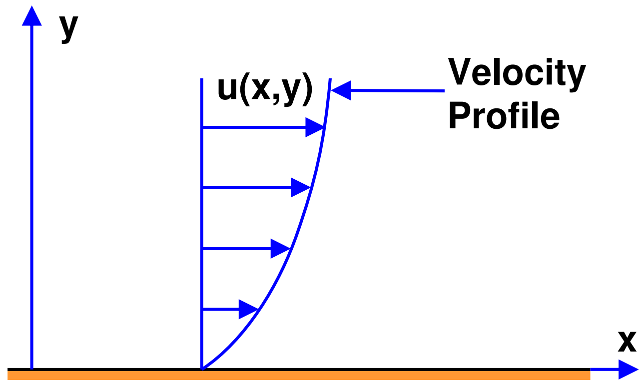
ME19a. **HOMEWORK.** **Oct. 20, 2009. Due Oct. 29**

PROBLEM 13

The velocity, u, in the x direction for a planar incompressible shear flow near a wall as shown in the following sketch,



and is given by the expression

$$u = U\left(\frac{2y}{ax} - \frac{y^2}{a^2x^2}\right)$$

where a is a constant. Find the corresponding expression for the velocity, v, assuming that $v = 0$ at the wall, $y = 0$.

PROBLEM 14

A particular planar, incompressible flow is given by:

$$\psi = Axyt$$

where A is constant in time and space.

(a) Sketch the streamlines for this flow at a particular instant in time (say $t = 1$). What is the typical equation for such a streamline?

(b) Write down expressions for the velocity components, $u(x, y, t)$ and $v(x, y, t)$.

(c) Find the parametric equations, $x(x_0, y_0, t)$ and $y(x_0, y_0, t)$, for the pathline of a particle whose position at time $t = 0$ is (x_0, y_0).

PROBLEM 15

In spherical coordinates, (r, θ, ϕ), the equations of motion for an inviscid fluid, Euler's equations, become:

$$\rho\left(\frac{\mathrm{D}u_r}{\mathrm{D}t} - \frac{u_\theta^2 + u_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + f_r$$

$$\rho\left(\frac{\mathrm{D}u_\theta}{\mathrm{D}t} + \frac{u_\theta u_r}{r} - \frac{u_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta$$

$$\rho\left(\frac{\mathrm{D}u_\phi}{\mathrm{D}t}+\frac{u_\phi u_r}{r}+\frac{u_\theta u_\phi \cot\theta}{r}\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+f_\phi$$

where u_r, u_θ, u_ϕ are the velocities in the r, θ, ϕ directions, p is the pressure, ρ is the fluid density and f_r, f_θ, f_ϕ are the body force components. The Lagrangian or material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

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For an incompressible fluid the equation of continuity in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}=0$$

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An underwater explosion creates a purely radial flow ($u_\theta = u_\phi = 0$ and $\partial/\partial\theta = 0$ and $\partial/\partial\phi = 0$) in water surrounding a bubble whose radius, denoted by $R(t)$, is increasing with time. Since the u_r velocity at the surface of the bubble must be equal to $\mathrm{d}R/\mathrm{d}t$ show that the equation of continuity requires that

$$u_r=\frac{R^2}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t}$$

Assume that the water is incompressible. Also note that, since R is a function only of time, there is no ambiguity about its time derivative and hence $\mathrm{d}R/\mathrm{d}t$ is just an ordinary time derivative.

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Now use the equations of motion to find the pressure, $p(r,t)$, at any position, r, in the water. Neglect all body forces. One integration step has to be performed which introduces an integration constant; this can be evaluated by assuming the pressure far from the bubble ($r \to \infty$) is known (denoted by p_∞).

Finally show that, if one neglects surface tension so that the pressure in the bubble, p_B, is the same as the pressure in the water at $r = R$, then

$$p_B - p_\infty = \rho\left[R\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right]$$

This is known as the Rayleigh equation for bubble dynamics.

PROBLEM 16

The following is the streamfunction for a particular steady, planar, incompressible and inviscid flow:

$$\psi = A(x^2y - y^3/3)$$

where A is a known constant.

(a) Find expressions for the velocity components u and v in this flow.

(b) Find an expression for the vorticity.

(c) Make a rough sketch of the streamlines of this flow.

(d) Find an expression for the pressure in this flow assuming that the pressure, p , at the origin is known. Denote the fluid density by ρ and neglect all body forces. What shape are the lines of constant pressure (isobars) ?