location 1 (because $\delta_{D2} > \delta_{D1}$). From mass conservation, this tells us that the uniform velocity at 2 will be higher.

$$U_1 A_1 = U_2 A_2,$$

$$U_1(L - 2\delta_{D1})^2 = U_2(L - 2\delta_{D2})^2,$$

so

$$\frac{U_2}{U_1} = \frac{A_1}{A_2} = \frac{(L - 2\delta_{D1})^2}{(L - 2\delta_{D2})^2}.$$

(b) Since $U_2 > U_1$, Bernoulli's equation should tell us that the pressure at 2 is lower than the pressure at 1.

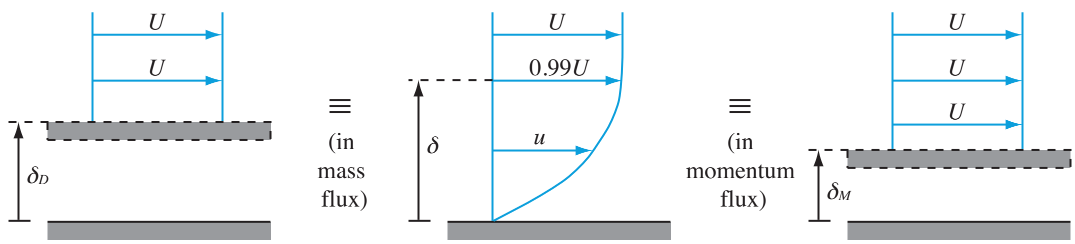
$$p_1 + \frac{1}{2}\rho U_1^2 = p_2 + \frac{1}{2}\rho U_2^2,$$

$$p_1 - p_2 = \frac{1}{2}\rho(U_2^2 - U_1^2) = \frac{1}{2}\rho U_1^2\left[\left(\frac{U_2}{U_1}\right)^2 - 1\right],$$

giving

$$\frac{p_1 - p_2}{\frac{1}{2}\rho U_1^2} = \left(\frac{U_2}{U_1}\right)^2 - 1 = \left(\frac{A_1}{A_2}\right)^2 - 1 = \frac{(L - 2\delta_{D1})^4}{(L - 2\delta_{D2})^4} - 1.$$

(c) The *disturbance thickness*, δ, is usually defined as the vertical distance from the surface of the plate at which the velocity is within 1% of the free stream. That is, the point at which $u \approx 0.99U$. The other two definitions are based on the notion that the boundary layer retards the fluid, so the mass flux and momentum flux are both less than they would be in the absence of the boundary layer. We imagine that the flow remains at uniform velocity U, but the surface of the plate is moved upwards to reduce either the mass or momentum flux by the same amount that the boundary layer actually does.



(*a*) Displacement thickness, δ_D (*b*) Disturbance thickness, δ (*c*) Momentum thickness, δ_M

The *displacement thickness*, δ_D, is the distance the plate would be moved so that the loss of mass flux (due to reduction in uniform flow area) is equivalent to the loss the boundary layer causes. For a semi-infinite plate, the mass flux in the absence of a boundary layer would be $\int_0^\infty \rho U b dy$, where b is the width of the plate perpendicular to the flow (into the page). Since the boundary layer induces a velocity profile, u, the actual mass flux is $\int_0^\infty \rho u b dy$. Hence, the loss due to the boundary layer is the difference, $\int_0^\infty \rho(U - u) b dy$. If we imagine keeping the velocity at a constant U, and instead move the plate up a distance δ_D (as shown in the figure above, on the left), the loss of mass flux would be $\rho U b \delta_D$. Setting these losses equal gives

$$\rho U b \delta_D = \int_0^\infty \rho(U - u) b dy,$$

so that for incompressible flow

$$\delta_D = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U}\right) dy.$$

Since $u \approx U$ at $y = \delta$, the integrand is essentially zero for all $y \geq \delta$. The **Blasius solution for laminar flow over a flat plate** gives

$$\delta_D = 1.72\left(\frac{\nu x}{U}\right)^{1/2}.$$

Similarly, the *momentum thickness*, δ_M, is the distance the plate would be moved so that the loss of momentum flux is equal to the loss the boundary layer actually causes. The momentum flux if we had no boundary layer would be $\int_0^\infty \rho u U b dy$ (the actual mass flux is $\int_0^\infty \rho u b dy$ and the momentum per unit mass flux of the uniform flow is U itself). The actual momentum flux of the boundary layer is $\int_0^\infty \rho u^2 b dy$. The loss of momentum in the boundary layer is therefore, $\int_0^\infty \rho u(U-u) b dy$. If we imagine keeping the velocity constant at U, and instead move the plate up a distance δ_M (as shown in the figure above, on the right), the loss of momentum flux is $\int_0^{\delta_M} \rho U^2 b dy = \rho U^2 b \delta_M$. Equating the losses gives

$$\rho U^2 \delta_M = \int_0^\infty \rho u(U-u) dy,$$

and

$$\delta_M = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right) dy \approx \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy.$$

The **Blasius solution for laminar flow over a flat plate** gives

$$\delta_M = 0.664 \left(\frac{\nu x}{U}\right)^{1/2}.$$

For completeness, the simplifying assumptions usually made for engineering analyses of boundary-layer development are:

1. $u \to U$ at $y = \delta$.
2. $\partial u / \partial y \to 0$ at $y = \delta$.
3. $v \ll U$ within the boundary layer.
4. Pressure variation across the thin boundary layer is negligible. The free-stream pressure distribution is impressed on the boundary layer.

EXAMPLE PROBLEM 2

This isn't really a problem, but it is something you should *definitely* read and understand. The **Karman Momentum Integral Equation** (KMIE) has appeared on every problem set since the midterm. It is very important that you understand where this comes from and the significance of each term. It also offers a perfect opportunity to combine a control-volume approach using the mass and momentum integral equations from the first half of the term to the boundary-layer analysis from the second half.

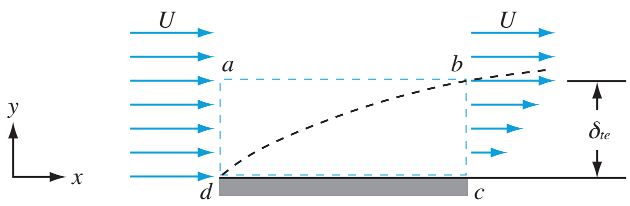
That said, please carefully read the additional document posted with this review at,
http://www.its.caltech.edu/~mefm/me19b/handouts/scan_fox9-4_momentumintegraleqn.pdf.
The material was scanned from Fox, R.W., McDonald, A.T., and Pritchard, P.J., *Introduction to Fluid Mechanics*, 6th ed., John Wiley & Sons, 2004, §9-4, pp. 415-420. It is very well written, easy to understand, and presented in a manner that is easy to follow.

SOLUTION 2

Go read the document!

EXAMPLE PROBLEM 3

A fluid, with density ρ, flows with speed U over a flat plate with length L and width w (assume that $L > w$). At the trailing edge, the boundary-layer thickness is δ_{te}. Assume the velocity profile within the boundary layer is linear, as shown, and that the flow is two-dimensional (flow conditions are independent of z).



(a) Using control volume *abcd*, shown by dashed lines, compute the mass flow rate across surface *ab*. Determine the drag force on the upper surface of the plate. Explain how this (viscous) drag can be computed from the given data even though we do not know the fluid viscosity.

(b) The flat plate is turned so that w is now the length parallel to the flow, and L perpendicular to it (recall $L > w$). Is the boundary-layer thickness at the trailing edge greater or less than the previous case? Should we expect the drag to increase or decrease?

(c) Now, assuming we know the fluid viscosity, $\nu = \mu/\rho$, compute the drag using boundary layer equations.

SOLUTION 3

(a) The general integral form of the continuity equation is (remember this from the first half of the term!)

$$\frac{\partial}{\partial t}\int_{cv} \rho dV + \oint_{cs} \rho \vec{u} \cdot \hat{n} dA = 0.$$

Applying this to *abcd* and noting steady, two-dimensional flow yields

$$\oint_{cs} \rho \vec{u} \cdot \hat{n} dA = 0 = \dot{m}_{da} + \dot{m}_{ab} + \dot{m}_{bc},$$

so

$$\dot{m}_{ab} = -\dot{m}_{da} - \dot{m}_{bc},$$

where

$$\dot{m}_{da} = -\int_0^{\delta_{te}} \rho U w dy = -\rho U w \delta_{te},$$

$$\dot{m}_{bc} = \int_0^{\delta_{te}} \rho \left(\frac{Uy}{\delta_{te}}\right) w dy = \frac{1}{2}\rho U w \delta_{te},$$

to give

$$\dot{m}_{ab} = -(-\rho U w \delta_{te}) - \frac{1}{2}\rho U w \delta_{te} = \frac{1}{2}\rho U w \delta_{te}.$$

The drag force can be found from the balance of momentum flux. The general integral form of the momentum conservation equation applied to a control volume is

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{u} dV + \oint_{cs} \rho \vec{u}(\vec{u} \cdot \hat{n}) dA,$$

where $\sum \vec{F}$ represents the sum of the forces acting on the control volume or control surface (which means they act on the fluid). Applying this to *abcd* and noting that there are no pressure forces (since $dU/dx = 0$), so the only force acting on the control volume is equal and opposite the drag on the plate, D,

$$-D = \oint_{cs} \rho \vec{u}(\vec{u} \cdot \hat{n}) dA = U\dot{m}_{da} + U\dot{m}_{ab} + \int_0^{\delta_{te}} \rho \left(\frac{Uy}{\delta_{te}}\right)^2 w dy.$$

Substituting the results from above

$$\begin{aligned} -D &= -\rho U^2 w \delta_{te} + \frac{1}{2}\rho U^2 w \delta_{te} + \frac{1}{3}\rho U^2 w \delta_{te} \\ &= -\frac{1}{6}\rho U^2 w \delta_{te}. \end{aligned}$$

So the drag force on the upper surface of the plate is simply

$$D = \frac{1}{6}\rho U^2 w \delta_{te}.$$

The force imparted on the plate by the fluid tends to pull the plate in the direction of the flow, hence D is in the positive x direction. The fluid viscosity appears implicitly through the boundary-layer thickness δ_{te}. The higher the viscosity, the larger the boundary layer.

(b) If we rotate the plate, and assume $w < L$, then the boundary-layer thickness at the trailing edge will be less than the original case. The drag depends on the dimension of the plate perpendicular to the flow as well as the height of the boundary layer. We must evaluate this height as a function of distance along the plate to determine if the drag increases or decreases. The KMIE gives

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2 \delta_M) + \delta_D U \frac{dU}{dx},$$

which simplifies for this flow to

$$\frac{\tau_w}{\rho} = U^2 \frac{d\delta_M}{dx}.$$

Since the drag is given by $D = \int \tau_w dA$ over the surface of the plate, we can integrate both sides of the above equation

$$\frac{D}{\rho} = \frac{\int \tau_w dA}{\rho} = \int U^2 \frac{d\delta_M}{dx} dA,$$

or

$$D_1 = \rho U^2 \int_0^L \frac{d\delta_M}{dx} w dx \quad \text{and} \quad D_2 = \rho U^2 \int_0^w \frac{d\delta_M}{dx} L dx,$$

where the subscripts refer to the two different cases (1 for the dimension L parallel to the flow, and 2 for w parallel to the flow, along the x direction). The resulting integrations give $D_1 = \rho U^2 w \delta_M(L)$ and $D_2 = \rho U^2 L \delta_M(w)$. Taking the ratio of the two drag forces

$$\frac{D_1}{D_2} = \frac{w}{L} \frac{\delta_M(L)}{\delta_M(w)}.$$

The laminar Blasius solution gives a drag ratio of

$$\frac{D_1}{D_2} = \frac{w}{L} \frac{L^{1/2}}{w^{1/2}} = \left(\frac{w}{L}\right)^{1/2} < 1 \quad \text{for} \quad w < L,$$

so the drag in the second case (side of dimension w aligned with the flow) will result in a higher drag on the plate.

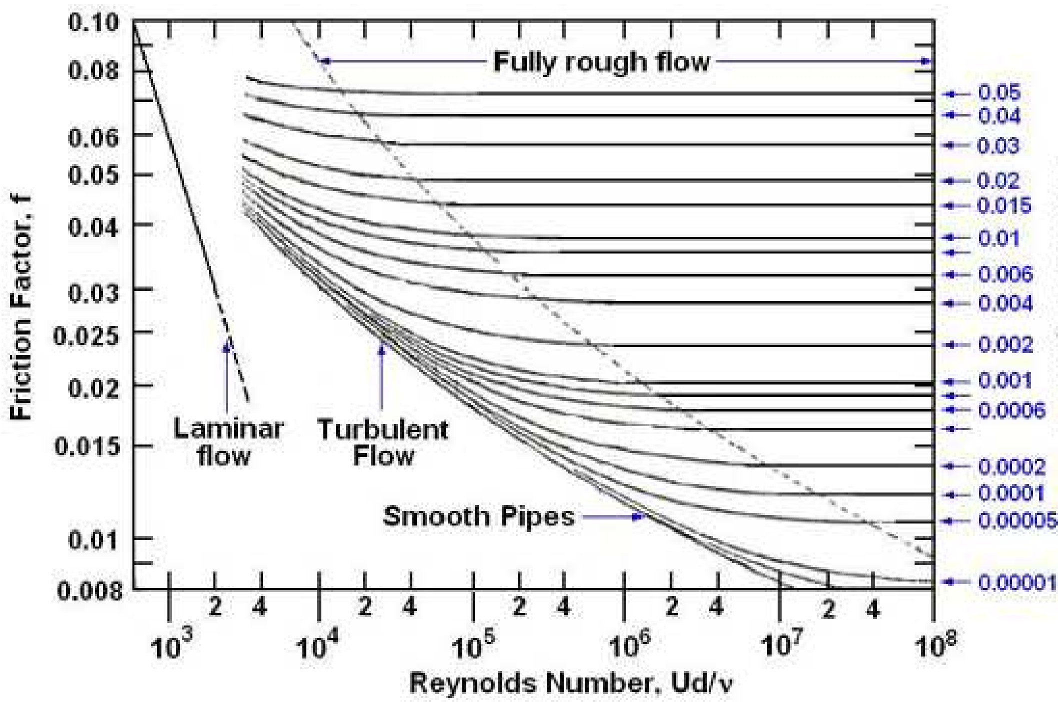
(c) Using the result from part (b) from the KMIE, and assuming the original orientation of the plate, the drag is given by

$$D = \rho U^2 w \delta_M(L) = \rho U^2 w \left[0.664 \left(\frac{\nu L}{U} \right)^{1/2} \right]$$
$$= 0.664 \rho^{1/2} \mu^{1/2} U^{3/2} w L^{1/2}$$

where we have substituted the laminar-flow result from the Blasius solution, $\delta_M = 0.664(\nu x/U)^{1/2}$. Alternatively, you could have assumed a turbulent boundary layer since this was not specified in the problem. The second orientation would cause a change in the positions of w and L in the equation.

EXAMPLE PROBLEM 4

Heated air at 1 atm and 35°C is to be transported in a 150 m long circular duct at a rate of 0.35 m^3/s. If the head loss in the pipe is not to exceed 20 m, determine the minimum diameter of the duct.



SOLUTION 4

The flow rate and the head loss in the duct are given. The diameter of the duct is to be determined.

Assumptions:

1. The flow is steady and incompressible.

2. The entrance effects are negligable and the pipe is smooth.

3. The duct involves no components such and bends or valves.

4. Air is an ideal gas.

5. The flow is intially assumed to be turbulent. This should be verified at the end of the problem.

Properties: The density, dynamic, and kinematic viscosity of air at $35°C$ and 1 atm are $\rho = 1.145\ kg/m^3$, $\mu = 1.895 \times 10^{-5}\ kg/(m \cdot s)$, and $\nu = 1.665 \times 10^{-5}\ m^2/s$.

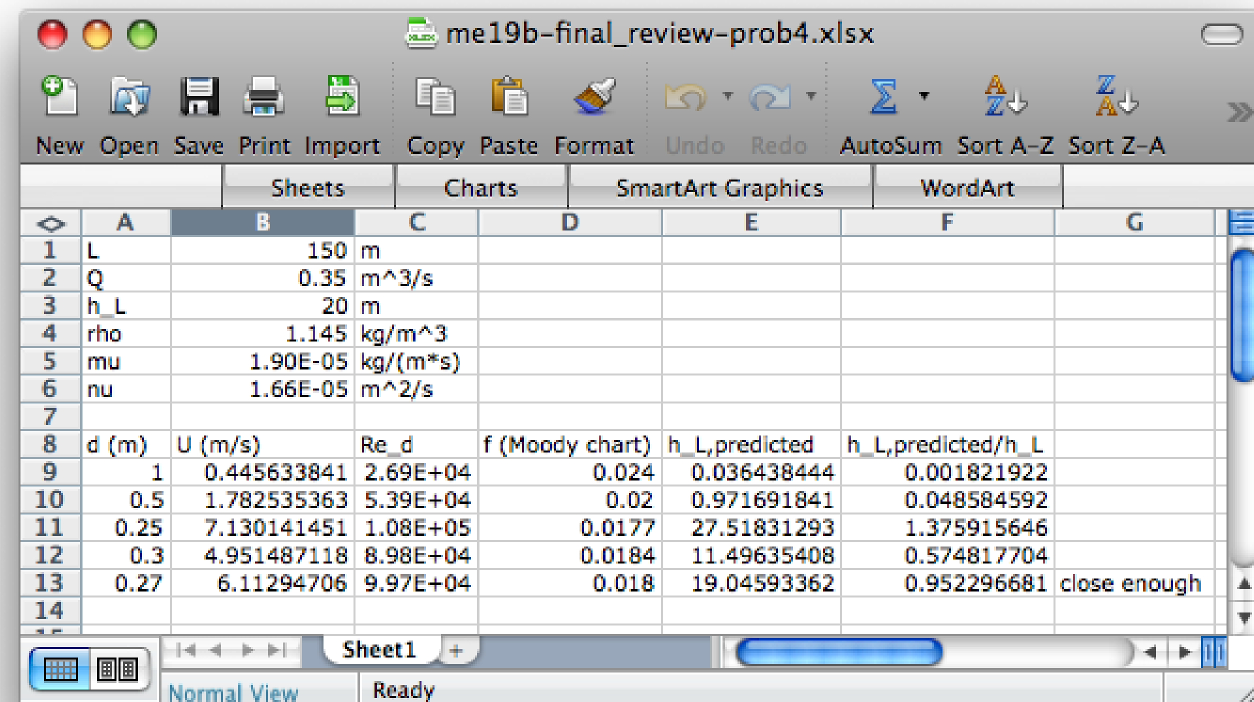
We can solve this problem by iteration using the Moody chart. First, assume a pipe diameter and use this to calculate the volume-averaged flow velocity in the pipe and the Reynolds number. Use the Reynolds number to estimate a friction factor. Then use these numbers to calculate the head loss, and compare the result to the actual head loss. Repeat the calculation until the calculated head loss matches the specified value.

$$U = \frac{Q}{A} = \frac{Q}{\pi d^2/4}$$

$$Re_d = \frac{Ud}{\nu}$$

$$h_L = f\frac{L}{d}\frac{U^2}{2g}$$

The iteration should lead to a solution of $d \approx 0.267\ m$.



	A	B	C	D	E	F	G
1	L	150	m				
2	Q	0.35	m^3/s				
3	h_L	20	m				
4	rho	1.145	kg/m^3				
5	mu	1.90E-05	kg/(m*s)				
6	nu	1.66E-05	m^2/s				
7							
8	d (m)	U (m/s)	Re_d	f (Moody chart)	h_L,predicted	h_L,predicted/h_L	
9	1	0.445633841	2.69E+04	0.024	0.036438444	0.001821922	
10	0.5	1.782535363	5.39E+04	0.02	0.971691841	0.048584592	
11	0.25	7.130141451	1.08E+05	0.0177	27.51831293	1.375915646	
12	0.3	4.951487118	8.98E+04	0.0184	11.49635408	0.574817704	
13	0.27	6.11294706	9.97E+04	0.018	19.04593362	0.952296681	close enough
14							