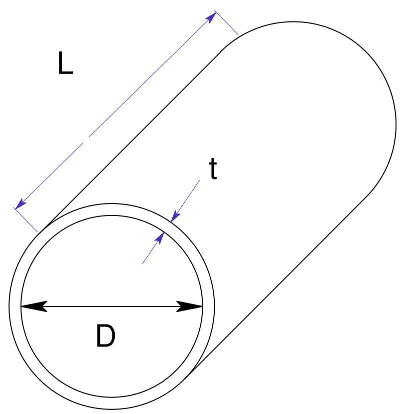
ME19a. **SOLUTIONS.** **Set 1. Due Oct. 6**

PROBLEM 1

A shaft which is 2 cm in diameter and 20 cm long is concentrically located in a journal of the same length and 2.02 cm in diameter. The gap between the shaft and the journal is filled with oil whose dynamic viscosity is 0.1 $kg/m\ s$. Find the torque (in $kg\ m^2/s^2$) and the power (in $kg\ m^2/s^3$; note that 1 $watt$ = 1 $kg\ m^2/s^3$) required to turn the shaft at 6000 rpm (revolutions per minute). Note: treat this as a Couette flow neglecting the curvature of the gap geometry.

SOLUTION 1



It is assumed that the flow is laminar and the fluid is Newtonian, such that the fluid displays a linear dependance between the shear stress and the shear rate. The experiment is conducted with fluid in between two concentric cylinders with one cylinder fixed. As the radius $D/2$ is much larger than the gap width $t << D/2$, Couette flow applies.
In the assignment, the parameters are given as in Table 1.

Parameter	Value	Units
diameter D	0.02	m
length L	0.2	m
thickness t	1×10^{-4}	m
dynamic viscosity μ	0.1	$kg/m\ s$
rotational speed n	6000	rpm

Table 1: Given parameters.

The torque T is defined as the force F multiplied by the radius $D/2$:

$$T = F\frac{D}{2} = \tau A\frac{D}{2}$$

where the shear stress τ acts only in the circumferential direction for Couette flow. The stress τ for this flow can be rewritten as:

$$\tau = \mu\frac{\mathrm{d}U}{\mathrm{d}y} = \mu\frac{U}{t}$$

given the linear velocity profile. The velocity U in this case can be calculated by:

$$U = n\frac{\pi D}{60} = 2\pi\ m/s \approx 6.283\ m/s$$

Thus, the torque is given by:

$$
\begin{aligned}
T &= \tau A \frac{D}{2} \\
&= \left(\mu \frac{U}{t}\right)(\pi D L)\left(\frac{D}{2}\right) \\
&= \mu \frac{n\pi^2 D^3 L}{120 t} \approx 0.790 \ \frac{kg\ m^2}{s^2}
\end{aligned}
$$

The power P is defined as the torque T multiplied by the angular velocity ω:

$$
\begin{aligned}
P &= T\omega \\
&= T \frac{U}{D/2} \\
&= \mu \ \frac{n^2 \pi^3 D^3 L}{3600 t} \approx 496 \ \frac{kg\ m^2}{s^3}
\end{aligned}
$$

PROBLEM 2

A body with a typical length, L, is dragged through a viscous fluid (viscosity, μ, and density, ρ) at a velocity, U. By utilizing only the known dimensions of these quantities (in terms of kg, m and s if you wish) construct two groupings of these quantities which have the units of force. One should contain μ but not ρ; the second should include ρ but not μ.

It could then be argued that the force required to drag the body through the fluid should be related to these two "typical forces". The one which includes μ is a viscous force (F_v) and the other is an inertial force (F_i). Identify the parameter which we can use to determine the conditions under which either F_v or F_i are dominant.

SOLUTION 2

The characteristic parameters of this assignment are given in Table 2.

Parameter	Units
length L	m
dynamic viscosity μ	$kg/m\ s$
density ρ	kg/m^3
velocity U	m/s

Table 2: Given parameters.

The unit of force, Newton, is defined as:

$$N \equiv \frac{kg\ m}{s^2}$$

By combination of the units of the parameters giving in table 2, there are two combinations possible which results to the units of force. The parameter related to the viscous force F_v is:

$$P_1 = U\mu L = \left[\frac{m}{s}\right]\left[\frac{kg}{m\ s}\right][m] = \left[\frac{kg\ m}{s^2}\right]$$

while the parameter related to the inertial force F_i is defined as:

$$P_2 = U^2 \rho L^2 = \left[\frac{m}{s}\right]^2 \left[\frac{kg}{m^3}\right][m]^2 = \left[\frac{kg\ m}{s^2}\right]$$

By dividing the parameter for the inertial force by the one for the viscous force, the Reynolds-number is obtained which is defined as:

$$Re = \frac{P_2}{P_1} = \frac{U^2 \rho L^2}{U \mu L} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

with kinematic viscosity $\nu \equiv \mu/\rho$.

PROBLEM 3

It is often conjectured that the earth was, at one time, comprised of molten material. If the acceleration due to gravity, $g(r)$, at a radius, r, within this fluid sphere (radius, $R = 6440\ km$) varied linearly with r, if the density of the fluid was uniformly 5600 kg/m^3 and if $g(R) = 9.81\ m/s^2$, find the pressure at the center of this fluid earth.

SOLUTION 3

Find the pressure at the center of the once-molten Earth.

- $R = 6.44 \times 10^6\ m,\ \rho = 5600\ kg/m^3$
- Assume the acceleration due to gravity is linear function of the radius ($g(r) = Ar + B$) with $g(0) = 0$ and $g(R) = 9.81\ m/s^2$.

$$\Rightarrow g(r) = \frac{g(R)}{R} r$$

For a fluid at rest:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g$$

Integrating:

$$\begin{aligned} p(r) = \int \frac{\mathrm{d}p}{\mathrm{d}r}\mathrm{d}r &= \int -\rho \frac{g(R)}{R} r \mathrm{d}r \\ &= -\frac{1}{2}\rho \frac{g(R)}{R} r^2 + C \end{aligned}$$

Evaluating the constant:

$$p(R) = p_A = -\rho \frac{g(R)}{2} R$$

$$\Rightarrow C = p_A + \rho \frac{g(R)}{2} R$$

So the pressure is given by:

$$p(r) = p_A + \frac{\rho R g(R)}{2} \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

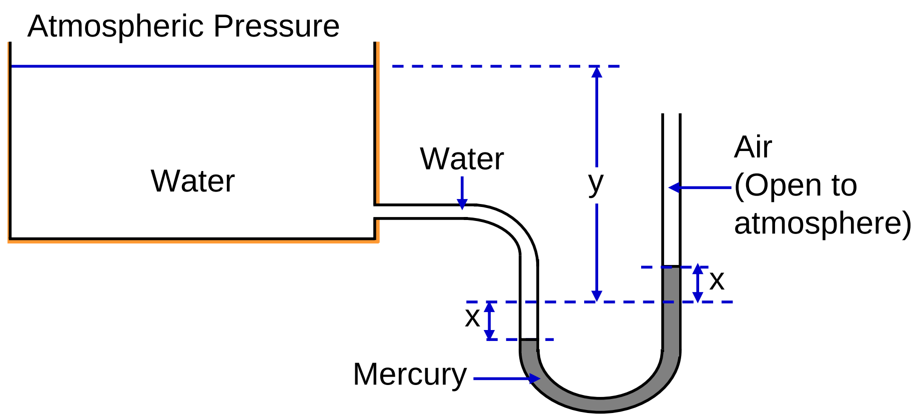
Now looking at the center of the molten Earth:

$$\begin{aligned} p(0) &= p_A + \frac{\rho R g(R)}{2} \\ &= 101.325 \times 10^3\ Pa + \frac{(5600\ kg/m^3)(6.44 \times 10^6\ m)(9.81\ kg\ m/s^2)}{2} \\ &= 1.77 \times 10^{11}\ Pa \end{aligned}$$

PROBLEM 4

A mercury manometer is connected to a large reservoir of water as shown in the figure. The difference in elevation between the free surface of the reservoir and the mean position of the two mercury levels in the manometer is denoted by y. The difference in the elevation between the two mercury levels is denoted by $2x$. Determine the ratio x/y if mercury is 13.6 times more dense than water.

SOLUTION 4



As the two fluids (water, mercury) are assumed to be incompressible, the hydrostatic relation can be used:

$$\frac{\mathrm{d}p}{\mathrm{d}y} = +\rho g$$

Note: y is pointing down. The density of water will be denoted ρ_w, the density of mercury by ρ_m and it is given that:

$$\frac{\rho_m}{\rho_w} = 13.6$$

The pressure of the air is atmospheric pressure P_A. The pressure in the mercury must be the same for both sides of the manometer, thus P_1 can be calculated in two different ways by using the hydrostatic equation:

$$\begin{aligned} p_1 = p_A + \rho_w g\,(x+y) &= p_A + \rho_m g\,(2x) \\ \rho_w\,(x+y) &= \rho_m\,(2x) \\ \frac{x}{y} &= \left[2\frac{\rho_m}{\rho_w} - 1\right]^{-1} \end{aligned}$$

and by using the relation, between the densities of water and mercury it can be solved that:

$$\frac{x}{y} = \frac{1}{26.2} = 3.82 \times 10^{-2}$$