ME19a. **SOLUTIONS.** **Nov. 3, 2009. Due Nov. 10**

PROBLEM 17

A manufacturer advertises a line of centrifugal pumps of various sizes given by the radius of the impeller (the rotating part) denoted by R. These can be run by a motor at any rotating speed, N (in radians per sec). The manufacturer also states that the "operating condition" for this line of pumps is given by specific values of the "flow coefficient" and "head coefficient". The flow coefficient, ϕ, is defined by $\phi = Q/\pi N R^3$ where Q is the volume flow rate and the head coefficient, ψ, is defined by $\psi = \Delta p/\rho N^2 R^2$ where Δp is the total pressure rise across the pump and ρ is the fluid density. We denote the manufacturer's specific design values for the operating condition by ϕ_D and ψ_D and regard them as given constants.

We now wish to choose one of these pumps for a system in which we want a particular flow rate, Q, with a particular total pressure rise, Δp. How do we decide on the necessary size of pump (R) and the necessary speed, N?

SOLUTION 17

The flow coefficient, ϕ, and the head coefficient, ψ, are defined as

$$\phi = \frac{Q}{\pi N R^3}, \qquad \psi = \frac{\Delta p}{\rho N^2 R^2}.$$

The pump is designed such that these two coefficients are known and are equal to ϕ_D and ψ_D, respectively

$$\phi_D = \frac{Q}{\pi N R^3}, \qquad \psi_D = \frac{\Delta p}{\rho N^2 R^2}.$$

The two unknowns in these two equations are the size of the pump R and the rotating speed N. By manipulating the equations, these parameters can be expressed in terms of the known variables. From the equation for the flow coefficient, it is known that

$$N R^3 = \frac{Q}{\pi \phi_D}. \tag{1}$$

The equations for the head coefficient gives

$$N^2 R^2 = \frac{\Delta p}{\rho \psi_D}. \tag{2}$$

Squaring equation 1 and dividing it by equation 2 gives

$$R^4 = \frac{\frac{Q^2}{\pi^2 \phi_D^2}}{\frac{\Delta p}{\rho \psi_D}}.$$

Simplifying this result to solve for R

$$R = \left(\frac{Q}{\pi \phi_D}\right)^{\frac{1}{2}} \left(\frac{\rho \psi_D}{\Delta p}\right)^{\frac{1}{4}}. \tag{3}$$

Plugging equation 3 into equation 1 gives an expression for the rotating speed N

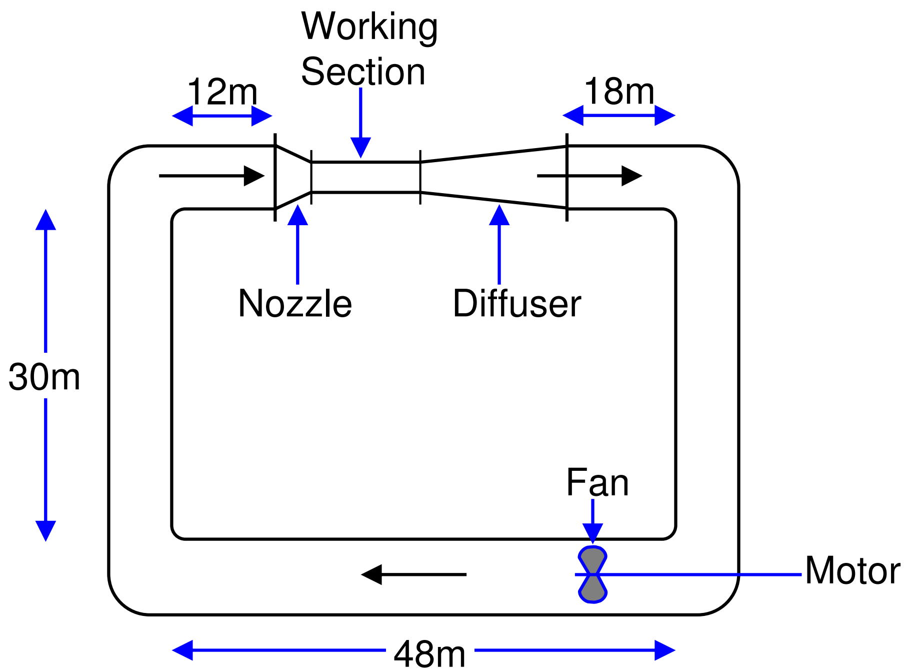
$$N \left(\frac{Q}{\pi \phi_D}\right)^{\frac{3}{2}} \left(\frac{\rho \psi_D}{\Delta p}\right)^{\frac{3}{4}} = \frac{Q}{\pi \phi_D}.$$

Rewriting and solving for N gives

$$N = \left(\frac{\pi \phi_D}{Q}\right)^{\frac{1}{2}} \left(\frac{\Delta p}{\rho \psi_D}\right)^{\frac{3}{4}}.$$

PROBLEM 18

A wind tunnel is constructed primarily of 6 m diameter piping arranged with four 90° elbows as shown in the sketch below.



The working section is 3 m in diameter and is preceded by a nozzle and followed by a diffuser. A fan is installed to create the flow and is 80% efficient. If the tunnel is to achieve an air velocity of 80 m/s in the working section, find the power which must be provided to the fan (in HP where 1 $HP = 746\ kg\ m^2/s^3$). Air at these speeds can be assumed essentially incompressible with a density of 1.2 kg/m^3. Assume the following losses occur in the tunnel:

1. A loss in each of the four corner bends equivalent to a length of 20 diameters of the large piping.
2. A friction factor, f, of 0.02 in the 138 m of 6 m diameter pipe.
3. A total loss in the nozzle, working section and diffuser equivalent to one fifth of the velocity head (the $\frac{1}{2}\rho u^2$) in the working section.

SOLUTION 18

The total pressure losses in the circuit are composed of several contributions:

- Pressure loss due to the bends in the pipe Δp_b
- Pressure loss due to length of pipe Δp_l

- Pressure loss in nozzle, working section and diffuser Δp_{ws}

The power provided to the fan, P, must offset the total pressure losses, $\Delta p = \Delta p_b + \Delta p_l + \Delta p_{ws}$ and account for the efficiency of the fan itself.

The pressure loss in the bends of the pipe can be quantified by the effective length

$$L_b^\star = 4 \text{ bends} \cdot 20D_{pipe} = 80D_{pipe} = 480 \ m$$

where $D_{pipe} = 6 \ m$. The pressure loss due to the length of the pipe has contributions

$$L_l = 48 + 30 + 30 + 12 + 18 = 138 \ m$$

so that the total pressure loss due to the bends and length of pipe is given as

$$\Delta p_b + \Delta p_l = \frac{f(L_b^\star + L_l)}{D_{pipe}} \left(\frac{1}{2}\rho U^2\right) = 1.03\rho U^2$$

The total pressure loss in the nozzle, working section and diffuser can be calculated as

$$\Delta p_{ws} = \left(\frac{1}{5}\right)\left(\frac{1}{2}\rho u^2\right) \qquad (4)$$

where the velocity in working section $u = 80$ m/s. Mass conservation requires constant flow rate

$$Q = uA_{ws} = UA_{pipe} = 180\pi \ m^3/s$$

where the velocity in the pipe U is found from the above relation as

$$U = \frac{D_{ws}^2}{D_{pipe}^2}u = \frac{u}{4}.$$

Rewriting equation 4 in terms of the velocity U gives

$$\Delta p_{ws} = \frac{8}{5}\rho U^2 = 1.6\rho U^2$$

such that the total pressure loss is

$$\Delta p = \Delta p_b + \Delta p_l + \Delta p_{ws} = (1.03 + 1.6)\rho U^2 = 2.63\rho U^2$$

The total power input, P, to the fan is given by

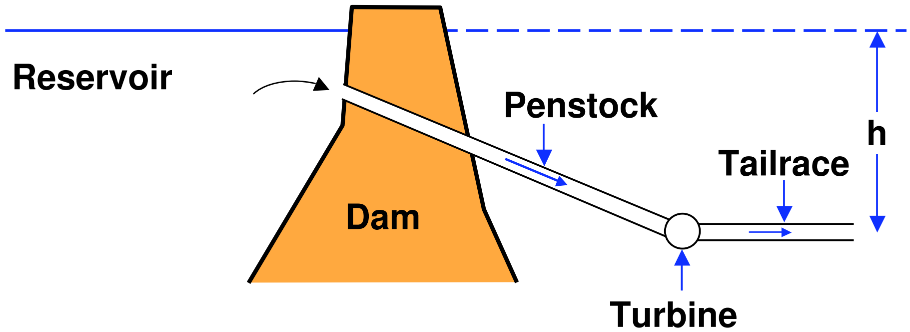
$$P = \frac{Q\Delta p}{\eta} = 8.923 \times 10^5 \ kg \ m^2/s^2$$

with flow rate Q, total pressure loss Δp and efficiency $\eta = 0.8$. We know $1 \ HP = 746 \ kg \ m^2/s^2$ such that the total power input is

$$P = 1.196 \times 10^3 \ HP$$

PROBLEM 19

The tailrace (discharge pipe) of a hydro-electric turbine installation is at an elevation, h, below the water level in the reservoir:
The frictional losses in the penstock (the pipe leading to the turbine) and the tailrace are represented by the loss coefficient, k, based on the mean velocity, U, in those pipes (which have the same cross-sectional area, A). The flow discharges to atmospheric pressure at the exit from the tailrace. The water density is denoted by ρ and the acceleration due to gravity by g.



(a) What is the drop in total head across the turbine in terms of U, h, k and g?

(b) What is the power developed by the turbine assuming that it is 90% efficient? (Answer in terms of U, h, k, ρ, A and g.)

(c) What is the optimum velocity, U, which will produce the maximum power output from the turbine assuming that h, k, A, ρ and g are constant? (Answer in terms of h, k, and g.)

SOLUTION 19

(a) Labelling a point on the reservoir surface as 1 and the discharge from the tailrace as 2, we can use Bernoulli's equation

$$\frac{p_a}{\rho g} + h = \frac{p_a}{\rho g} + \frac{U^2}{2g} + \text{Head losses from 1 to 2}$$

to obtain the total head loss

$$\text{Head losses from 1 to 2} = h - \frac{U^2}{2g}$$

The total head loss is a combination of losses due to friction in the pipes and a head drop across the turbine

$$\text{Head losses from 1 to 2} = h - \frac{U^2}{2g} = \text{Head loss in pipes due to friction} + \text{Head drop through the turbine}$$

Noting that the head loss due to friction in the pipes is equal to $kU^2/2g$ it follows that

$$\text{Head drop through the turbine} = h - \frac{U^2}{2g}(1+k)$$

(b) If the turbine were 100% efficient the power it would produce would be the total pressure drop through the turbine (total head drop times ρg) multiplied by the volume flow rate through the turbine, namely UA. However, since the turbine is only 90% efficient we only obtain a fraction of that power

$$P = 0.9\rho g U A \left[h - \frac{U^2}{2g}(1+k)\right]$$

(c) Differentiating the above expression for the power gives

$$\frac{\partial P}{\partial U} = 0.9\rho g A \left[h - \frac{3U^2}{2g}(1+k)\right]$$

Maximum power will occur when $\frac{\partial P}{\partial U} = 0$ which can be solved for the optimum velocity

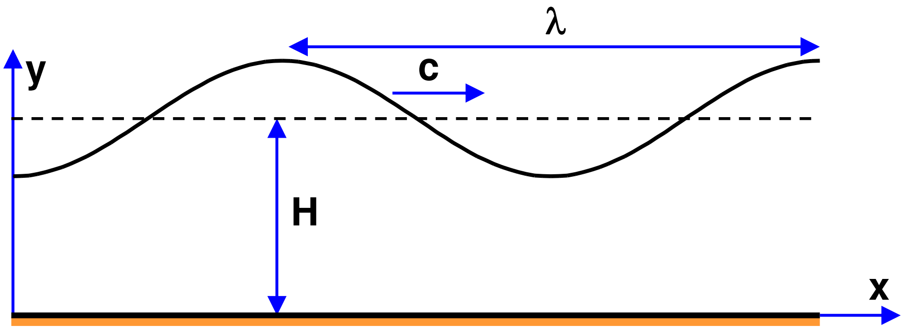
$$U = \left[\frac{2gh}{3(k+1)}\right]^{\frac{1}{2}}$$

PROBLEM 20

Find the speed of propagation, c , of small amplitude, traveling water waves of wavelength, λ, on an ocean of infinite extent but finite depth, H. The answer is an expression involving H, λ, and the acceleration due to gravity, g.

SOLUTION 20

Find the speed of propagation, c, of small amplitude, traveling water waves of wavelength, λ, on an ocean of infinite extent but finite depth, H. The answer is an expression involving H, λ, and the acceleration due to gravity, g.



Solutions to the wave equation need to satisfy Laplace's equation

$$\nabla^2\phi = 0$$

The wave of interest is a traveling wave (as opposed to a standing wave). A wave traveling to the right with velocity c can be described by

$$\phi = \left(Ae^{ny} + Be^{-ny}\right)\cos\left(n(x - ct)\right)$$

with wave number $n = 2\pi/\lambda$. This equation can be verified by substituting it into the Laplace equation. There are two constants (A and B) in this equation and an unknown velocity c which needs to be determined by boundary conditions.

There are several appropriate boundary conditions needed for water with a finite depth. The first boundary condition is formed by the requirement that the velocity component in the y-direction must go to zero at the channel bottom

$$v = 0 \text{ at } y = 0 \tag{5}$$

Furthermore, the kinematic condition tells us that the velocity at the water surface is approximately the rate of change of the wave amplitude

$$v = \frac{\partial h}{\partial t} \text{ at } y = H$$

where the amplitude of the wave $h(x, t)$ is small such that $h << H$. In these equations (and the ones to follow), we recall that the velocity vector is $\mathbf{u} = \nabla\phi$, such that the velocity components are $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$. We know the flow is irrotational (because we're using a velocity potential!), so we can use Bernoulli's equation to obtain another boundary condition. The dynamic boundary condition is obtained from the non-steady Bernoulli equation and tells us that the pressure at the surface is constant

$$\rho\frac{\partial\phi}{\partial t}\bigg|_{y=H} + p_a + \frac{1}{2}\rho|\vec{u}|^2\bigg|_{y=H} + \rho g(H + h) = f(t) \text{ at } y = H$$

If the wave amplitude is small, we can neglect the kinetic energy term involving $|\vec{u}|^2$ relative to the other terms, namely the potential energy (because a fluid particle is being displaced only a very small amount). Also, because this equation is true on the surface at any time, the function $f(t)$ must be a constant. Hence, we obtain

$$\rho \frac{\partial \phi}{\partial t}\bigg|_{y=H} + p_a + \rho g(H+h) = const \text{ at } y = H \tag{6}$$

Differentiation of equation 6 with respect to time gives

$$\rho \frac{\partial^2 \phi}{\partial t^2}\bigg|_{y=H} + \rho g \frac{\partial h}{\partial t}\bigg|_{y=H} = 0$$

and we know from the kinematic condition that $\partial h/\partial t = v = \partial \phi/\partial y$ at $y = H$ so that the equation above becomes

$$\rho \frac{\partial^2 \phi}{\partial t^2}\bigg|_{y=H} + \rho g \frac{\partial \phi}{\partial y}\bigg|_{y=H} = 0 \tag{7}$$

Equation 7 is the second boundary condition which needs to be satisfied (the first is equation 5).

By using the first boundary condition as described in equation 5, you will find

$$A - B = 0$$

hence $B = A$ which leads to:

$$\phi = A(e^{ny} + e^{-ny}) \cos(n(x - ct))$$

By using equation 7, it can be shown that the constant A is arbitrary and an expression for c can be found from

$$-An^2c^2(e^{nH} + e^{-nH}) + gAn(e^{nH} - e^{-nH}) = 0$$

where ρ and $\cos(n(x - ct))$ have divided out of both terms. This can be rearranged to solve for the propagation speed

$$c = \sqrt{\frac{g}{n}\left(\frac{e^{nH} - e^{-nH}}{e^{nH} + e^{-nH}}\right)} = \sqrt{\frac{g}{n}\tanh(nH)}$$

where it should be noted that

$$\cosh(nH) = \frac{e^{nH} + e^{-nH}}{2}$$

$$\sinh(nH) = \frac{e^{nH} - e^{-nH}}{2}$$

By filling in the expression for the wave number $n = 2\pi/\lambda$, the speed of propagation is given by

$$c = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi H}{\lambda}\right)}$$