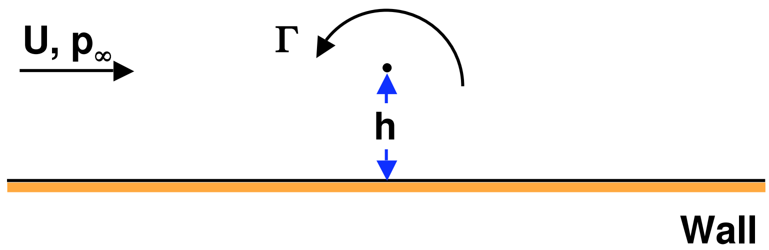
PROBLEM 25

A two-dimensional potential (or free) vortex is located near an infinite wall at a distance h from the wall:



The velocity and pressure of the flow far from the vortex are U and p_∞ respectively. The flow is irrotational and planar and the fluid is incompressible. The strength or circulation of the vortex is Γ. Find the velocity potential for this flow and make a sketch of the streamlines and the equipotentials. Find the force on the wall per unit depth normal to the sketch if the pressure on the underside of the wall is p_∞.

SOLUTION 25

The origin is chosen along the wall such that $y = 0$ and between the vortices such that $x = 0$. To create the desired flow, a combination of velocity potentials should be made:

- Uniform stream, $\phi = Ux$
- 2-D counterclockwise free vortex at $y = h$, $\phi = \frac{\Gamma}{2\pi}\theta_1$
- 2-D clockwise free vortex at $y = -h$, $\phi = \frac{-\Gamma}{2\pi}\theta_2$

with

$$\theta_1 = \arctan\left(\frac{y-h}{x}\right)$$

and

$$\theta_2 = \arctan\left(\frac{y+h}{x}\right)$$

The combined velocity potential is

$$\phi = Ux + \frac{\Gamma}{2\pi}\arctan\left(\frac{y-h}{x}\right) - \frac{\Gamma}{2\pi}\arctan\left(\frac{y+h}{x}\right)$$

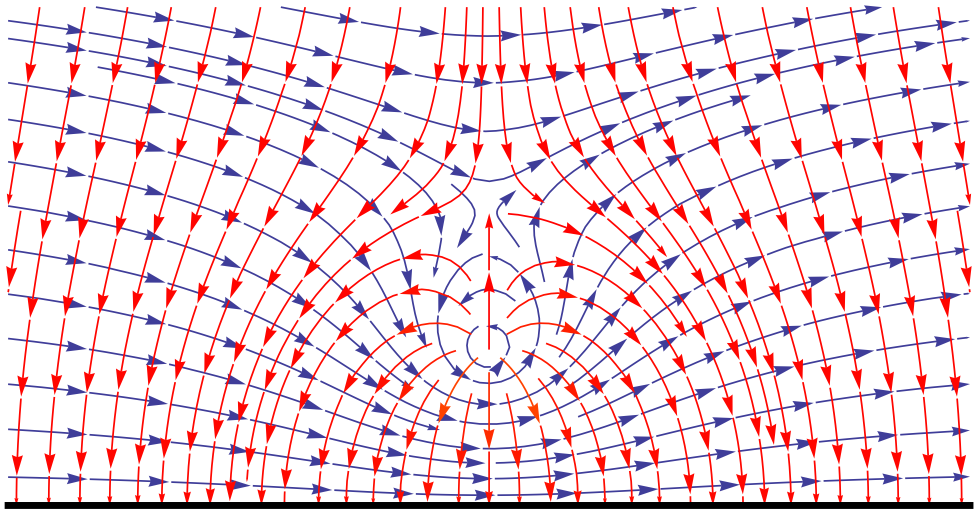
In order to graph the streamlines, we can take advantage of the fact that lines of constant ψ will be orthogonal to lines of constant ϕ, so that

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

which gives the streamfunction,

$$\psi = Uy - \frac{\Gamma}{2\pi}\ln\left(\sqrt{(y-h)^2 + x^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\sqrt{(y+h)^2 + x^2}\right)$$

In the plot, the streamlines are dark blue lines, the equipotentials are the red lines and the horizontal wall is given by the thick black line.



The velocity component in the x-direction along the line $y = 0$ is given as

$$u|_{y=0} = \left.\frac{\partial \phi}{\partial x}\right|_{y=0} = U + \frac{\Gamma h}{\pi(x^2 + h^2)}$$

and the y-component of the velocity is zero. This is confirmed by the boundary conditions, as no flow through the wall can occur ($v|_{y=0} = 0$) and the x-component of the velocity far away from the vortex is equal to the free stream velocity ($u|_{x=\pm\infty, y=0} = U$). The pressure difference across the wall can be calculated with Bernoulli's equation,

$$p_\infty + \frac{1}{2}\rho U_\infty^2 = p_w + \frac{1}{2}\rho u|_{y=0}^2$$

such that

$$p_\infty - p_w = \frac{1}{2}\rho \left(u|_{y=0}^2 - U^2 \right) = \frac{\rho h \Gamma}{\pi(x^2 + h^2)} \left(U + \frac{h\Gamma}{2\pi(x^2 + h^2)} \right)$$

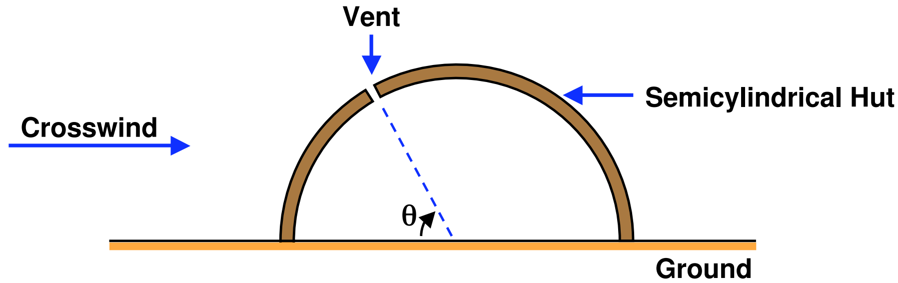
The total force in the y-direction on the wall is given by the pressure difference integrated from $x = (-\infty, \infty)$

$$\begin{aligned} F_y &= \int_{-\infty}^{\infty} (p_\infty - p_w) \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\rho h \Gamma}{\pi(x^2 + h^2)} \left(U + \frac{h\Gamma}{2\pi(x^2 + h^2)} \right) \mathrm{d}x \\ &= \rho U \Gamma + \frac{1}{4} \frac{\rho \Gamma^2}{\pi h} \end{aligned}$$

Thus, the lift due to the circulation ($\rho U \Gamma$) is modified by the wall term $\frac{1}{4}\frac{\rho\Gamma^2}{\pi h}$. Note that the force in the x-direction is equal to zero.

PROBLEM 26

A cylindrical artic hut is subjected to a crosswind as shown in the figure:



The interior of the hut is ventilated to the outside through a small vent at a position, θ, as indicated. Hence the pressure inside the hut (assumed uniform and constant) is the same as the pressure just outside the vent. Assuming potential flow over the hut find the angle, θ, at which the net vertical lift on the hut is zero. Neglect the thickness of the wall of the hut and assume that the vent has no effect on the exterior flow. Also assume that the air is incompressible.

SOLUTION 26

As derived in class, the pressure distribution on the surface of a cylinder in potential flow is given by $p(\alpha)$ where:

$$p(\alpha) = p_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\alpha)$$

where α is the angle measured from the front or rear stagnation point. Note that as $p(\alpha)$ is symmetric about $\alpha = \pi/2$ and it does not matter whether the angle is measured from the front or rear stagnation point. The above expression describes therefore the pressure acting on the outside or the arctic hut where p_∞ is the pressure far away, ρ is the (incompressible) fluid density and U is the velocity of the cross-wind. The pressure inside of the hut is p_I where

$$p_I = p_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta)$$

where θ is the constant angle of the vent location. Any force on the hut will be given by integrating the difference in pressure between the inside and outside over the arc.

$$F = \int_0^\pi (p - p_I)\, R\mathrm{d}\alpha$$

where

$$p - p_I = 2\rho U^2\left(\sin^2\theta - \sin^2\alpha\right)$$

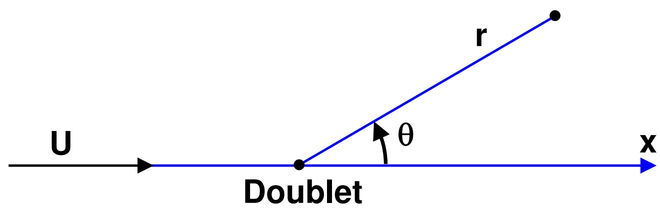
Since the hut is stationary, the lift on the hut must be zero so the forces in the y direction must balance.

$$\begin{aligned} F_y &= \int_0^\pi 2\rho R U^2\left(\sin^2\theta - \sin^2\alpha\right)\sin\alpha\mathrm{d}\alpha \\ &= 2\rho R U^2\left(2sin^2\theta - \frac{4}{3}\right) \\ &= 0 \\ \rightarrow &\quad \sin^2\theta = \frac{2}{3} \end{aligned}$$

hence $\theta = 54.7^\circ$ or 125.3°

PROBLEM 27

The incompressible, axisymmetric potential flow around a sphere can be generated by superposition of a uniform stream ($\phi = Ux$) and a three-dimensional doublet whose potential is given by $A\cos\theta/r^2$ where A is a constant representing the doublet strength. The coordinates r, θ are centered on the doublet and the direction x ($x = r\cos\theta$) is in the direction of the uniform stream:



On the basis of this information construct the velocity potential for potential flow around a sphere of radius R in terms of U, R and the coordinates r, θ. What is the maximum velocity on the surface of the sphere?

SOLUTION 27

The velocity potential for flow around a sphere of radius R is created using the superposition of uniform flow and a 3-D doublet:

$$\begin{aligned}\phi &= \underbrace{Ux}_{\text{Uniform stream}} + \underbrace{A\frac{\cos\theta}{r^2}}_{\text{3-D Doublet}} \\ &= \left(Ur + \frac{A}{r^2}\right)\cos\theta\end{aligned}$$

The differentiation of the velocity u to the radius r is given as:

$$u_r = \frac{\partial\phi}{\partial r} = \left(U - \frac{2A}{r^3}\right)\cos\theta$$

The impenetrable condition ($u_r = 0$) must hold at the surface of the sphere ($r = R$) such that:

$$u_r|_{r=R} = \left(U - \frac{2A}{R^3}\right)\cos\theta = 0$$

Therefore the constant A can be determined as:

$$A = \frac{UR^3}{2}$$

The total velocity potential for flow over a sphere is given as:

$$\phi = Ur\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$

The only component of the velocity along the surface of the sphere will be the one in the tangential direction, u_θ. Therefore, the maximum velocity on the surface of the sphere will be the position of maximum velocity in tangential direction u_θ:

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

on the surface of the sphere $r = R$

$$u_\theta|_{r=R} = -\frac{3}{2}U\sin\theta$$

The magnitude of the velocity will be maximized when

$$|\sin\theta| = 1$$

such that:

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Thus, the maximum velocity is given as

$$\max\left(|u_\theta|_{r=R}\right) = \frac{3}{2}U$$

PROBLEM 28

For the purposes of estimating the drag force on a spherical body (radius, R) in a uniform stream (velocity, U, and density, ρ) it is assumed that the pressure distribution over the upstream side (facing the oncoming stream) is the same as in potential flow whereas the pressure on the downstream side is constant simulating the conditions in a wake. Moreover the pressures match at the "equator", $\theta = \pi/2$ (where θ is the angle measured from the front stagnation point). Find the drag, F_D, on the sphere as a function of ρ, R and U. Evaluate the "drag coefficient" defined as $C_D = F_D/(0.5\rho U^2 \pi R^2)$.

SOLUTION 28

From the previous problem, the velocity potential for the flow past a sphere is given as

$$\phi(r,\theta) = U\cos\theta\left(r + \frac{1}{2}\frac{R^3}{r^2}\right)$$

The angular velocity component is:

$$u_\theta(R,\theta) = -U\sin\theta\left(1 + \frac{1}{2}\frac{R^3}{r^3}\right)\bigg|_{r=R} = -\frac{3}{2}U\sin\theta$$

By using Bernoulli's equation on a streamline between the surface and infinity, it can be determined that:

$$p_\infty + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho\left(-\frac{3}{2}U\sin\theta\right)^2$$

$$p_\infty - p_s = -\frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)$$

The pressure should match at the equator $\theta = \pi/2$,

$$p_\infty - p_w = (p_s - p_\infty)|_{\theta=\pi/2} = \frac{5}{8}\rho U^2$$

The drag is defined as the force acting on the surface parallel to the oncoming stream such that:

$$\begin{aligned} D &= \int_{dA}(p_\infty - p_s)\cos\theta \mathrm{d}A \\ &= \int_0^{2\pi}\int_0^{\pi}(p_\infty - p_s)R^2\sin\theta\cos\theta \mathrm{d}\theta\mathrm{d}\alpha \\ &= 2\pi R^2\int_0^{\pi}(p_\infty - p_s)\sin\theta\cos\theta \mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&= 2\pi R^2 \int_0^{\frac{\pi}{2}} \left(\frac{5}{8}\rho U^2\right) \sin\theta\cos\theta d\theta + 2\pi R^2 \int_{\frac{\pi}{2}}^{\pi} \left(-\frac{1}{2}\rho U^2\left(1-\frac{9}{4}\sin^2\theta\right)\right)\sin\theta\cos\theta d\theta \\
&= \frac{5}{8}\rho U^2 \pi R^2 - \frac{1}{16}\rho U^2 \pi R^2 \\
&= \frac{9}{16}\rho U^2 \pi R^2
\end{aligned}
$$

By defining a non-dimensional drag, the drag coefficient C_D as drag divided by the frontal projected area of the body $A_p = \pi R^2$ and by the term $\frac{1}{2}\rho U_\infty^2$, the drag coefficient is:

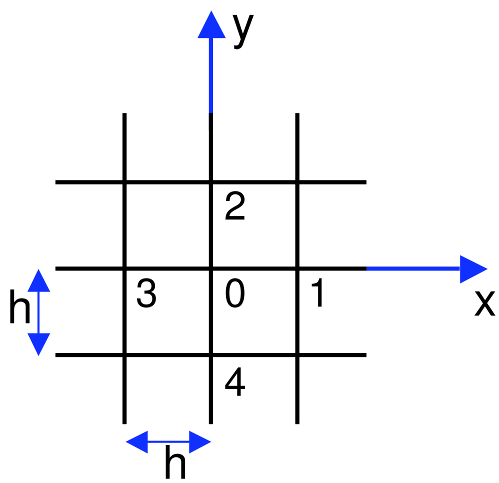
$$C_D = \frac{9}{8}$$

PROBLEM 29

A finite difference method is to be used with a mesh having a uniform node spacing, h, in the x and y directions to solve for the quantity, f, which is governed by the following partial differential equation:

$$f\frac{\partial^2 f}{\partial x^2} = -4\left(\frac{\partial f}{\partial y}\right)^2 \tag{1}$$

Determine the finite difference form of this equation at the node 0 that utilizes values of f at the nodes 0, 1, 2, 3 and 4 as shown below:



SOLUTION 29

First find an expression for $\frac{\partial^2 f}{\partial x^2}$ at the point 0. Find the Taylor series expansion (around 0) for f_1 and f_3:

$$
\begin{aligned}
f_1 &= f_0 + \frac{h}{1!}\left(\frac{\partial f}{\partial x}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_0 + O(h^4) \\
f_3 &= f_0 - \frac{h}{1!}\left(\frac{\partial f}{\partial x}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 - \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_0 + O(h^4)
\end{aligned}
$$

Add the expressions for f_1 and f_3, and solve for $\left(\frac{\partial^2 f}{\partial x^2}\right)_0$:

$$
\begin{aligned}
f_1 + f_3 &= 2f_0 + h^2 \left(\frac{\partial^2 f}{\partial x^2}\right)_0 + O(h^4) \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_0 &= \frac{f_1 + f_3 - 2f_0 - O(h^4)}{h^2} \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_0 &= \frac{f_1 + f_3 - 2f_0}{h^2} + O(h^2)
\end{aligned}
$$

Now find an expression for $\frac{\partial f}{\partial y}$ at the point 0 using the same method, except this time subtract f_4 from f_2 because we want to keep the $\left(\frac{\partial f}{\partial y}\right)_0$ term.

$$
\begin{aligned}
f_2 &= f_0 + \frac{h}{1!}\left(\frac{\partial f}{\partial y}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial y^2}\right)_0 + \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial y^3}\right)_0 + O(h^4) \\
f_4 &= f_0 - \frac{h}{1!}\left(\frac{\partial f}{\partial y}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial y^2}\right)_0 - \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial y^3}\right)_0 + O(h^4) \\
f_2 - f_4 &= 2h\left(\frac{\partial f}{\partial y}\right)_0 + O(h^3) \\
\left(\frac{\partial f}{\partial y}\right)_0 &= \frac{f_2 - f_4 + O(h^3)}{2h} \\
\left(\frac{\partial f}{\partial y}\right)_0 &= \frac{f_2 - f_4}{2h} + O(h^2)
\end{aligned}
$$

Square the expression for $\frac{\partial f}{\partial y}$ at the point 0, and be sure to keep track of the error terms:

$$
\begin{aligned}
\left(\frac{\partial f}{\partial y}\right)_0^2 &= \left(\frac{f_2 - f_4}{2h} + O(h^2)\right)^2 \\
\left(\frac{\partial f}{\partial y}\right)_0^2 &= \left(\frac{f_2 - f_4}{2h}\right)^2 + O(h^4) + \frac{f_2 - f_4}{h}O(h^2) \\
\left(\frac{\partial f}{\partial y}\right)_0^2 &= \left(\frac{f_2 - f_4}{2h}\right)^2 + O(h)
\end{aligned}
$$

Putting everything together, and dropping our error terms $O(h)$ and higher (the terms we keep are $O(h^{-2})$):

$$
\begin{aligned}
f_0\left(\frac{\partial^2 f}{\partial x^2}\right)_0 &= -4\left(\frac{\partial f}{\partial y}\right)_0^2 \\
f_0\frac{f_1 + f_3 - 2f_0}{h^2} &= -4\left(\frac{f_2 - f_4}{2h}\right)^2 \\
f_0(f_1 + f_3 - 2f_0) &= -(f_2 - f_4)^2
\end{aligned}
$$