Spatiotemporal Chaos in Rayleigh-Bénard Convection

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June 2006

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Support: DOE
Computer time: NSF, NERSC, NCSA, ANL

Outline

1. Rayleigh-Bénard Convection

2. Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3. Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4. Conclusions

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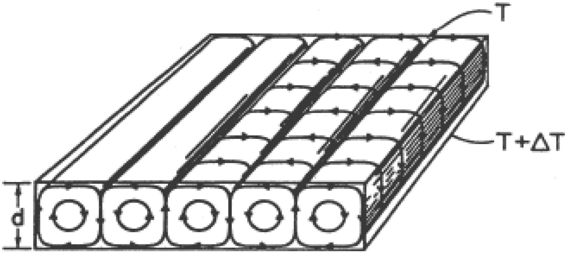
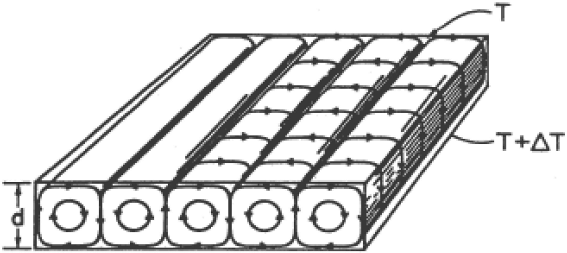
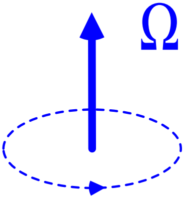
Rayleigh-Bénard Convection



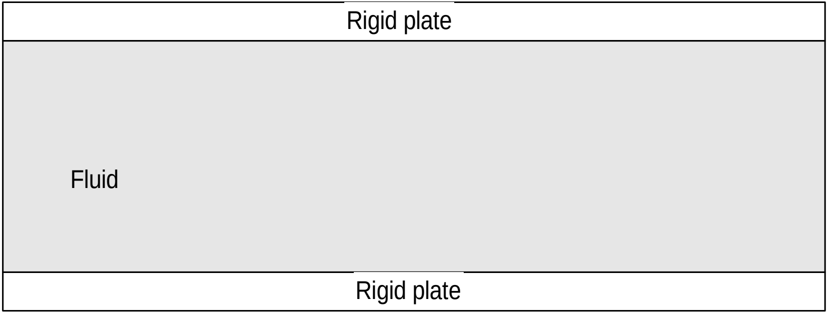
Rayleigh

Rayleigh-Bénard Convection





Rayleigh-Bénard Instability



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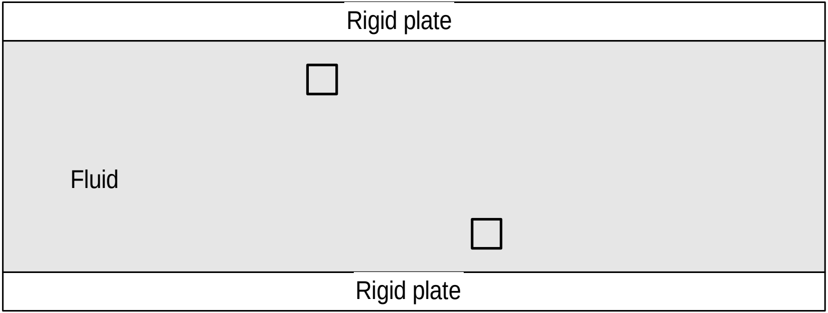

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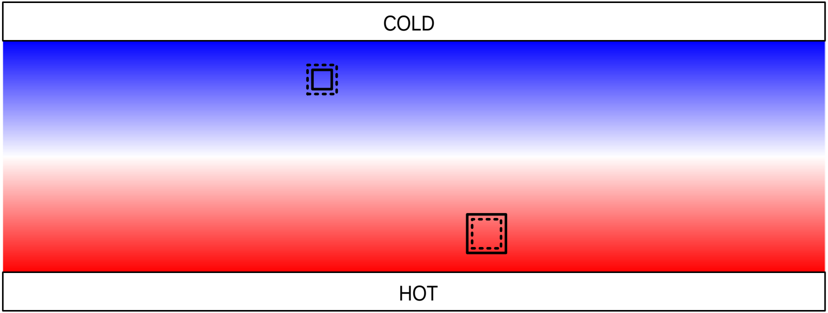

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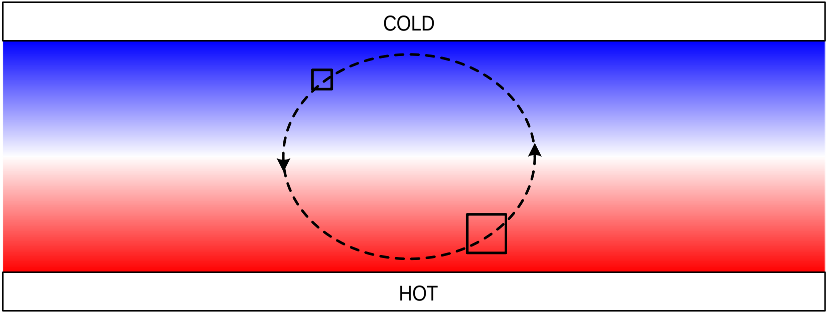

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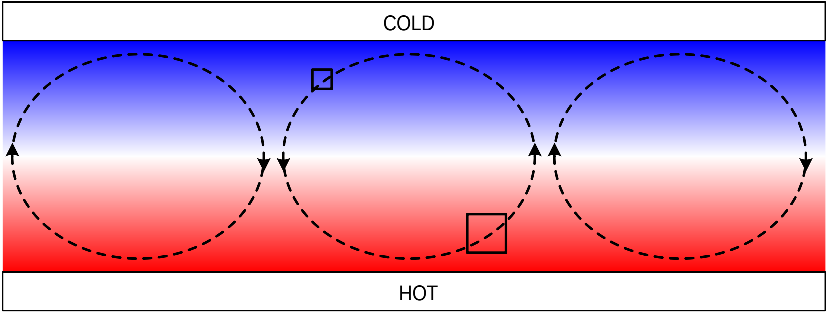

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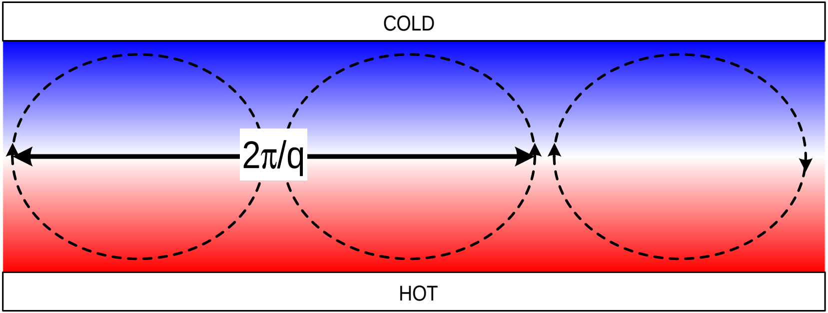

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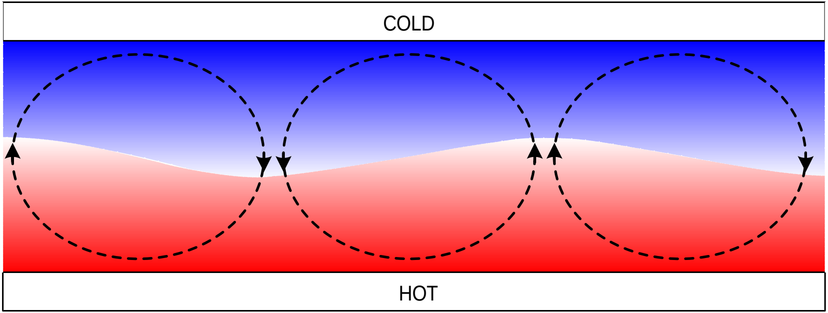


Convection Patterns

From the website of Eberhard Bodenschatz

Equations for Convection

- Momentum Conservation

$$\frac{1}{\sigma}\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right] = -\vec{\nabla}p + R\,T\hat{e}_z + \nabla^2\vec{u} + 2\Omega\hat{e}_z \times \vec{u}$$

- Energy Conservation

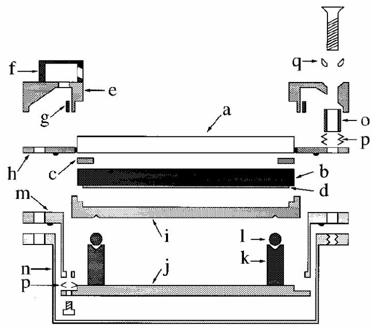
$$\frac{\partial T}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)T = \nabla^2 T$$

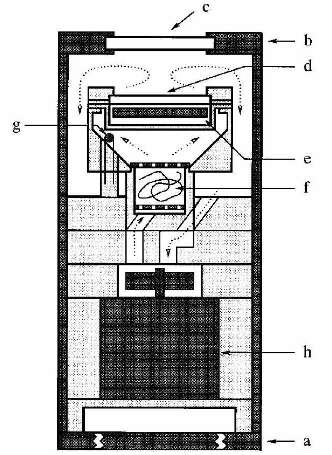
- Mass Conservation

$$\vec{\nabla}\cdot\vec{u} = 0$$

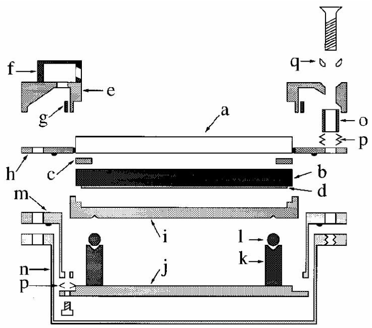
- BC: no-slip boundaries at $z = 0, 1$ with $T(z=0) = 1$, and $T(z=1) = 0$
- Aspect Ratio: $\Gamma = r/d$

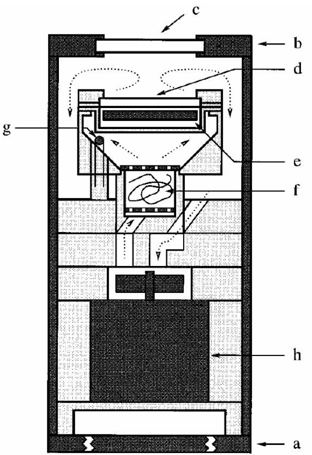
Modern Convection Apparatus





From de Bruyn et al., Rev. Sci. Instr. (1996)





From de Bruyn et al., Rev. Sci. Instr. (1996)

Allows a quantitative comparison between theory and experiment.

Outline

Outline

What Is It?

- Definitions
  - dynamics, disordered in time and space, of a large, uniform system
  - collective motion of many chaotic elements
  - breakdown of pattern to dynamics
- Natural examples:
  - atmosphere and ocean (weather, climate etc.)
  - arrays of nanomechanical oscillators
  - heart fibrillation

Cultured monolayers of cardiac tissue (from Gil Bub, McGill)

A new paradigm of unpredictable dynamics

- Simplifications over small-system chaos
  - Perhaps smooth dependence on parameters
  - Statistical rather than geometrical description
  - $N \to \infty$ limit
- Not as difficult as fully developed turbulence!

Spatiotemporal Chaos

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Issues

- Role of space as well as time in sensitivity to initial conditions
- Insightful diagnostics
- Specificity and universality of behavior
- Quantitative descriptions
  - Scaling behavior near transitions
  - Reduced long wavelength description (cf. "hydrodynamics")
- Control …

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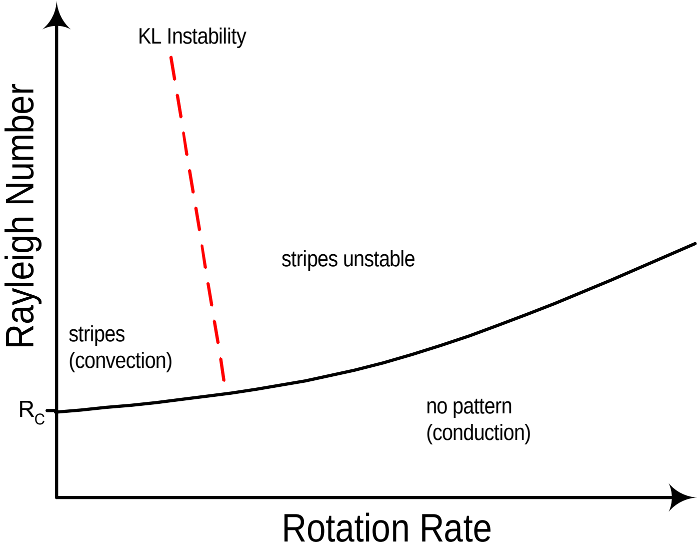
Spiral Chaos in Rayleigh-Bénard Convection

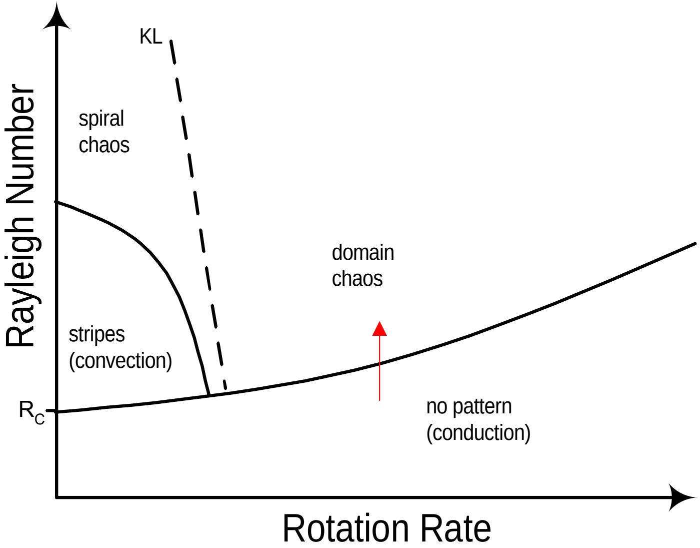
…and from experiment

Domain Chaos in Rayleigh-Bénard Convection

Spiral and Domain Chaos in Rayleigh-Bénard Convection



Spiral and Domain Chaos in Rayleigh-Bénard Convection



Outline

Summary of Results

- Amplitude equation theory predicts

Length scale	$\xi \sim \varepsilon^{-1/2}$
Time scale	$\tau \sim \varepsilon^{-1}$
Velocity scale	$v \sim \varepsilon^{1/2}$

  with $\varepsilon = (R - R_c(\Omega))/R_c(\Omega)$
- Numerical Tests
  - generalized Swift-Hohenberg equations ✓
  - full fluid dynamic simulations ✓
- Experiment × (but now we understand why)

Outline

Amplitudes

- Rayleigh's linear stability analysis gives

$$u = u_0 e^{\gamma t} \cos(qx) \ldots = A(t) \cos(qx) \ldots$$

so that in the linear approximation and for R near R_c

$$\frac{dA}{dt} = \gamma A \qquad \text{with} \qquad \gamma \propto \varepsilon = \frac{R - R_c}{R_c}$$

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- Use ε as small parameter in expansion about threshold
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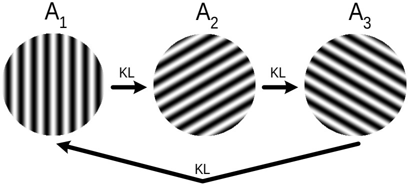
$$\frac{dA}{dt} = (\varepsilon - A^2)A$$

- Spatial variation

$$\frac{\partial A}{\partial t} = \varepsilon A - A^3 + \frac{\partial^2 A}{\partial x^2}$$

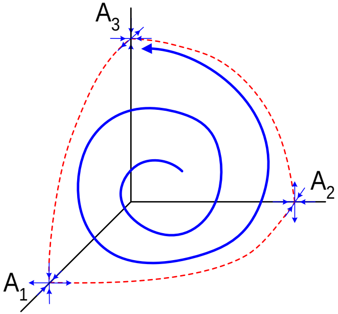
Amplitude Equations for KL Instability

(Busse-Heikes, May-Leonard)



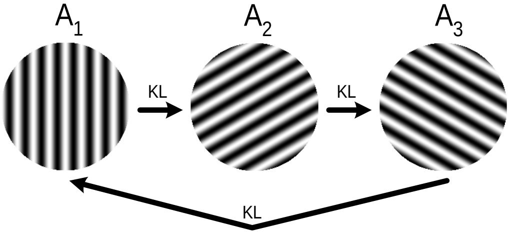
$$dA_1/dt = \varepsilon A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2)$$
$$dA_2/dt = \varepsilon A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2)$$
$$dA_3/dt = \varepsilon A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2)$$

give a heteroclinic cycle



Three Amplitudes + Rotation + Spatial Variation

(Tu and MCC, 1992)

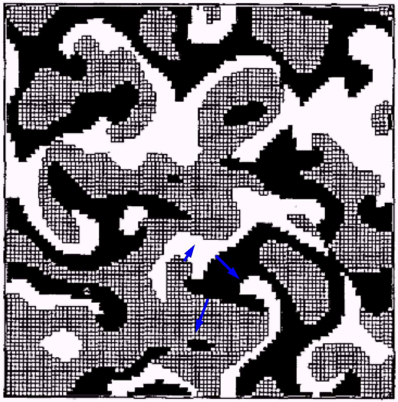


$$\partial A_1/\partial t = \varepsilon A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2) + \partial^2 A_1/\partial x_1^2$$
$$\partial A_2/\partial t = \varepsilon A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2) + \partial^2 A_2/\partial x_2^2$$
$$\partial A_3/\partial t = \varepsilon A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2) + \partial^2 A_3/\partial x_3^2$$

gives chaos!

Simulations of Amplitude Equations

(Tu and MCC, 1992)



Grey: A_1 largest; White: A_2 largest; Black: A_3 largest

Scaling

Rescale $X = \varepsilon^{1/2} x,\ T = \varepsilon t,\ \bar{A} = \varepsilon^{-1/2} A$

$$\partial_T \bar{A}_1 = \bar{A}_1 - \bar{A}_1(\bar{A}_1^2 + g_+ \bar{A}_2^2 + g_- \bar{A}_3^2) + \partial_{X_1}^2 \bar{A}_1$$
$$\partial_T \bar{A}_2 = \bar{A}_2 - \bar{A}_2(\bar{A}_2^2 + g_+ \bar{A}_3^2 + g_- \bar{A}_1^2) + \partial_{X_2}^2 \bar{A}_2$$
$$\partial_T \bar{A}_3 = \bar{A}_3 - \bar{A}_3(\bar{A}_3^2 + g_+ \bar{A}_1^2 + g_- \bar{A}_2^2) + \partial_{X_3}^2 \bar{A}_3$$

Numerical simulations show chaotic dynamics with $O(1)$ length and time scales
Therefore in unscaled (physical) units

Length scale	$\xi \sim \varepsilon^{-1/2}$
Time scale	$\tau \sim \varepsilon^{-1}$
Velocity scale	$v \sim \varepsilon^{1/2}$

Issues

Important Issues

- Validity of scaling results from truncated expansions
- Validity of “mean field” results in nonlinear fluctuating state

Other Approximations

- Restriction to 3 roll orientations
- Amplitudes assumed real
  - No wave number variation
  - No dislocations or phase grain boundaries
- No perpendicular derivative terms

$$\left(\partial_{x_i} - \frac{i}{2q_c}\partial_{y_i}^2\right)^2 \longrightarrow \partial_{x_i}^2$$

Tests of the Theory

- Simulations of generalized Swift-Hohenberg equations in periodic geometries show results consistent with predictions [MCC, Meiron, and Tu (1994)]
- Experiments give results that are consistent either with finite values of ξ, τ at onset, or much smaller power laws $\xi \sim \varepsilon^{-0.2}$, $\tau \sim \varepsilon^{-0.6}$ [Hu et al. (1995) + many others]
- Simulations of generalized Swift-Hohenberg equations in circular geometries of radius Γ gave results similar to experiment but also consistent with finite size scaling

$$\xi_M = \xi f(\Gamma/\xi) \qquad \text{with} \qquad \xi \sim \varepsilon^{-1/2}$$

  [MCC, Louie, and Meiron (2001)]
- Fluid simulations ...

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Generalized Swift-Hohenberg Simulations

MCC, Meiron, and Tu (1994)

Real field of two spatial dimensions $\psi(x, y; t)$

$$\frac{\partial \psi}{\partial t} = \varepsilon\psi + (\nabla^2 + 1)^2\psi - \psi^3 \qquad \text{gives stripes}$$

Generalized Swift-Hohenberg Simulations

MCC, Meiron, and Tu (1994)

Real field of two spatial dimensions $\psi(x, y; t)$

$$\frac{\partial \psi}{\partial t} = \varepsilon\psi + (\nabla^2 + 1)^2\psi - \psi^3$$
$$+ g_2 \hat{\mathbf{z}} \cdot \nabla \times [(\nabla\psi)^2 \nabla\psi] + g_3 \nabla \cdot [(\nabla\psi)^2 \nabla\psi]$$

gives domain chaos!

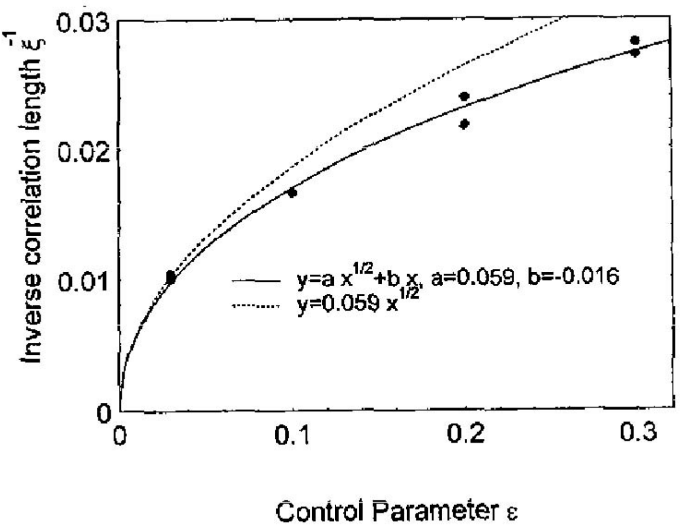
Stripes

Orientations

Domain Walls

Scaling of Correlation Length

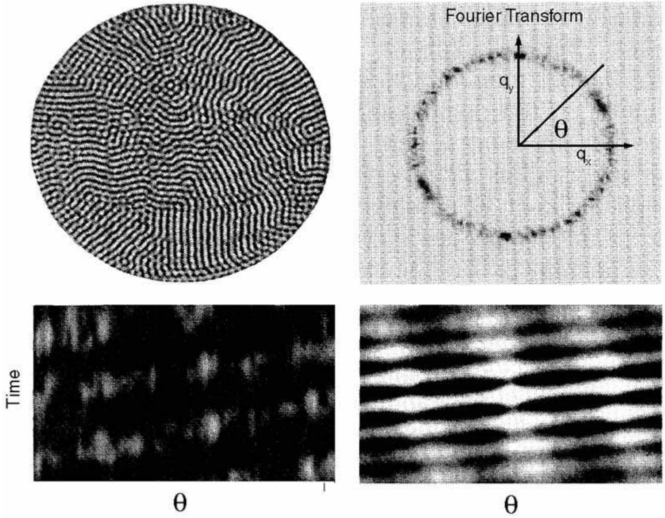
MCC, Meiron, and Tu (1994)



Outline

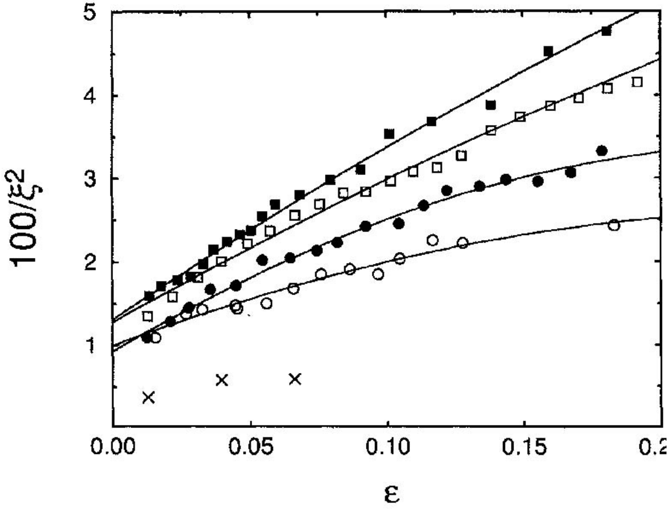
Experiment and Diagnosis

Hu et al. (1995)



Experimental Results for Correlation Length

Hu et al. (1995)

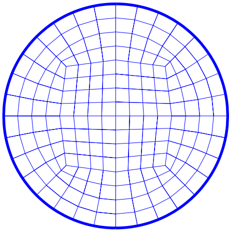


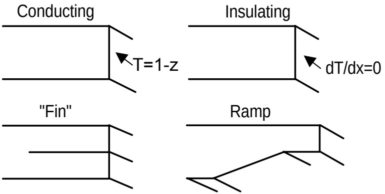
Outline

Spectral Element Numerical Solution

MCC, Greenside, Fischer et al.

- Accurate simulation of long-time dynamics
- Exponential convergence in space, third order in time
- Efficient parallel algorithm, unstructured mesh
- Arbitrary geometries, realistic boundary conditions





Simulations Complement Experiments

- Knowledge of full flow field and other diagnostics (e.g. total heat flow)
- No experimental/measurement noise (roundoff "noise" very small)
- Measure quantities inaccessible to experiment e.g. Lyapunov exponents and vectors
- Readily tune parameters
- Turn on and off particular features of the physics (e.g. centrifugal effects, realistic v. periodic boundary conditions)

Full Fluid Dynamic Simulations

Scheel, Caltech thesis (2006)

Periodic Boundaries

Realistic Boundaries

Lyapunov Exponent

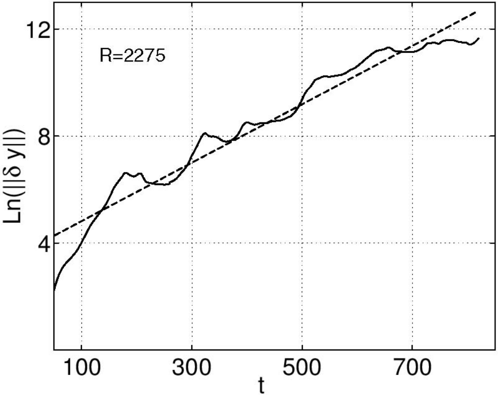
Jayaraman et al. (2005)

Temperature

Temperature Perturbation

Lyapunov Exponent

(Jayaraman et al., 2005)



Aspect ratio $\Gamma = 40$, Prandtl number $\sigma = 0.93$, rotation rate $\Omega = 40$

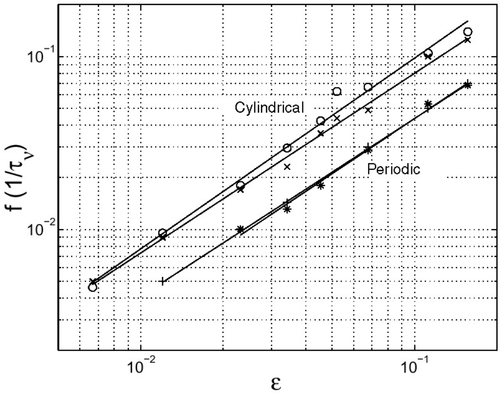
Full Fluid Simulations for Domain Chaos

Summary of results of full 3d fluid simulations:

- Simulations of Rayleigh-Bénard convection with Coriolis forces give $\tau \sim \varepsilon^{-1}$ for small enough ε. For larger ε a slower growth is seen perhaps consistent with $\tau \sim \varepsilon^{-0.7}$
  [Scheel and MCC (2005)]
- Scaling of largest Lyapunov exponent consistent with $\lambda \sim c + \varepsilon^1$ with c comparable to the finite size shift in onset
  [Jayaraman et al. (2006)]
- Role of centrifugal force
  [Becker, Scheel, MCC, and Ahlers (2006)]

Frequency Scaling

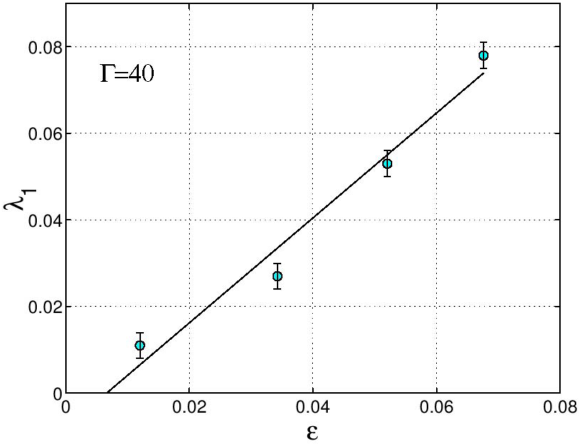
Scheel and MCC (2005)



Slopes give frequency $\propto \varepsilon^{1.07}$ ($\Gamma = 40$ cylinder) and $\varepsilon^{1.04}$ (periodic)

Scaling of Lyapunov Exponent

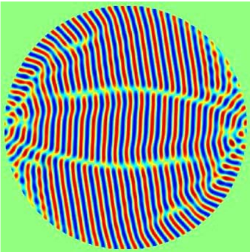
Jayaraman et al. (2005)



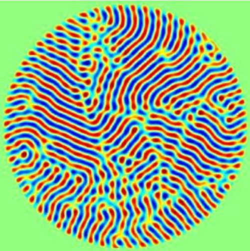
Importance of Centrifugal Force

Becker, Scheel, MCC, and Ahlers (2006)

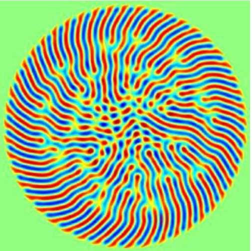
Aspect ratio $\Gamma = 20$, $\varepsilon \simeq 1.05$, $\Omega = 17.6$



Centrifugal force 0



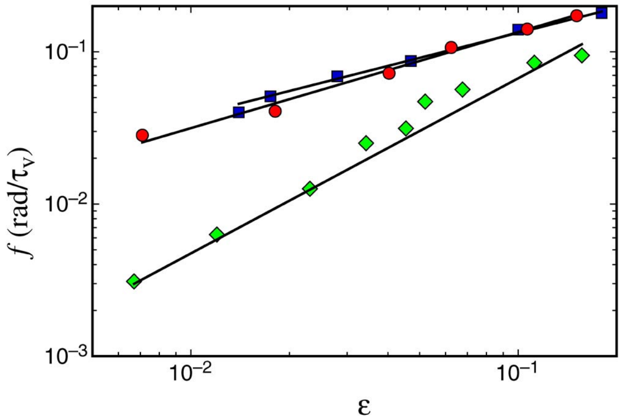
Centrifugal force x4



Centrifugal force x10

Time Scaling

Becker, Scheel, MCC, and Ahlers (2006)



o simulations $\Gamma = 20$ with centrifugal force $\times 2$; □ experiment $\Gamma = 40$

◇ simulations $\Gamma = 40$ no centrifugal force

Current Picture

- Simulations of the full fluid equations near onset without centrifugal forces are consistent with predictions of scaling of times as ε^{-1}; not yet able to probe scaling of lengths.
- Centrifugal forces are important in experiment, enhancing the finite size effects and limiting size of region of domain chaos.
- Maximum centrifugal force cf. Coriolis force $\sim (\alpha \Delta T)\Omega\Gamma/u$. (Near threshold $\Omega_{KL} \sim 10^1$, $u \sim \varepsilon^{1/2}$).

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Conclusions

- Spatiotemporal chaos is a third paradigm of complex dynamics (cf. chaos, turbulence)
- Rotating convection shows spatiotemporal chaos in the weakly nonlinear regime near onset where there is hope for a quantitative understanding.
- Numerical simulations of realistic experimental geometries are now feasible

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- Rotating convection shows spatiotemporal chaos in the weakly nonlinear regime near onset where there is hope for a quantitative understanding.
- Numerical simulations of realistic experimental geometries are now feasible
- Truncated amplitude equation model makes predictions for scaling of lengths $\propto \varepsilon^{-1/2}$ and times $\propto \varepsilon^{-1}$
- Scalings and features of dynamics predicted by truncated amplitude model confirmed by GSH simulations and full fluid simulations
- Disagreement between experiment and predictions resolved (finite size, centrifugal effects)

To Do

- More precise experimental tests of the (homogeneous) theory
- Understand theoretically how good the truncated amplitude equation model should be
- Relate to lattice systems of coupled heteroclinic oscillators
- Understand the origins of chaos in the system and models

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THE END