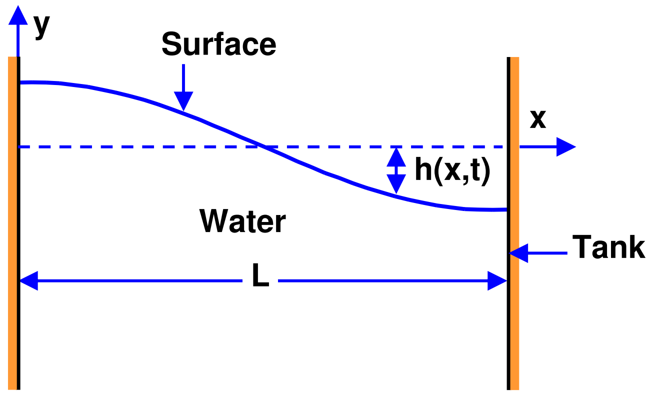
ME19a. SOLUTIONS. Nov. 10, 2009. Due Nov. 17

PROBLEM 21

Water is sloshing back and forth between two infinite vertical walls separated by a distance L:



The flow is assumed to be planar, incompressible, inviscid potential flow. The free surface is devoid of surface tension and is at constant atmospheric pressure. Its position is described by $h(x,t)$ as indicated in the sketch. The wave height, $h(x,t)$, is small so that the assumptions of linear water wave theory may be used. An appropriate velocity potential for this flow is

$$\phi = Ae^{ky}\cos kx \sin \omega t$$

where A, k and ω are undetermined constants.

(a) What are the four boundary conditions which a solution to this flow must satisfy ?

(b) Find the series of values which are possible for the wavelength, λ ($\lambda = 2\pi/k$), of the free surface waves. Each of these wavelengths corresponds to a particular mode of sloshing.

(c) Use the kinematic condition on the free surface to determine the shape of the free surface, $h(x,t)$, as a function of A, k, ω, x and t.

(d) Use the dynamic condition on the free surface to determine the frequency, f ($f = \omega/2\pi$), for each of the modes of sloshing. Denote the acceleration due to gravity by g.

SOLUTION 21

(a) The flow must satisfy the following four boundary conditions,

1. $u = 0$ at $x = 0$,
2. $u = 0$ at $x = L$,
3. On the free surface ($y = h$),

$$v|_{y=h} = \frac{\partial h}{\partial t}$$

but for small amplitude waves, $v|_{y=h} \approx v|_{y=0}$, so the kinematic condition is

$$v|_{y=0} = \frac{\partial h}{\partial t}$$

4. The dynamic condition on the free surface, namely that the pressure is constant and is equal to the atmospheric pressure.

(b) The velocity potential is given as

$$\phi = Ae^{ky}\cos kx \sin\omega t$$

where A, k and ω are undetermined constants. The velocity in the x-direction, u, is

$$u = \frac{\partial\phi}{\partial x} = -Ake^{ky}\sin kx\sin\omega t$$

The boundary condition at $x = 0$ is automatically satisfied by the above equation, but for $u = 0$ at $x = L$,

$$kL = n\pi$$

where n is an integer. From the relationship between the wave number, k, and the wavelength, λ,

$$\lambda = \frac{2\pi}{k} = \frac{2L}{n}, \quad n = \text{integer}$$

Thus, there can be a half wave ($n = 1$), full wave ($n = 2$), etc. trapped between the walls.

(c) The velocity v in the y-direction is,

$$v = \frac{\partial\phi}{\partial y} = Ake^{ky}\cos kx\sin\omega t$$

and the kinematic condition gives

$$\frac{\partial h}{\partial t} = v|_{y=0} = Ak\cos kx\sin\omega t$$

Integrating the above equation yields

$$h(x,t) = -\frac{Ak}{\omega}\cos kx\cos\omega t$$

where the constant of integration, some unknown function $f(x)$, becomes a constant since this is linear wave theory and is taken to be zero (so that the x-axis extends through the center of the waves).

(d) The unsteady Bernoulli equation requires

$$\rho\frac{\partial\phi}{\partial t} + p + \frac{1}{2}\rho\left(u^2 + v^2\right) + \rho g y = const.$$

On the free surface, the dynamic condition gives that pressure p is constant. The kinetic energy terms ($\frac{1}{2}\rho u^2$ and $\frac{1}{2}\rho v^2$) are of higher order than the other terms and are thus negligible. Finally, substituting the height for y yields,

$$\rho\left.\frac{\partial\phi}{\partial t}\right|_{y=h} + \rho g h = const.$$

The small amplitude assumption allows the approximation,

$$\left.\frac{\partial\phi}{\partial t}\right|_{y=h} \approx \left.\frac{\partial\phi}{\partial t}\right|_{y=0} = A\omega\cos kx\cos\omega t$$

which, when substituted into Bernoulli's equation yields

$$A\omega\cos kx\cos\omega t - \frac{Agk}{\omega}\cos kx\cos\omega t = const$$

The only constant which will satisfy this equation is zero (due to the oscillating time and distance terms). This can be confirmed by evaluating these properties at one of the nodes.

$$A\omega \cos kx \cos \omega t - \frac{Agk}{\omega} \cos kx \cos \omega t = 0,$$

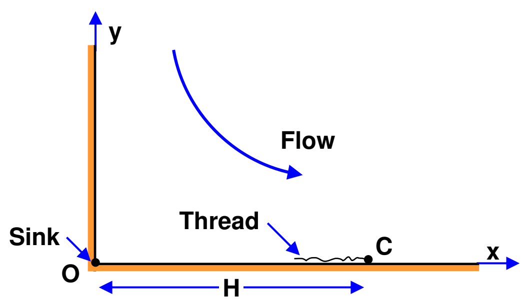
yields,

$$A\omega - \frac{Agk}{\omega} = 0$$
$$\rightarrow \omega = \sqrt{gk}$$

Thus, the frequency, f ($f = \omega/2\pi$) is given by

$$f = \frac{1}{2\pi} \left(\frac{n\pi g}{L} \right)^{1/2}$$

PROBLEM 22



The flow in the neighborhood of a corner in a rectangular ventilation duct is to be modelled as a planar potential flow of an incompressible, inviscid fluid and is therefore given by the streamfunction, $\psi = Axy$, where A is assumed known:

This flow is then changed by withdrawing fluid through pipes connected to the walls at the origin, O; fluid is thereby withdrawn at a volumetric rate of q per unit depth normal to the sketch. Construct the velocity potential for the modified flow and find expressions for the velocity components in terms of x, y, A and q.

A piece of thread is attached by one end to a point, C, which is at a distance, H, from the origin. The flow will extend the free end of this thread either toward the origin or toward $x = \infty$. Find the condition under which it will extend toward the origin.

SOLUTION 22

The stream function for the corner flow, ψ_c, is given as

$$\psi_c = Axy$$

The stream function and velocity potential of a given potential flow are related by

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Using this relationship to calculate the velocity potential of the corner flow from the known stream function yields

$$\frac{\partial \psi_c}{\partial x} = Ay = -\frac{\partial \phi_c}{\partial y}$$
$$\phi_c = -\frac{1}{2}Ay^2 + f(x),$$

and

$$\frac{\partial \psi_c}{\partial y} = Ax = \frac{\partial \phi_c}{\partial x}$$
$$\phi_c = \frac{1}{2}Ax^2 + g(y)$$

Combining the two equations yields

$$\phi_c = \frac{1}{2}A\left(x^2 - y^2\right)$$

where the constant that arises is taken to be zero.

The velocity potential for a sink, is defined as

$$\phi_s = k \ln r$$

where the strength k of the sink is to be determined. The volumetric rate per unit depth removed from the flow is given as q which can be related to k by integrating over the corner for which the sink acts

$$q = \int_A \mathbf{u} \cdot \mathrm{d}\mathbf{A}$$
$$-q = \int_0^{\frac{\pi}{2}} u_r r \mathrm{d}\theta$$

where

$$u_r = \frac{\partial \phi}{\partial r} = \frac{k}{r}$$

Thus,

$$k = -\frac{2q}{\pi}$$

Substituting k into the expression for the velocity potential of the sink yields

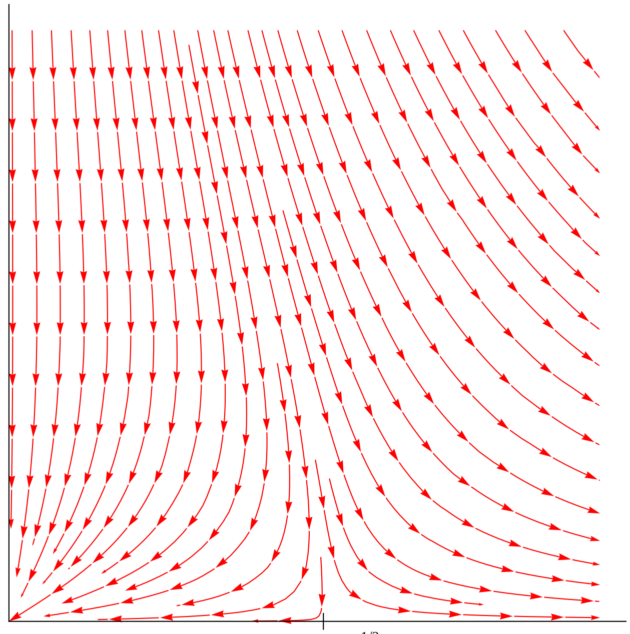
$$\phi_s = -\frac{2q}{\pi}\ln r = -\frac{q}{\pi}\ln r^2 = -\frac{q}{\pi}\ln\left(x^2 + y^2\right)$$

The velocity for the entire flow is the velocity potential of the corner flow, ϕ_c, plus the velocity potential of the sink, ϕ_s,

$$\phi = \underbrace{\frac{1}{2}A\left(x^2 - y^2\right)}_{\text{Corner Flow}} \underbrace{-\frac{q}{\pi}\ln\left(x^2 + y^2\right)}_{\text{Sink}}$$

The string will be pushed toward the origin as long as the velocity vector of the flow along the wall points toward the origin (i.e. $u|_{y=0} < 0$). The velocity of the flow in the x-direction is

$$u = \frac{\partial \phi}{\partial x} = Ax - \frac{2qx}{\pi\left(x^2 + y^2\right)}$$



Evaluating this expression along the wall gives

$$u|_{y=0} = Ax - \frac{2q}{\pi x}$$

Solving for when the velocity changes sign ($u_{y=0} = 0$)

$$\begin{aligned} Ax &= \frac{2q}{\pi x} \\ x^2 &= \frac{2q}{\pi A} \\ \therefore \quad x &= \left(\frac{2q}{\pi A}\right)^{1/2} \end{aligned}$$

Since the string is positioned at $x = H$, the string will extend toward the origin if

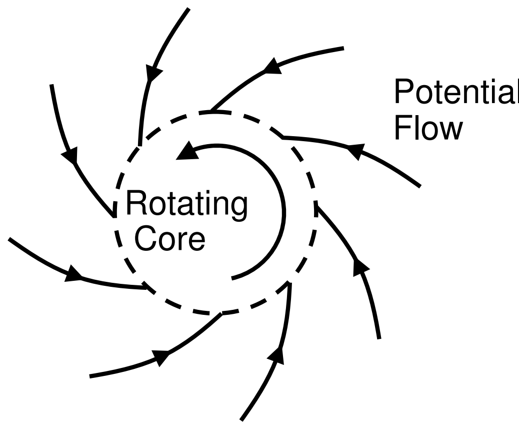
$$H < \left(\frac{2q}{\pi A}\right)^{1/2}.$$

PROBLEM 23

A hurricane can be visualized as a planar incompressible flow consisting of a rotating circular core surrounded by a potential flow:

A particular hurricane has a core of radius 40 m and air is sucked into this core at a volume flow rate per unit depth perpendicular to the diagram of 5000 m^2/s. Furthermore the pressure difference between the air far away from the hurricane and the air at the edge of the core is 1500 $kg/m^2\ s$. The velocity of the air far from the core is assumed to be negligible. The density of the air is assumed uniform and constant at 1.2 kg/m^3. Find the angular rate of rotation of the hurricane and the velocity of the wind at the edge of the core.

SOLUTION 23



Find the angular rate of rotation of the hurricane and the velocity of the wind at the edge of the core. The properties of the problem are:

$$R = 40\ m \qquad Q = 5000\ \frac{m^2}{s} \qquad p_\infty - p_R = \Delta p = 1500\ \frac{kg}{m^2 s} \qquad \rho = 1.2\ \frac{kg}{m^3}$$

The flow outside of the core of the hurricane can be modelled as the combination of a sink and a vortex.

$$\phi = \underbrace{-\frac{Q}{2\pi}\ln r}_{\text{Sink}} + \underbrace{\frac{\Gamma}{2\pi}\theta}_{\text{Vortex}}$$

The velocity components will then be:

$$u_r = \frac{\partial \phi}{\partial r} = -\frac{Q}{2\pi r} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

Evaluating the velocity at the edge of the core ($r = R$):

$$u_r = -\frac{Q}{2\pi R} \qquad u_\theta = \frac{\Gamma}{2\pi R}$$

$$\Rightarrow |u|^2_{r=R} = {u_r}^2 + {u_\theta}^2 = \frac{Q^2}{4\pi^2 R^2} + \frac{\Gamma^2}{4\pi^2 R^2}$$

Since the flow is irrotational outside of the core, we can apply Bernoulli's Equation. We apply the equation at the edge of the core and evaluate the constant far from the core ($r \to \infty$):

$$\begin{aligned} \frac{1}{2}\rho|u|^2_R + p_R &= \frac{1}{2}\rho|u_\infty|^2 + p_\infty \\ \frac{1}{2}\rho|u|^2_R &= p_\infty - p_R = \Delta p \\ |u|^2_{r=R} &= \frac{2\Delta p}{\rho} = 50\ \frac{m}{s} \end{aligned}$$

Now solve for Γ:

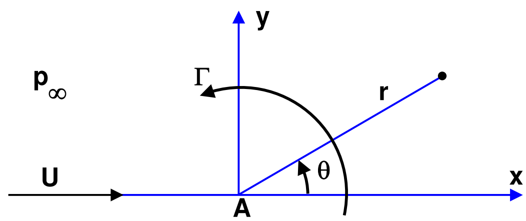
$$\begin{aligned} \frac{2\Delta p}{\rho} &= \frac{Q^2}{4\pi^2 R^2} + \frac{\Gamma^2}{4\pi^2 R^2} \\ \Gamma &= \sqrt{\frac{8\pi^2 R^2 \Delta p}{\rho} - Q^2} \\ &= \sqrt{\frac{8\pi^2 (40\ m)^2 (1500\ \frac{kg}{m^2 s})}{1.2\ \frac{kg}{m^3}} - \left(5000\ \frac{m^2}{s}\right)^2} = 11500\ \frac{m^2}{s} \end{aligned}$$

The angular rate of rotation of the hurricane:

$$\omega = \frac{u_\theta}{R} = \frac{\Gamma}{2\pi R^2} = 1.147\ \frac{rad}{s}$$

PROBLEM 24

A planar incompressible potential flow is generated by superposition of:



1. A uniform stream with velocity potential Ux.
2. A doublet with velocity potential $UR^2\cos\theta/r$ at the point A in the sketch above.
3. A potential vortex at the point A with circulation, Γ, and velocity potential, $\Gamma\theta/2\pi$.

This generates the flow around a cylinder of radius, R, whose center is at A; the cylinder is also spinning in the counterclockwise direction. Find the velocity and pressure on the surface of the cylinder as a function of angular position, θ. Neglecting shear stresses and considering only the pressures on the surface of the cylinder, find the total force on the cylinder per unit depth normal to the sketch. This is probably most readily done by separately evaluating the drag (the component of the force in the direction of the uniform stream, in other words the direction x) and the lift (the component of the force in the direction, y, perpendicular to the uniform stream). Denote the fluid density by ρ and the pressure far from the cylinder by p_∞.

SOLUTION 24

Flow around a cylinder of radius, R, is generated by the superposition of a uniform stream, doublet, and potential vortex:

$$\phi = \underbrace{Ur\cos\theta}_{\text{Uniform Stream}} + \underbrace{U\frac{R^2}{r}\cos\theta}_{\text{Doublet}} + \underbrace{\frac{\Gamma\theta}{2\pi}}_{\text{Vortex}}$$

The velocity components will be:

$$\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \\ u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} \end{aligned}$$

On the surface of the cylinder ($r = R$):

$$\begin{aligned} u_r|_{r=R} &= U\left(1 - \frac{R^2}{R^2}\right)\cos\theta = 0 \\ u_\theta|_{r=R} &= -2U\sin\theta + \frac{\Gamma}{2\pi R} \end{aligned}$$

The radial velocity is naturally zero since there is no flow through the cylinder. To find the corresponding pressure on the surface of the cylinder, we apply Bernoulli's Equation (since the flow is irrotational):

$$\frac{1}{2}\rho|u(R,\theta)|^2 + p(R,\theta) = \frac{1}{2}\rho|u_\infty|^2 + p_\infty$$

$$\therefore\ p(R,\theta) = \frac{1}{2}\rho\left(U^2 - \left[-2U\sin\theta + \frac{\Gamma}{2\pi R}\right]^2\right) + p_\infty$$

The total force on the cylinder per unit depth normal to the sketch is given as the integral of the pressure over the surface. The components of the force in the horizontal and vertical directions, the drag and lift, respectively, will be given by:

$$D = F_x = -\int_0^{2\pi} p(R,\theta)\cos\theta R\mathrm{d}\theta$$

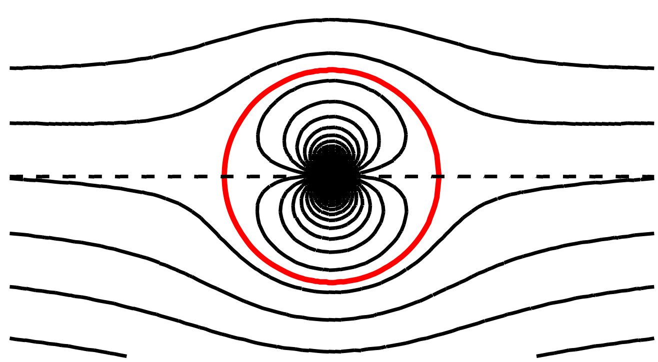
$$L = F_y = -\int_0^{2\pi} p(R,\theta)\sin\theta R\mathrm{d}\theta$$

Evaluating the drag:

$$\begin{aligned} D &= -\int_0^{2\pi}\left[\frac{1}{2}\rho U^2 R\left(\cos\theta - 4\sin^2\theta\cos\theta + \frac{2\Gamma}{\pi UR}\sin\theta\cos\theta - \frac{\Gamma^2}{4\pi^2R^2U^2}\cos\theta\right) + p_\infty R\cos\theta\right]\mathrm{d}\theta \\ &= -\left[\frac{1}{2}\rho U^2 R\left(\sin\theta\left[1 - \frac{\Gamma^2}{4\pi^2R^2U^2}\right] - \frac{4}{3}\sin^3\theta + \frac{\Gamma}{\pi UR}\sin^2\theta\right) + p_\infty R\sin\theta\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

since $\sin 0 = \sin 2\pi = 0$. Evaluating the lift:

$$\begin{aligned} L &= -\int_0^{2\pi}\left[\frac{1}{2}\rho U^2 R\left(\sin\theta - 4\sin^3\theta + \frac{2\Gamma}{\pi UR}\sin^2\theta - \frac{\Gamma^2}{4\pi^2R^2U^2}\sin\theta\right) + p_\infty R\sin\theta\right]\mathrm{d}\theta \\ &= -\left[\frac{1}{2}\rho U^2 R\left(-\cos\theta\left[1 - \frac{\Gamma^2}{4\pi^2R^2U^2}\right] + \frac{4}{3}\sin^2\theta\cos\theta + \frac{8}{3}cos\theta + \frac{\Gamma}{\pi UR}\theta - \frac{\Gamma}{2\pi UR}\sin 2\theta\right) - p_\infty R\cos\theta\right]_0^{2\pi} \\ &= -\rho\Gamma U \end{aligned}$$



The lack of drag can be seen by looking at the symmetry of the streamlines front to back on the cylinder. Since the pressure is directly linked to the velocity through Bernoulli's equation, symmetric streamlines imply that there is no pressure imbalance front to back and thus no force (d'Alembert's paradox). For the lift, the circulation, Γ, breaks the top to bottom symmetry of the streamlines and creates a net force, the lift. The presence of lift based on the magnitude of the circulation is consistent with the Kutta-Joukowski Theorem.