Matrix analysis of planar frames

A frame is a network of beams connected together. In a planar frame, the beams and loads lie in a single plane. Each beam can carry axial force, shear force and bending couple (no twisting). Along a beam, E, I and A are assumed to be constant. However, any beam can be subdivided along its length into multiple beams, each with its own E, I and A values. Distributed loads are not considered; however, a distributed load can be approximately represented using a subdivided beam with point loads at the beam connection points.

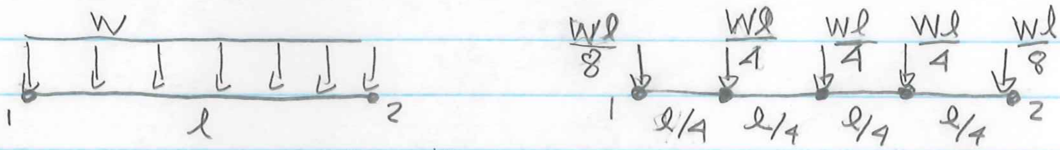


Fig. 1

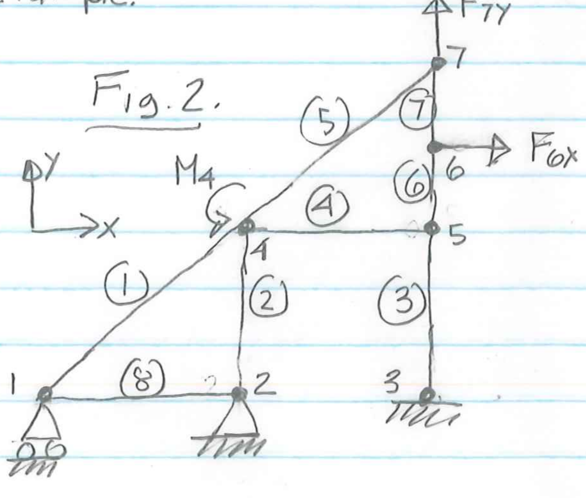
beam with distributed load — approximate representation

A beam connection point is called a node. Similarly, nodes should be introduced if a beam has actual point loads along its length.

General concepts

The general concepts are best presented with an example.

Fig. 2.

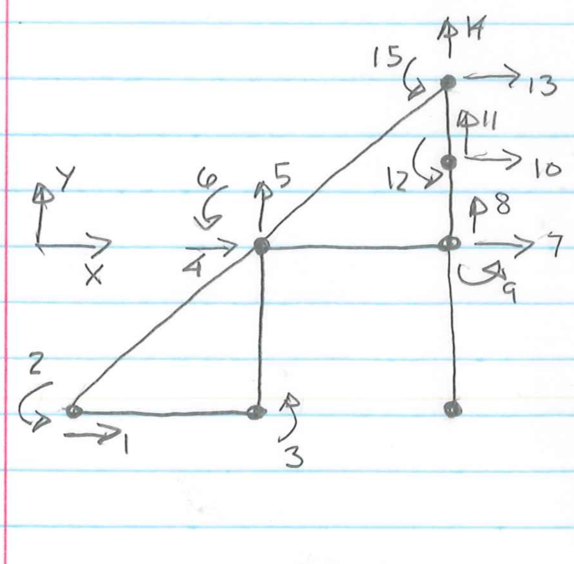


There are seven nodes numbered 1 to 7. There are eight elements (a beam connected to two nodes) numbered (1) to (8). Three point loads (nodal loads) are shown: an X force at node 6, a Y force at node 7, and a couple at node 4.

Nodes 1, 2 and 3 are support nodes, and a variety of support conditions are present. Node 6 would not exist if it were not for the presence of the horizontal force there. (Without this force, there would be six nodes, and node 6 would replace node 7.)

All elements connected at a node are assumed to be rigidly connected there, which means they have equal end rotations (angles between the elements remain the same). Support pins can exist as at nodes 1 and 2. For example, at node 2, elements (2) and (8) are connected together rigidly, but the support can not supply a reaction couple. Elements can also have individual pins at their ends, but these are handled in the context of the element formulations.

Every node has up to three degrees of freedom, which can consist of a horizontal translation, vertical translation, and rotation. The support nodes 1, 2 and 3 have two, one and zero degrees of freedom, respectively. The example structure has fifteen degrees of freedom, which can be numbered sequentially beginning with node 1.



Shows the fifteen degrees of freedom of the structure.

Fig. 3

The degrees of freedom are associated with nodal displacements (translations and rotations), which are denoted by $a_1, a_2, a_3 \ldots$, and the nodal loads (forces and couples), which are denoted by $f_1, f_2, f_3 \ldots$. For example, the horizontal translation of node 6 is a_{10}, and the horizontal force applied at node 6 (previously denoted by F_{6x}) is f_{10}. The subscript on an a_i or f_i is the degree of freedom number.

The nodal displacements are grouped into a vector $\{a\}$ and the nodal loads into a vector $\{f\}$. Those are $N \times 1$ vectors, where N is the total number of degrees of freedom (equal to 15 for the example structure). $\{a\}$ is unknown and $\{f\}$ is known.

$\{a\}$ is computed by solving a matrix equation

$$[K]\{a\} = \{f\} \qquad (1)$$

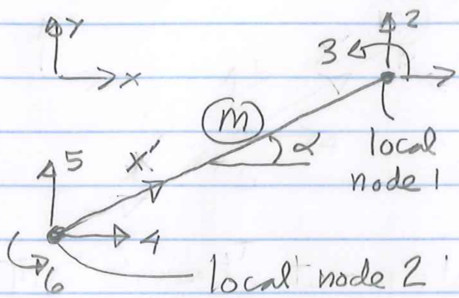
where $[K]$ is the $N \times N$ stiffness matrix of the structure. The terms of $[K]$ can be interpreted as follows. Consider $\{a\}$ to contain zeros except for a 1 in the jth location. Equation (1) says that the nodal loads that would cause such a displacement profile are equal to the terms in the jth column of $[K]$. K_{jj} is the force or couple applied to dof j and the other K_{ij} are applied to the other dof to keep them fixed.

This interpretation of the terms in $[K]$ suggests a way to compute them. It should be evident that $[K]$ can be constructed from contributions from the elements. This process is known as assembly and can be written as

$$[K] = \underset{m=1}{\overset{M}{\mathrm{A}}} [k]^m \qquad (2)$$

where $\mathcal{A}$ is an assembly operator, M is the total number of elements, m denotes the mth element, and $[K]^m$ is the stiffness matrix of element m.

An element has six degrees of freedom, which are always numbered from 1 to 6 (local numbering scheme). Associated with these element degrees of freedom are the are the elements nodal displacements and nodal loads, as shown below.



$$\{a\}^m = \begin{Bmatrix} U_1 \\ V_1 \\ \theta_1 \\ U_2 \\ V_2 \\ \theta_2 \end{Bmatrix} \qquad \{f\}^m = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ F_{x2} \\ F_{y2} \\ M_2 \end{Bmatrix}$$

Fig. 4

The element's two nodes are always numbered locally as 1 and 2, and the elements x' axis points to local node 1.

The element stiffness matrix $[K]^m$ relates the element nodal displacements $\{a\}^m$ to the element nodal forces $\{f\}^m$:

$$[K]^m \{a\}^m = \{f\}^m, \qquad (3)$$

where $[K^m]$ is a 6×6 matrix. Since no support conditions are included in $[K]^m$, it is singular with rank 3.

<u>Computation of $[K]^m$</u>

A general formula for $[K]^m$ can be derived as a function of E, I, A, element length L and angle α that the element x' axis makes with global x. The first step is to consider the form of equation (3) in local element coordinates:

$$[K']^m \{a'\}^m = \{f'\}^m \qquad (4)$$

With respect to local element coordinates, the nodal displacements and loads are defined below.

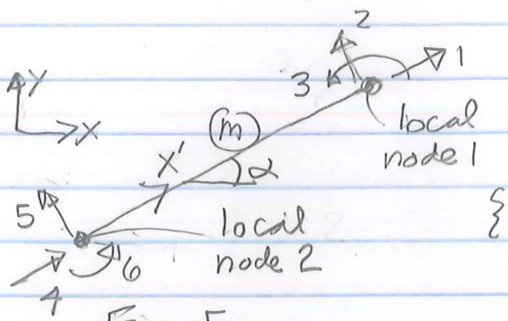
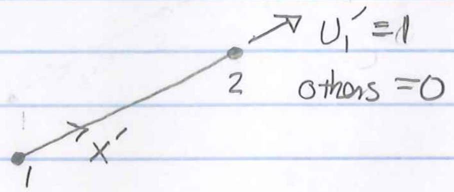


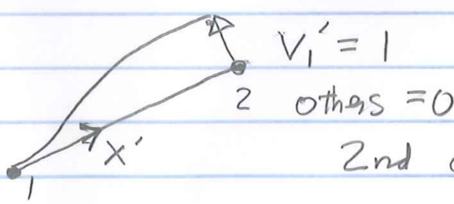
Fig. 5

$$\{a'\}^m = \begin{Bmatrix} U_1' \\ V_1' \\ \theta_1 \\ U_2' \\ V_2' \\ \theta_2 \end{Bmatrix} \qquad \{f'\}^m = \begin{Bmatrix} P_1 \\ Q_1 \\ M_1 \\ P_2 \\ Q_2 \\ M_2 \end{Bmatrix}$$

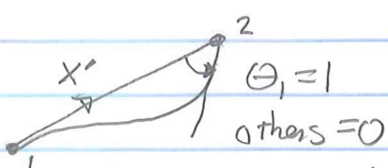
where P_i and Q_i denote axial and shear forces at node i, respectively. Now, terms of $[K']^m$ are derived column by column. The 1st and 4th columns are based on the definition of axial stiffness, while terms in the other four columns result from solving the beam differential equation.



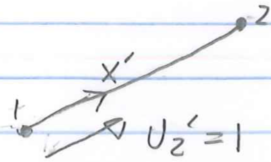
$$\text{1st col of } [K']^m = \begin{Bmatrix} AE/L \\ 0 \\ 0 \\ -AE/L \\ 0 \\ 0 \end{Bmatrix}$$



$$\text{2nd col. of } [K']^m = \begin{Bmatrix} 0 \\ 12EI/L^3 \\ -6EI/L^2 \\ 0 \\ -12EI/L^3 \\ -6EI/L^2 \end{Bmatrix}$$

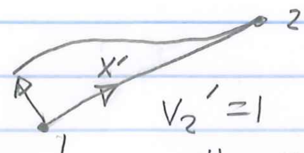


$$\text{3rd col. of } [K']^m = \begin{Bmatrix} 0 \\ -6EI/L^2 \\ 4EI/L \\ 0 \\ 6EI/L^2 \\ 2EI/L \end{Bmatrix}$$



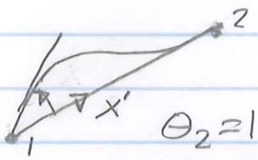
$U_2' = 1$
others $= 0$

$$4\text{th col of } [K']^m = \begin{Bmatrix} -AE/L \\ 0 \\ 0 \\ AE/L \\ 0 \\ 0 \end{Bmatrix}$$



$V_2' = 1$
others $= 0$

$$5\text{th col. of } [K']^m = \begin{Bmatrix} 0 \\ -12EI/L^3 \\ 6EI/L^2 \\ 0 \\ 12EI/L^3 \\ 6EI/L^2 \end{Bmatrix}$$



$\theta_2 = 1$
others $= 0$

$$6\text{th col. of } [K']^m = \begin{Bmatrix} 0 \\ -6EI/L^2 \\ 2EI/L \\ 0 \\ 6EI/L^2 \\ 4EI/L \end{Bmatrix}$$

Thus,

$$[K']^m = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \qquad (5)$$

Note that this matrix is symmetric and that the axial terms are uncoupled from the terms that involve bending.

The next step is to transform equation (4) into equation (3). This transformation is based on the following equations:

$$\{a'\}^m = [T]\,\{a\}^m \qquad (6)$$

$$\{f\}^m = [T]^T \{f'\}^m \qquad (7)$$

where

$$[T] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

Equation (6) is based on geometry while equation (7) is based on static equivalence.

To do the transformation, pre-multiply equation (4) by $[T]^T$:

$$[T]^T [K']^m \{a'\}^m = [T]^T \{f'\}^m, \qquad (9)$$

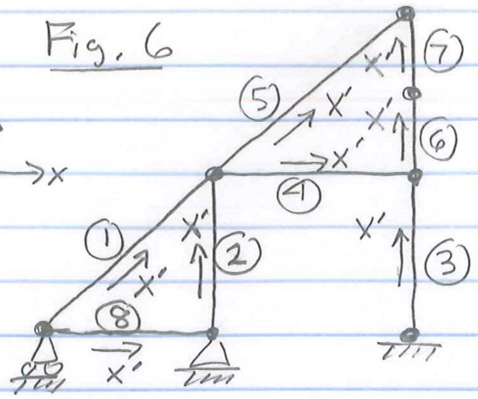
and substitute from equations (6) and (7). Equation (3) results, where

$$[K]^m = [T]^T [K']^m [T] \qquad (10)$$

Assembly process

The stiffness matrix $[K]$ of the structure in equation (1) is assembled from the element stiffness matrices $[K]^m$ as indicated by equation (2). The assembly process makes use of assembly arrays. An assembly array for an element lists the six global degrees of freedom in the order of the six local degrees of freedom. Since an assembly array depends on which of an element's two nodes are designated local 1 and 2, the x' axis for an element (which points to local node 1) must be defined before its assembly array can be constructed. For the example structure, the x' axis for each element is shown below. Note that the direction of the x' axis is arbitrary.

Fig. 6



Shows the X' direction for each element.

Referring to Figs. 3, 4 and 6, the assembly arrays are as follows.

(1)	< 4	5	6	1	0	2 >
(2)	< 4	5	6	0	0	3 >
(3)	< 7	8	9	0	0	0 >
(4)	< 7	8	9	4	5	6 >
(5)	< 13	14	15	4	5	6 >
(6)	< 10	11	12	7	8	9 >
(7)	< 13	14	15	10	11	12 >
(8)	< 0	0	3	1	0	2 >

Note that a zero is used in an assembly array if the corresponding degree of freedom is fixed.

The assembly process is illustrated below for element (1). For reference, the assembly array is written across the top and down the side of the element stiffness matrix.

$$[K]^{(1)} \quad \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 0 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 0 & 2 \end{array} \\ \left[\begin{array}{cccccc} X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \end{array} \right] \end{array}$$

assembled to row 4 col 6 of [K]

assembled to row 6, col. 2 of [K]

No terms from this row are assembled

No terms from this column are assembled.

Note that the assembly operation automatically accounts for the boundary conditions. If these boundary conditions are sufficient to prevent rigid body modes, then the assembled [K] for the structure will be non-singular (in fact, symmetric and positive definite).

Terms of the structure load vector {f} are assembled directly as each term can be identified with a particular degree of freedom.

Pinned connections

The presence of a pin at one or both ends of an element is handled in the computation of the element stiffness matrix $[K]^m$, or rather $[K']^m$ since the [T] transformation is the same. Only the terms in $[K']^m$ associated with bending are affected (not the AE/L terms). These terms are different depending on whether one pin is present at local node 1, or one pin at local node 2, or a pin at each location.

As an example, consider the case of one pin located at local node 1. The AE/L part of $[K']^m$ is the same. The 2nd, 4th and 5th columns of $[K']^m$ are computed as before but with the $M_1=0$ boundary condition replacing $\theta_1=0$. Terms in the 3rd column of $[K']^m$ are all zero. The result is

$$[K']^m = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & -\frac{3EI}{L^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{3EI}{L^3} & 0 & 0 & \frac{3EI}{L^3} & \frac{3EI}{L^2} \\ 0 & -\frac{3EI}{L^2} & 0 & 0 & \frac{3EI}{L^2} & \frac{3EI}{L} \end{bmatrix}$$

for one pin at local node 1. Symmetry is still present.

If two pins are present, then only the $\frac{AE}{L}$ terms remain; that is,

$$[K']^m = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for a pin at both local nodes 1 and 2. Note that if all elements connected to a node have a pin there, then the global rotational degree of freedom will not have any stiffness, so it should be fixed to avoid $[K]$ being singular.

Additional steps

Once $[K]$ and $\{f\}$ are computed, equation (1) is solved for $\{a\}$, such as by a Gauss elimination routine. Then, terms of $\{a\}$ are used to fill $\{a\}^m$ for each element; equation (6) is used to transform to $\{a'\}^m$ for each element; then equation (4) is used to obtain the end forces (axial and shear) and couples for each element. Then bending couple and shear diagrams can be drawn.

Note that displacements of all nodes are found once equation (1) is solved. If a displacement profile along an element is needed, then the beam differential equation can be solved using the terms in $\{a'\}^m$ as boundary conditions.

Input data for computer program

The matrix analysis routine described previously can be efficiently programmed for computer application.

Below is a description of the input data.

Number of elements, number of nodes

For each node:

x coordinate
y coordinate
Coordinates

translation in x fixed or free
translation in y fixed or free
rotation fixed or free
fixity

applied force in x
applied force in y
applied couple
loads

For each element:

global node corresponding to local node 1
global node corresponding to local node 2
connectivity

E valve
I valve
A valve
section properties

pin or not at local node 1
pin or not at local node 2
pinned connections