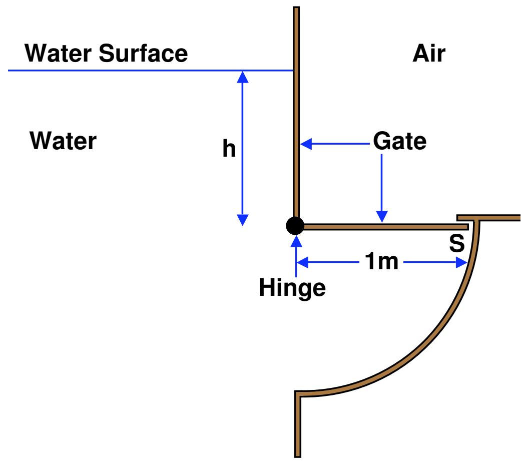
PROBLEM 5

The L-shaped gate shown in the figure can rotate about the hinge. As the water level rises the gate will automatically open when the water level rises to a certain height, h. If the length of the lower arm of the gate is 1 m find the critical height, h. (Assume that when the gate is closed, as shown in the figure, there is a seal at the point S. Neglect the weight of the gate.)



SOLUTION 5

There are two forces acting on the L-shaped gate both due to pressure:

(1) For the horizontal part, p_A acts down from the air side, and a pressure of $p_A + \rho g h$ acts up from the water side. The net pressure is therefore, $p_1 = \rho g h$ acting up. A differential force, $\mathrm{d}F_1$, on the horizontal part follows from the pressure

$$\mathrm{d}F_1 = p_1 b \mathrm{d}x = \rho g h b \mathrm{d}x,$$

where $\mathrm{d}x$ is a differential length along the horizontal section, and b is the dimension into the page (this is arbitrary).

The moment about the hinge due to this force acts in the counter-clockwise direction and is

$$M_1 = \int x \mathrm{d}F_1 = \rho g h b \int_0^L x \mathrm{d}x = \rho g h b \frac{L^2}{2},$$

where $L = 1$ m is the length of the horizontal section

(2) For the vertical section of the gate, p_A acts to the left from the air side, and a pressure that varies with depth $p_A + \rho g y$ acts to the right from the water side. The net pressure at depth y (measured from the water surface) is $p_2 = \rho g y$ pushing to the right. The associated force is

$$\mathrm{d}F_2 = p_2 b \mathrm{d}y = \rho g y b \mathrm{d}y,$$

The moment about the hinge due to this force acts in the clockwise direction (opposite from M_1) and can be found from the force as

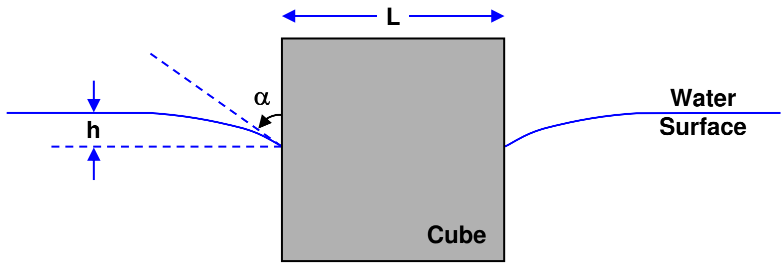
$$M_2 = \int (h-y) \mathrm{d}F_2 = \rho g b \int_0^h (h-y) y \mathrm{d}y = \rho g b (h^3/2 - h^3/3) = \rho g b \frac{h^3}{6}$$

As h increases, M_2 increases faster than M_1 (h^3 versus h, respectively). The gate opens when $M_2 = M_1$, so that the critical height, h, is:

$$h = \sqrt{3}L = 1.73\ m$$

PROBLEM 6

A cube with sides, L, and density, ρ_S, floats in a pool of water whose density is ρ_L and whose surface tension is S. The acceleration due to gravity is denoted by g. The cube is made of hydrophobic material with a contact angle of $\pi - \alpha$ so that it floats in the following configuration:



Because the material is hydrophobic the density of the cube can be greater than that of the water and it will still float. Assuming that

[1] $\alpha = \pi/4$

[2] The surface tension, S, is such that $S/(\rho_L g L^2) = 0.1$

[3] The elevation difference, h, between the line of contact on the sides of the cube and the water surface far from the cube is given by $h = (S/(\rho_L g))^{1/2} \cot\alpha$.

[4] It is stipulated that the water surface can only contact the cube along the vertical faces of that cube.

determine the maximum specific gravity of the cube (ρ_S/ρ_L) for which the cube will still float.

SOLUTION 6

Three vertical forces act on the cube:

- The weight of the cube, $\rho_s g L^3$
- The vertical component of the surface tension force acting along the contact line on the sides of the cube, $4LS\cos\alpha$
- The combination of the atmospheric pressure on the top of the cube and the water pressure on the bottom of the cube. The pressure on the bottom is atmospheric pressure plus $\rho_L g(h + x)$ where x is the distance from the bottom up the side to the contact line. The upward force will be greatest when x is greatest and this condition will support the heaviest cube. Thus the heaviest cube will be supported when $x = L$. Under this condition, the upward force resulting from the pressures on the top and bottom is $\rho_L g(h+L)L^2$

The balance of these three forces yields

$$\rho_L g(h + L)L^2 + 4LS\cos\alpha = \rho_S g L^3$$

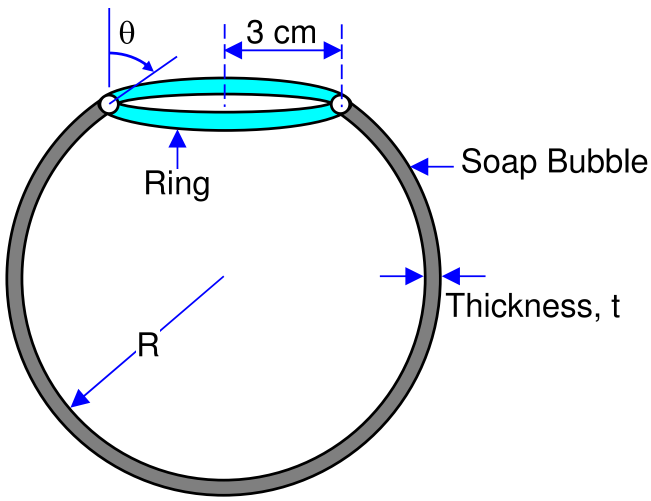
Substituting for h and α and doing some algebra will give

$$\rho_S/\rho_L = 1 + (S/(\rho_L g L^2))^{1/2} + 4(S/(\rho_L g L^2))/2^{1/2}$$

which for $S/(\rho_L g L^2) = 0.1$ yields $\rho_S/\rho_L = 1.6$

PROBLEM 7

A soap bubble hangs from a horizontal circular ring of radius equal to 3 cm:



The mass of the soapy water comprising the bubble is 0.0014 kg. Assuming the bubble is spherical and neglecting any contact angle effects at the junction of the ring and the bubble, find:

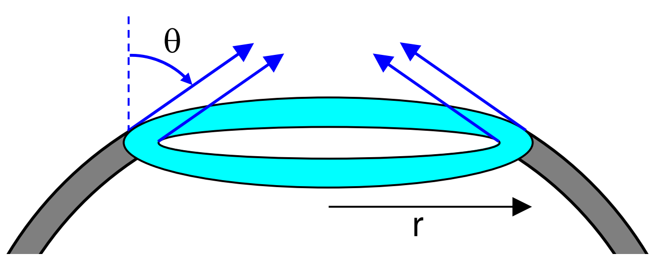
[1] The angle, θ, between a tangent to the bubble at the ring and the vertical if the surface tension of the soapy water is 0.05 kg/s^2. Assume the acceleration due to gravity is 9.8 m/s^2.

[2] The radius of the soap bubble.

[3] The thickness of the soap bubble given that the density of the soapy water is 1000 kg/m^3.

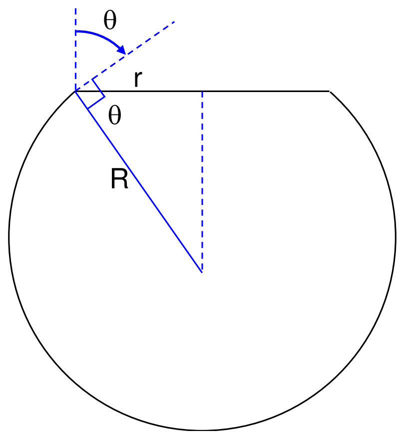
SOLUTION 7

[1] The weight of the soap film must balance the component of the surface tension force, S. Note that there are two surfaces to consider.



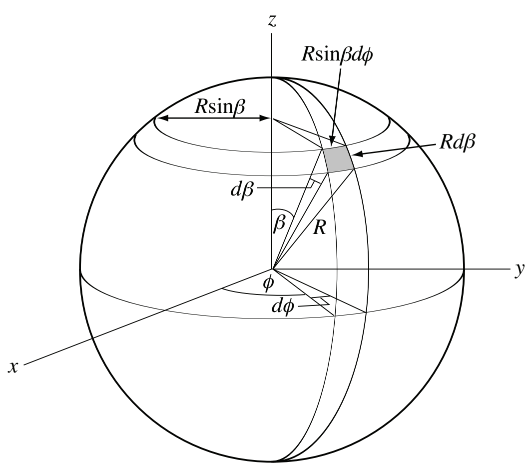
$$
\begin{aligned}
mg &= 2(2\pi r S\cos\theta) \\
\theta &= \cos^{-1}\left[\frac{mg}{4\pi r S}\right] \\
&= \cos^{-1}\left[\frac{(0.0014\ kg)(9.81\ m/s^2)}{4\pi(0.03\ m)(0.05\ kg/s^2}\right] \\
\rightarrow \theta &\approx 43.3^\circ
\end{aligned}
$$

[2] The radius of the soap bubble can be found with a bit of geometry:



$$
\begin{aligned}
\cos\theta &= \frac{r}{R} \\
R &= \frac{r}{\cos\theta} \\
&= \frac{3\ cm}{\cos 43.3^\circ} \\
\rightarrow R &\approx 4.12\ cm
\end{aligned}
$$

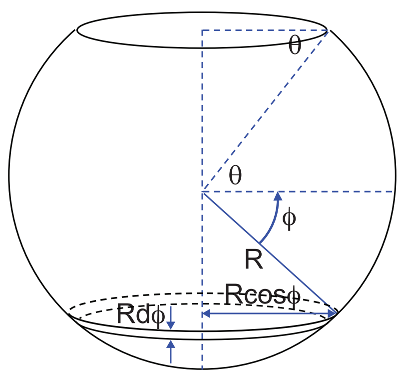
[3] The thickness of the soap bubble can be found by using the density of the soapy water ($\rho = 1000\ kg/m^3$) and the mass ($m = .0014\ kg$). By assuming $t \ll R$, the approximate volume is given by the (surface area)$\times$(thickness).



Consider the system of spherical polar coordinates given in the figure. Typically, to cover the entire surface of the sphere, the range of angles in the figure are $0 \leq \beta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The differential area of the shaded surface element is $R^2 \sin\beta \mathrm{d}\phi \mathrm{d}\beta$. For the bubble geometry, we want to integrate over the entire range of ϕ, but we only want to integrate β from $\pi/2 - \theta$, to the maximum angle π. Thus,

$$
\begin{aligned}
A(\theta) &= \int_{\frac{\pi}{2}-\theta}^{\pi} \int_0^{2\pi} R^2 \sin\beta \mathrm{d}\phi \mathrm{d}\beta \\
&= 2\pi R^2 \int_{\frac{\pi}{2}-\theta}^{\pi} \sin\beta \mathrm{d}\beta \\
&= 2\pi R^2 \left(\cos\left[\frac{\pi}{2} - \theta\right] - \cos\pi\right) \\
&= 2\pi R^2 (\sin\theta + 1) \\
\therefore m &= \rho V \approx \rho A(\theta) t \\
\rightarrow t &= \frac{m}{\rho A(\theta)} = \frac{0.0014\ kg}{(1000\ kg/m^3)(2\pi)(0.0412\ m)^2(\sin[43.3°] + 1)} \\
t &= 7.78 \times 10^{-5}\ m
\end{aligned}
$$

Alternatively (and perhaps a bit more confusing), the surface area can be found by a similar integration over the angle ϕ defined in the figure below. The horizontal rings have circumference $2\pi R \cos\phi$ and differential thickness $R\mathrm{d}\phi$ where the integration angle ϕ varies from $-\pi/2$ to the contact angle θ:



$$
\begin{aligned}
A(\theta) &= 2\pi R^2 \int_{-\pi/2}^{\theta} \cos\phi\ \mathrm{d}\phi \\
&= 2\pi R^2 (\sin\theta + 1) \\
\therefore m &= \rho V \approx \rho A(\theta) t \\
\rightarrow t &= \frac{m}{\rho A(\theta)} = \frac{0.0014\ kg}{(1000\ kg/m^3)(2\pi)(0.0412\ m)^2(\sin[43.3°] + 1)} \\
t &= 7.78 \times 10^{-5}\ m
\end{aligned}
$$

Yet another method (and perhaps the most accurate, but definitely more math-intensive) is to find the difference in volume between the outer and inner parts of the bubble. Using the same spherical coordinates as in the first method, we now include integration over the radius to give

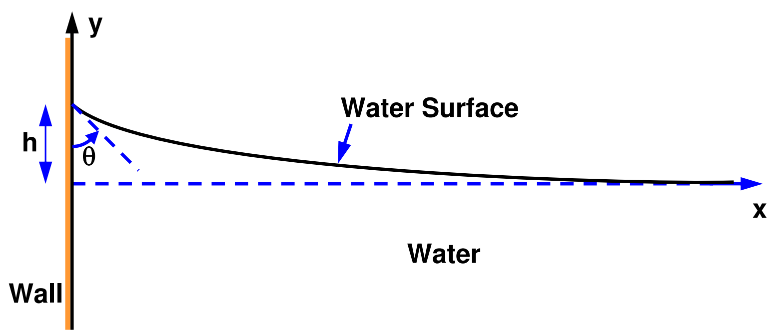
$$
\begin{aligned}
V_{out}(\theta) &= \int_{\frac{\pi}{2}-\theta}^{\pi} \int_0^{2\pi} \int_0^{R+t} r^2 \sin\beta \mathrm{d}r \mathrm{d}\phi \mathrm{d}\beta \\
&= 2\pi \frac{(R+t)^3}{3} \int_{\frac{\pi}{2}-\theta}^{\pi} \sin\beta \mathrm{d}\beta
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi\frac{(R+t)^3}{3}(\sin\theta+1) \\
V_{in}(\theta) &= \int_{\frac{\pi}{2}-\theta}^{\pi}\int_0^{2\pi}\int_0^R r^2\sin\beta \mathrm{d}r\mathrm{d}\phi\mathrm{d}\beta \\
&= 2\pi\frac{R^3}{3}\int_{\frac{\pi}{2}-\theta}^{\pi}\sin\beta\mathrm{d}\beta \\
&= 2\pi\frac{R^3}{3}(\sin\theta+1) \\
\therefore m &= \rho V = \rho(V_{out}(\theta)-V_{in}(\theta)) \\
&= \rho\frac{2\pi}{3}(\sin\theta+1)\left[(R+t)^3-R^3\right]
\end{aligned}
$$

Inverting this result to obtain a function for the thickness t is complicated, but it can be done. It can just as easily be solved numerically to give three possible solutions (because of the 3rd-order dependence on the thickness). Two of the solutions are imaginary, and the third gives $t = 7.777 \times 10^{-5}\ m$. This confirms that the volume is well approximated by (surface area)×(thickness).

PROBLEM 8

A plane wall is immersed in a large body of liquid of density ρ which is at rest:



The surface tension of the liquid surface is denoted by S and the contact angle with the wall by θ. Find the equation of the water surface in the form $y = y(x)$; the function should contain the quantities S, θ, ρ and the acceleration due to gravity, g. To simplify the problem assume that the curvature of the water surface can be approximated by $\mathrm{d}^2y/\mathrm{d}x^2$. Find the height, h, in terms of S, θ, ρ and g.

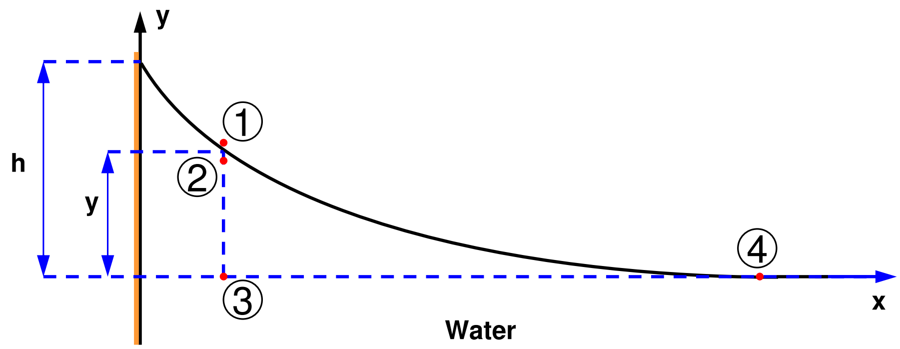
SOLUTION 8

Consider the four points in the figure:

1. On the surface
2. Just below the surface of point 1
3. Vertically below point 2 on the horizontal line
4. Bottom most point of curvature.

The pressure at each point is given as

$$
\begin{aligned}
p_1 &= p_A = p_2 + S/R = p_2 + S\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
p_2 &= p_3 - \rho g y \\
p_3 &= p_A \\
p_4 &= p_A
\end{aligned}
$$



where S is the surface tension and $y = y(x)$ is the equation of the meniscus. In the equation for p_1, we have used the following relationship between the radius of curvature R and the equation describing the meniscus $y = y(x)$

$$\frac{1}{R} = \frac{\mathrm{d}^2 y/\mathrm{d}x^2}{[1 + (\mathrm{d}y/\mathrm{d}x)^2]^{3/2}}.$$

If we assume the gradients are small on the meniscus, i.e. $\mathrm{d}y/\mathrm{d}x \ll 1$, then the denominator of the above relationship is unity and we are left with $1/R \approx \mathrm{d}^2 y/\mathrm{d}x^2$.

At point 4 as $R \to \infty$, $S\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{S}{R} \to 0$. From these equations,

$$\begin{aligned} p_1 = p_A &= p_2 + S\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ &= (p_3 - \rho g y) + S\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ &= p_A - \rho g y + S\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ \to \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\rho g y}{S} \end{aligned}$$

For a nth order, constant-coefficient, homogeneous linear differential equation of the form

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where the subscripts $^{(n)}$ indicate the nth derivative of the function y, we can use the *method of undetermined coefficients* to find a solution. Our equation is of the form $y'' + a_0 y = 0$ where $a_0 = -\rho g/S$, and we try a solution of the form $y = e^{\lambda x}$ to produce the *characteristic equation* (by substitution of $y = e^{\lambda x}$ into $y'' - (\rho g/S)y = 0$)

$$\lambda^2 - \frac{\rho g}{S} = 0.$$

The general solution $y = e^{\lambda x}$ will result in exponential functions for real eigenvalues λ, and trigonometric functions (i.e. cos, sin) for imaginary values of λ. We see here that the two eigenvalues (this is expected because we have a second-order differential equation!) are $\lambda = \pm\sqrt{\rho g/S}$. Each of these is real because ρ, g, and S are all positive quantities.

Therefore, the solution to the second order differential equation has the form:

$$y(x) = Ae^{(\sqrt{\rho g/S})x} + Be^{-(\sqrt{\rho g/S})x}$$

where the constants A and B can be found using the boundary conditions

$$@\ x = 0: \qquad \frac{dy}{dx} = -\cot(\theta)$$

$$\begin{aligned}
@\ x = \infty: &\qquad y = 0 \\
\rightarrow A = 0, &\qquad B = \sqrt{\frac{S}{\rho g}}\cot\theta \\
\therefore &\qquad y = \sqrt{\frac{S}{\rho g}}\cot\theta\ e^{-(\sqrt{\rho g/S})x}
\end{aligned}$$

and

$$h = \sqrt{\frac{S}{\rho g}}\cot\theta$$