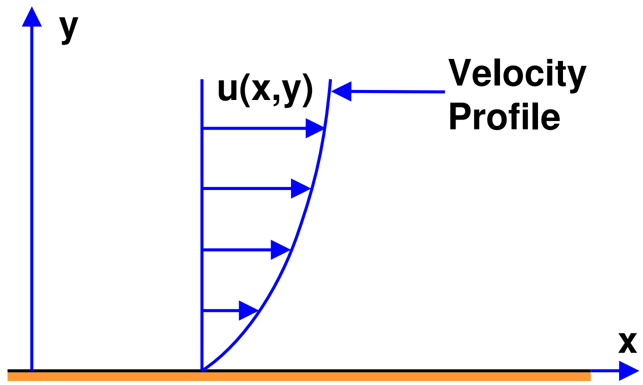
ME19a. **SOLUTIONS.** **Oct. 20, 2009. Due Oct. 29**

PROBLEM 13

The velocity, u, in the x direction for a planar incompressible shear flow near a wall as shown in the following sketch,



and is given by the expression

$$u = U\left(\frac{2y}{ax} - \frac{y^2}{a^2x^2}\right)$$

where a is a constant. Find the corresponding expression for the velocity, v, assuming that $v = 0$ at the wall, $y = 0$.

SOLUTION 13

Since the velocity, u, in the x direction for this planar incompressible flow is

$$u = U\left(\frac{2y}{ax} - \frac{y^2}{a^2x^2}\right)$$

where a is a constant. Since $u = \partial\psi/\partial y$, where ψ is the streamfunction, it follows that

$$\frac{\partial\psi}{\partial y} = U\left(\frac{2y}{ax} - \frac{y^2}{a^2x^2}\right)$$

and this can be integrated with respect to y to yield

$$\psi = U\left(\frac{y^2}{ax} - \frac{y^3}{3a^2x^2}\right) + c(x)$$

where $c(x)$ is the integration constant, an unknown function of x alone. Then, differentiating with respect to x we obtain the velocity, v, in the y direction:

$$v = -\frac{\partial\psi}{\partial x} = U\left(\frac{y^2}{ax^2} - \frac{2y^3}{3a^2x^3}\right) + \frac{dc}{dx}$$

where dc/dx will also just be a function of x.

But we also know that, at the wall $y = 0$, we must have zero velocity, v, normal to the wall and therefore, from the last equation, dc/dx must be zero at the wall, $y = 0$. But since dc/dx is only a function of x dc/dx must therefore be zero everywhere and hence

$$v = -\frac{\partial \psi}{\partial x} = U\left(\frac{y^2}{ax^2} - \frac{2y^3}{3a^2x^3}\right)$$

PROBLEM 14

A particular planar, incompressible flow is given by:

$$\psi = Axyt$$

where A is constant in time and space.

(a) Sketch the streamlines for this flow at a particular instant in time (say $t = 1$). What is the typical equation for such a streamline?

(b) Write down expressions for the velocity components, $u(x, y, t)$ and $v(x, y, t)$.

(c) Find the parametric equations, $x(x_0, y_0, t)$ and $y(x_0, y_0, t)$, for the pathline of a particle whose position at time $t = 0$ is (x_0, y_0).

SOLUTION 14

The streamfunction for planar incompressible flow is given by

$$\psi = Axyt$$

where A is a known constant in time and space.

a) For $t = 1$, we get $\psi = Axy$. These are hyperbolic functions, typically given by

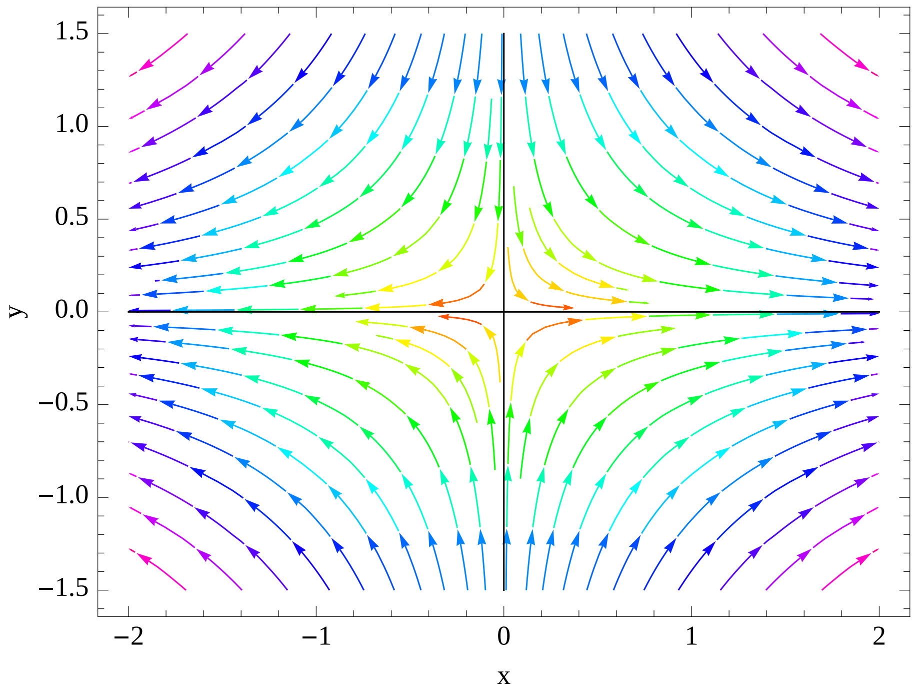
$$\psi = Axy = const$$
$$\rightarrow y = \frac{B}{x},$$

with constant B. The streamlines are shown below (for $A = 1$, $t = 1$)
b) Velocity:

$$u(x, y, t) = \frac{\partial \psi}{\partial y} = Axt$$
$$v(x, y, t) = -\frac{\partial \psi}{\partial x} = -Ayt$$

c) For a Lagrangian element,

$$u = \frac{dx}{dt} = Axt$$
$$v = \frac{dy}{dt} = -Ayt$$



Integrating from 0 to t and from x_0 or y_0 to x or y,

$$x = x_0 e^{At^2/2}$$
$$y = y_0 e^{-At^2/2}$$

PROBLEM 15

In spherical coordinates, (r, θ, ϕ), the equations of motion for an inviscid fluid, Euler's equations, become:

$$\rho \left(\frac{\mathrm{D}u_r}{\mathrm{D}t} - \frac{u_\theta^2 + u_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} + f_r$$

$$\rho \left(\frac{\mathrm{D}u_\theta}{\mathrm{D}t} + \frac{u_\theta u_r}{r} - \frac{u_\phi^2 \cot\theta}{r} \right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta$$

$$\rho \left(\frac{\mathrm{D}u_\phi}{\mathrm{D}t} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi \cot\theta}{r} \right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + f_\phi$$

where u_r, u_θ, u_ϕ are the velocities in the r, θ, ϕ directions, p is the pressure, ρ is the fluid density and f_r, f_θ, f_ϕ are the body force components. The Lagrangian or material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

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For an incompressible fluid the equation of continuity in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$$

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An underwater explosion creates a purely radial flow ($u_\theta = u_\phi = 0$ and $\partial/\partial\theta = 0$ and $\partial/\partial\phi = 0$) in water surrounding a bubble whose radius, denoted by $R(t)$, is increasing with time. Since the u_r velocity at the surface of the bubble must be equal to $\mathrm{d}R/\mathrm{d}t$ show that the equation of continuity requires that

$$u_r = \frac{R^2}{r^2}\frac{\mathrm{d}R}{\mathrm{dt}}$$

Assume that the water is incompressible. Also note that, since R is a function only of time, there is no ambiguity about its time derivative and hence $\mathrm{d}R/\mathrm{d}t$ is just an ordinary time derivative.

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Now use the equations of motion to find the pressure, $p(r,t)$, at any position, r, in the water. Neglect all body forces. One integration step has to be performed which introduces an integration constant; this can be evaluated by assuming the pressure far from the bubble ($r \to \infty$) is known (denoted by p_∞).

Finally show that, if one neglects surface tension so that the pressure in the bubble, p_B, is the same as the pressure in the water at $r = R$, then

$$p_B - p_\infty = \rho\left[R\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right]$$

This is known as the Rayleigh equation for bubble dynamics.

SOLUTION 15

Purely radial flow $\Rightarrow u_\theta = u_\phi = 0, \frac{\partial}{\partial\theta} = 0, \frac{\partial}{\partial\phi} = 0$
continuity: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$

Since the flow is purely radial, this reduces to:

$$\frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$$

Integrating with respect to r:

$$r^2 u_r = f(t)$$

At $r = R(t), u_r = \frac{\mathrm{d}R}{\mathrm{d}t}$ so:

$$f(t) = R^2\frac{\mathrm{d}R}{\mathrm{d}t} \qquad \Rightarrow \qquad u_r = \frac{R^2}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t}$$

For purely radial flow, Euler's equations in the θ and ϕ directions are automatically satisfied. In the r direction, the equation reduces to:

$$\rho\frac{\mathrm{D}u_r}{\mathrm{D}t} = \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r}\right) = -\frac{\partial p}{\partial r}$$

Substituting the expression derived for u_r:

$$-\frac{\partial p}{\partial r} = \rho\left(\left[\frac{2R}{r^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + \frac{R^2}{r^2}\frac{\mathrm{d}^2R}{\mathrm{d}t^2}\right] + \frac{R^2}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t}\left[-2\frac{R^2}{r^3}\frac{\mathrm{d}R}{\mathrm{d}t}\right]\right)$$

Separating and integrating:

$$\int \partial p = \int -\rho \left(\frac{1}{r^2} \left[2R \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 + R^2 \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} \right] - 2\frac{R^4}{r^5} \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 \right) \partial r$$

$$\Rightarrow p(r,t) = \rho \left(\frac{1}{r} \left[2R \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 + R^2 \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} \right] - \frac{1}{2} \frac{R^4}{r^4} \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 \right) + c(t)$$

The unknown function $c(t)$ is evaluated as $r \to \infty$:

$$p(r \to \infty, t) = c(t) = p_\infty$$

$$\Rightarrow p(r,t) = \rho \left(\frac{1}{r} \left[2R \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 + R^2 \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} \right] - \frac{1}{2} \frac{R^4}{r^4} \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 \right) + p_\infty$$

p_B is equal to $p(r,t)$ evaluated at $r = R$:

$$p_B = p(R,t) = \rho \left[2 \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 + R \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} - \frac{1}{2} \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 \right] + p_\infty$$

$$\Rightarrow p_B - p_\infty = \rho \left[\frac{3}{2} \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 + R \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} \right]$$

PROBLEM 16

The following is the streamfunction for a particular steady, planar, incompressible and inviscid flow:

$$\psi = A(x^2 y - y^3/3)$$

where A is a known constant.

(a) Find expressions for the velocity components u and v in this flow.

(b) Find an expression for the vorticity.

(c) Make a rough sketch of the streamlines of this flow.

(d) Find an expression for the pressure in this flow assuming that the pressure, p , at the origin is known. Denote the fluid density by ρ and neglect all body forces. What shape are the lines of constant pressure (isobars) ?

SOLUTION 16

The streamfunction for planar incompressible flow is given by

$$\psi = A(x^2 y - y^3/3)$$

where A is a known constant.

a) $u = \frac{\partial \psi}{\partial y} = A(x^2 - y^2)$
$v = -\frac{\partial \psi}{\partial x} = -2Axy$

b) Vorticity:

$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,$

c) Solve for $\psi = 0$:

$$y(x^2 - y^2/3) = 0$$
$$\to y = 0 \text{ or } y = \pm\sqrt{3}x,$$

Along $y = 0$:

$u = Ax^2$

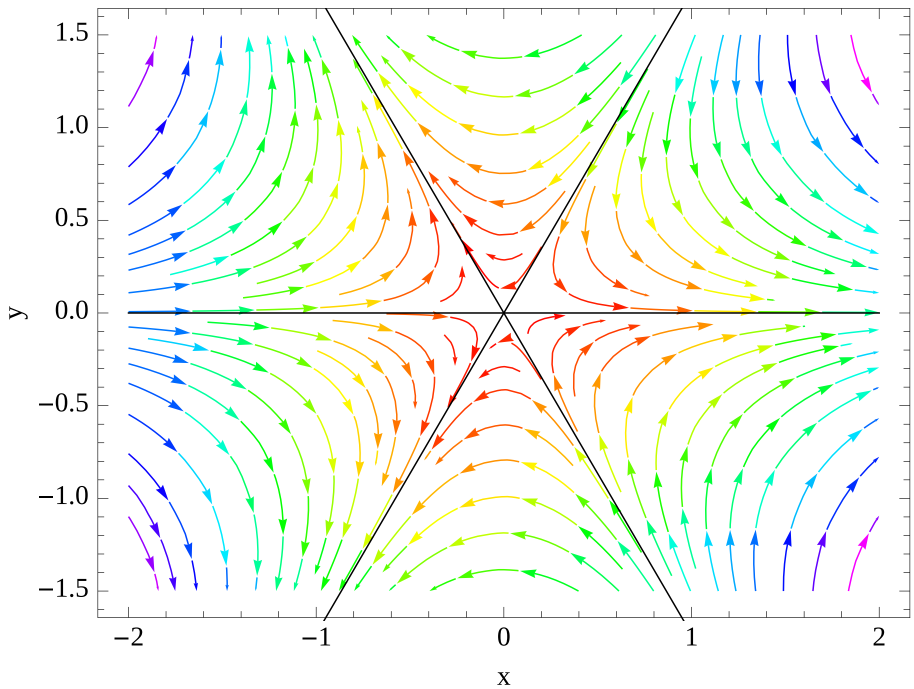
$v = 0 \to$ at $(0,0)$, $(u, v) = (0,0)$

Along $x = 0$:

$u = -Ay^2$

$v = 0 \to$ at $(0,0)$, $(u, v) = (0,0)$

The streamlines are shown below (for $A = 1$)



d) Pressure:

The flow is irrotational, inviscid and incompressible so we will use Bernoulli's eqn:

$$p + \frac{1}{2}\rho|u|^2 = const$$
$$|u|^2 = u^2 + v^2 = A^2(x^2 + y^2)^2$$
$$\therefore\ p + \frac{1}{2}\rho A^2(x^2 + y^2)^2 = const$$

Set $p = p_0$ at $(x, y) = (0,0)\ \to\ const = p_0$

$$p = p_0 - \frac{1}{2}\rho A^2(x^2 + y^2)^2$$

A line of constant pressure is a circle centered at the origin.

Alternatively, you can solve for the pressure from the equations of motions for an inviscid, incompressible fluid. The two of interest are:

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}$$
$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}$$

The steady-flow assumption means $\partial/\partial t = 0$ so that only convective terms are left in the Lagrangian derivative. The two equations become:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}$$
$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}$$

Now we use $u = A(x^2 - y^2)$ and $v = -2Axy$ from part a) and take appropriate derivatives to obtain two coupled partial differential equations for the pressure p. For the x component:

$$\rho\left[A(x^2 - y^2)(2Ax) + (-2Axy)(-2Ay)\right] = -\frac{\partial p}{\partial x}$$
$$\therefore\ \frac{\partial p}{\partial x} = -2\rho A^2\left(x^3 + xy^2\right)$$

For the y component:

$$\rho\left[A(x^2 - y^2)(-2Ay) + (-2Axy)(-2Ax)\right] = -\frac{\partial p}{\partial y}$$
$$\therefore\ \frac{\partial p}{\partial y} = -2\rho A^2\left(x^2 y + y^3\right)$$

How do we solve this system of coupled PDE's? Let's start by integrating the expression for $\partial p/\partial x$ with respect to x to obtain

$$p = -2\rho A^2\left(\frac{x^4}{4} + \frac{x^2y^2}{2}\right) + c(y)$$

We don't know what the function $c(y)$ is, but we can differentiate the expression for p above with respect to y and set this equal to the relation for $\partial p/\partial y$ we obtained from the equations of motion.

$$\frac{\partial p}{\partial y} = -2\rho A^2(x^2y) + c'(y) = -2\rho A^2\left(x^2y + y^3\right)$$

Immediately we see that $c'(y) = -2\rho A^2 y^3$ so that integration gives us

$$c(y) = -2\rho A^2\left(\frac{y^4}{4}\right) + c$$

where c now represents a constant to be determined from the boundary conditions. Substituting this into the equation for p

$$\begin{aligned} p &= -2\rho A^2\left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right) + c \\ &= -\frac{1}{2}\rho A^2(x^4 + 2x^2y^2 + y^4) + c \\ &= -\frac{1}{2}\rho A^2(x^2 + y^2)^2 + c \end{aligned}$$

Using $p = p_0$ at $(x, y) = (0, 0) \rightarrow c = p_0$ and the final expression is exactly the same as the one we obtained through Bernoulli's equation

$$p = p_0 - \frac{1}{2}\rho A^2(x^2 + y^2)^2$$

Remember, Bernoulli's equation only works here because the flow is irrotational (vorticity $\omega = 0$), inviscid (no viscous forces, shear layers, etc.), and incompressible (constant density ρ). You can always start with the full equations of motion, make the necessary assumptions, and proceed from there. Generally that is the best starting point.