ME19a. **SOLUTIONS.** **Oct. 13, 2009. Due Oct. 20**

PROBLEM 9

Consider the flow of a fluid in which the fluid elements are traveling with velocity, u, in the x direction (this is the only non-zero velocity of the fluid which it is necessary to consider in this problem). A succession of fluid elements travel through the Eulerian point, $x = x_o$ with a velocity $u = u_o$ and subsequently accelerate according to

$$u = (u_o/x_o^2)x^2$$

However the flow is **steady**. Chemical constituents within the fluid are reacting in such a way that the concentration, c, of one of the constituents is increasing with time at a rate denoted by α (a constant). If the concentration at the point $x = x_o$ has a known and constant value denoted by c_o find an expression for the concentration elsewhere as a function of x, x_o, u_o, c_o and α.

SOLUTION 9

Since chemical constituents are carried along with the fluid, they follow material path lines described by the Lagrangian time derivative

$$\frac{\mathrm{D}c}{\mathrm{D}t} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = \alpha.$$

Here, we see the total rate of change of c with respect to time depends explicitly on the time-derivative $\frac{\partial c}{\partial t}$ (nonzero for unsteady flow, for sources/sinks, etc.), and also depends on the convection of c at velocity u in the flow. But the flow is steady and therefore $\frac{\partial c}{\partial t} = 0$ and $c(x)$ is only a function of x. Therefore,

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{\alpha}{u} = \frac{\alpha}{u_0}\left(\frac{x_0}{x}\right)^2$$

Integrating,

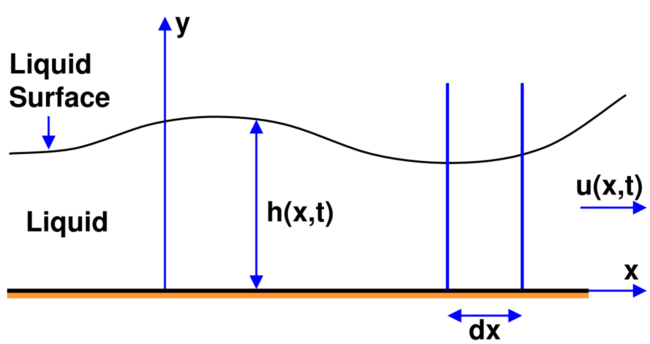
$$c = -\frac{\alpha x_0^2}{u_0}\frac{1}{x} + constant$$

But $c = c_0$ at $x = x_0$. Therefore

$$c - c_0 = \frac{\alpha x_0^2}{u_0}\left[\frac{1}{x_0} - \frac{1}{x}\right]$$

PROBLEM 10

Construct from first principles an equation for the conservation of mass which governs the planar flow (in the xy plane) of an incompressible liquid lying on a flat horizontal plane:



The depth, $h(x,t)$, is a function of x and time, t. Examine an Eulerian element of width, $\mathrm{d}x$, as shown above (it extends from $y = 0$ to $y = \infty$) and assume that the velocity, $u(x,t)$, of the water in the positive x direction is **independent of** y. Then utilize conservation of mass to obtain a partial differential equation connecting the depth, $h(x,t)$, and the velocity, $u(x,t)$. Neglect surface tension. [This is one of the equations of what is known as "shallow water wave theory".]

SOLUTION 10

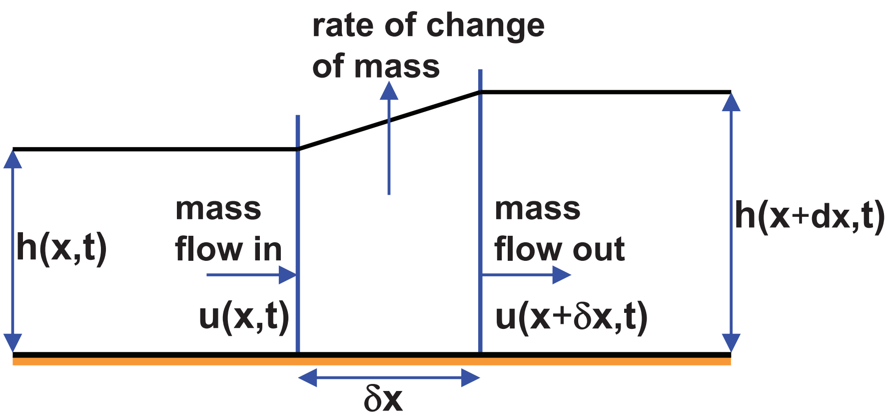
To find an equation for the shallow water wave, a mass balance for the element δx will be used. The mass balance is given by

$$mass\ flow\ in - mass\ flow\ out = rate\ of\ change\ of\ mass\ in\ \delta x$$

or

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\mathrm{d}\dot{m}_{\delta x}}{\mathrm{d}t}.$$

The three contributions can be identified as:



- $\dot{m}_{in}$, mass flow in per unit depth into the page:

  $\rho u(x,t)h(x,t) = \rho uh$

- $\dot{m}_{out}$, mass flow out per unit depth into the page:

  $\rho u(x+\delta x,t)h(x+\delta x,t) \approx \rho \left[u(x,t) + \frac{\partial u}{\partial x}\delta x\right] \left[h(x,t) + \frac{\partial h}{\partial x}\delta x\right] \approx \rho \left[uh + h\frac{\partial u}{\partial x}\delta x + u\frac{\partial h}{\partial x}\delta x\right] = \rho \left[uh + \frac{\partial (uh)}{\partial x}\delta x\right]$

  Note that the term involving $(\delta x)^2$ can be neglected for small δx.

- $\mathrm{d}\dot{m}_{\delta x}/\mathrm{d}t$, rate of change of mass in δx per unit depth into the page:

  $\rho \frac{\partial h}{\partial t}\delta x$

  Note that if we were sitting at fixed x position and watching the flow, the depth $h(x,t)$ would be changing in time. This change is a result of the difference of mass flow in and out of the control volume.

The three contributions can be substituted into the mass conservation relation to yield

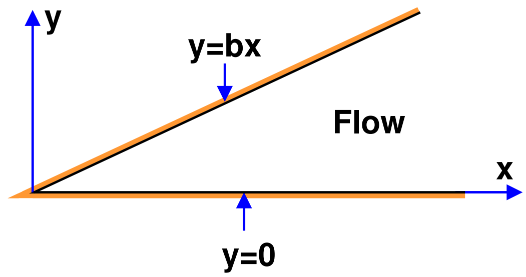
$$\begin{aligned} uh - \left[uh + \frac{\partial (uh)}{\partial x}\delta x\right] &= \frac{\partial h}{\partial t}\delta x \\ -\frac{\partial (uh)}{\partial x}\delta x &= \frac{\partial h}{\partial t}\delta x \end{aligned}$$

or

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

PROBLEM 11

A planar, incompressible flow within a wedge-shaped region bounded by solid walls at $y = 0$ and $y = bx$ has a velocity, u, in the x direction given by $u = A(y - ax)$ where A and a are constants:



Find expressions for the streamfunction, ψ, and the velocity, v, in the y direction. Determine the relation between a and b. Sketch some of the streamlines of the flow.

[Do **not** use the no-slip condition which is violated in the above problem. Later, in class, we shall discuss the issues associated with the no-slip condition.]

SOLUTION 11

Given $u = A(y - ax)$
Recall that $u = \frac{\partial \psi}{\partial y}$, which implies that $\frac{\partial \psi}{\partial y} = A(y - ax)$
Integrating the equation, we find that

$$\psi = \frac{Ay^2}{2} - Aaxy + c(x)$$

Solving for v,

$$v = -\frac{\partial \psi}{\partial x} = Aay + c'(x)$$

Apply the boundary condition that $v|_{y=0} = 0 \rightarrow c'(x) = 0$, so $c(x)$ is an arbitrary constant, which we can set to zero. The streamfunction and velocities are

$$\begin{aligned} \psi &= \frac{Ay^2}{2} - Aaxy \\ u &= A(y - ax) \\ v &= Aay \end{aligned}$$

Consider the effects of the sloping wall

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = b = \frac{v}{u} &= \frac{Aay}{A(y - ax)}, \text{ where } y = bx \\ b &= \frac{Aabx}{A(bx - ax)} \\ b &= \frac{ab}{(b - a)} \\ \therefore\ b &= 2a \end{aligned}$$

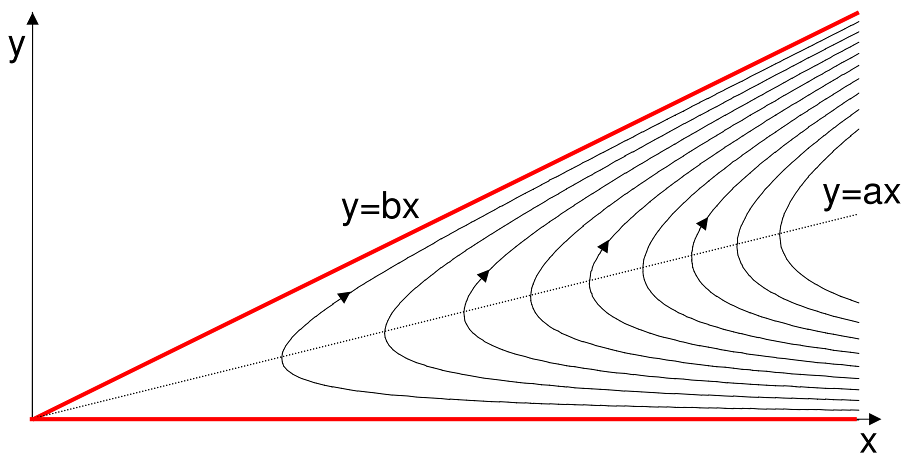
Or, we note that the stream function ψ is constant along boundaries, so that

$$\begin{aligned}\psi = constant &= \frac{Ay^2}{2} - Aaxy, \text{ where } y = bx \\ constant &= \frac{Ab^2x^2}{2} - Aabx^2 \\ constant &= Ab\left(\frac{b}{2}x^2 - ax^2\right)\end{aligned}$$

But this can only be true if the term in parenthesis is zero (otherwise, something that is supposed to be equal to a constant would depend on x^2).

$$\begin{aligned}0 &= \frac{b}{2}x^2 - ax^2 \\ \therefore\ b &= 2a\end{aligned}$$

The u velocity is zero everywhere on $y = ax$ (from the equation for u). The relationship between b and a tells us that u will always be zero along a line at the half-angle between the bottom wall and the line $y = bx$ that defines our wedge shape. Thus, streamlines must be vertical along this line.



PROBLEM 12

Consider the following streamfunction, ψ, for a planar incompressible flow:

$$\psi = Ur\left(1 - \frac{r_0^2}{r^2}\right)\sin\theta$$

where U and r_0 are constants and r, θ are polar coordinates.

(a) Find and sketch the streamline corresponding to $r = r_0$.

(b) Find and add to your sketch the streamlines for $\theta = 0, r > r_0$ and for $\theta = \pi, r > r_0$. Note on your sketch the value of ψ along these lines and along the streamline for $r = r_0$.

(c) Make a rough estimate of some other streamlines with $\psi > 0$ and show the form of these streamlines in your sketch.

(d) What is the magnitude and direction of the flow for $r \gg r_0$?

(e) Guided by your sketch, estimate what real flow might have the above streamfunction.

Note: In polar coordinates, the velocities in the r and θ directions, denoted respectively by u_r and u_θ, are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad ; \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

SOLUTION 12

The streamfunction for planar incompressible flow is given by

$$\psi = Ur\left(1 - \frac{r_0^2}{r^2}\right)\sin\theta$$

where U and r_0 are constants and r, θ are polar coordinates. The velocities, given by the derivatives of the streamfunction are

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}Ur\left(1 - \frac{r_0^2}{r^2}\right)\cos\theta = U\left(1 - \frac{r_0^2}{r^2}\right)\cos\theta$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\left[U\left(1 - \frac{r_0^2}{r^2}\right) + Ur\left(2\frac{r_0^2}{r^3}\right)\right]\sin\theta = -U\left(1 + \frac{r_0^2}{r^2}\right)\sin\theta$$

(a) $u_r|_{r=r_0} = 0$
$u_\theta|_{r=r_0} = -2U\sin\theta$

(b) $\theta = 0$, $r > r_0$: $\psi = 0$, $u_r = U\left(1 - \frac{r_0^2}{r^2}\right)$, $U_\theta = 0$
$\theta = \pi$, $r > r_0$: $\psi = 0$, $u_r = -U\left(1 - \frac{r_0^2}{r^2}\right)$, $U_\theta = 0$

(c) $\theta = \frac{\pi}{2}$: $u_r = 0$, $u_\theta = -U\left(1 + \frac{r_0^2}{r^2}\right)$

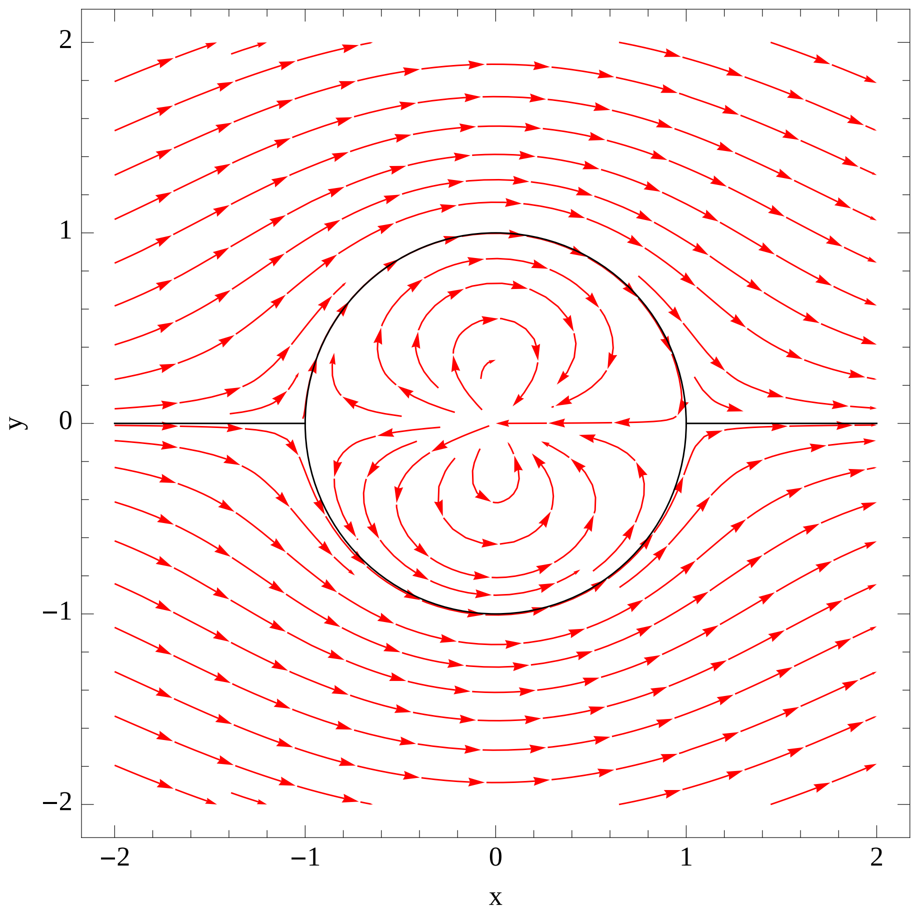
(d) $r \gg r_0$: $u_r \to U\cos\theta$ $u_\theta \to -U\sin\theta$

Magnitude: $|\vec{u}| = \sqrt{u_r^2 + u_\theta^2} = U$
Direction: transform into Cartesian coordinates

$u_x = u_r\cos\theta - u_\theta\sin\theta = U(\cos^2\theta + \sin^2\theta) = U$
$u_y = u_r\sin\theta + u_\theta\cos\theta = 0$

The far field looks like a uniform stream U in the x direction.

(e) The flow around a stationary cylinder. The streamlines are shown below (for $U = 1$, $r_0 = 1$).



2
1
0
−1
−2
y
−2
−1
0
1
2
x