Spatiotemporal Chaos in Rayleigh-Bénard Convection

Michael Cross

California Institute of Technology
Beijing Normal University

June 2006

Janet Scheel, Keng-Hwee Chiam, Mark Paul
Henry Greenside, Anand Jayaraman
Paul Fischer
Yuhai Tu, Dan Meiron, Michael Louie

Support: DOE
Computer time: NSF, NERSC, NCSA, ANL
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   ■ What is it?
   ■ Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   ■ Amplitude equation theory
   ■ Generalized Swift-Hohenberg simulations
   ■ Experiment
   ■ Simulations of full fluid equations

4 Conclusions
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
Rayleigh-Bénard Convection

Rayleigh

Spatiotemporal Chaos

June 2006

4 / 54
Rayleigh-Bénard Instability

Fluid

Rigid plate

Rigid plate
Rayleigh-Bénard Instability

![Diagram of Rayleigh-Bénard Instability]

- Fluid
- Rigid plate

Michael Cross (Caltech, BNU)
Rayleigh-Bénard Instability
Rayleigh-Bénard Instability
Rayleigh-Bénard Instability

\[ 2\pi/q \]

COLD

HOT
Rayleigh-Bénard Instability
Convection Patterns

From the website of Eberhard Bodenschatz
Equations for Convection

- **Momentum Conservation**

\[ \frac{1}{\sigma} \left[ \frac{\partial \vec{u}}{\partial t} + \left( \vec{u} \cdot \vec{\nabla} \right) \vec{u} \right] = -\vec{\nabla} p + RT\hat{e}_z + \nabla^2 \vec{u} + 2\Omega \hat{e}_z \times \vec{u} \]

- **Energy Conservation**

\[ \frac{\partial T}{\partial t} + \left( \vec{u} \cdot \vec{\nabla} \right) T = \nabla^2 T \]

- **Mass Conservation**

\[ \vec{\nabla} \cdot \vec{u} = 0 \]

- **BC:** no-slip boundaries at \( z = 0, 1 \) with \( T(z = 0) = 1 \), and \( T(z = 1) = 0 \)

- **Aspect Ratio:** \( \Gamma = r/d \)
Modern Convection Apparatus


Allows a **quantitative** comparison between theory and experiment.
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
What Is It?

- Definitions
  - dynamics, disordered in time and space, of a large, uniform system
  - collective motion of many chaotic elements
  - breakdown of pattern to dynamics

- Natural examples:
  - atmosphere and ocean (weather, climate etc.)
  - arrays of nanomechanical oscillators
  - heart fibrillation

Cultured monolayers of cardiac tissue (from Gil Bub, McGill)
A new paradigm of unpredictable dynamics

- Simplifications over small-system chaos
  - Perhaps smooth dependence on parameters
  - Statistical rather than geometrical description
  - $N \to \infty$ limit

- Not as difficult as fully developed turbulence!
A new paradigm of unpredictable dynamics

- Simplifications over small-system chaos
  - Perhaps smooth dependence on parameters
  - Statistical rather than geometrical description
  - $N \to \infty$ limit
- Not as difficult as fully developed turbulence!

Issues

- Role of space as well as time in sensitivity to initial conditions
- Insightful diagnostics
- Specificity and universality of behavior
- Quantitative descriptions
  - Scaling behavior near transitions
  - Reduced long wavelength description (cf. “hydrodynamics”)
- Control …
A new paradigm of unpredictable dynamics

- Simplifications over small-system chaos
  - Perhaps smooth dependence on parameters
  - Statistical rather than geometrical description
  - $N \to \infty$ limit
- Not as difficult as fully developed turbulence!

Issues

- Role of space as well as time in sensitivity to initial conditions
- Insightful diagnostics
- Specificity and universality of behavior
- Quantitative descriptions
  - Scaling behavior near transitions
  - Reduced long wavelength description (cf. “hydrodynamics”)
- Control …
Outline

1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
...and from experiment
Spiral and Domain Chaos in Rayleigh-Bénard Convection

Rotation Rate

Rayleigh Number

KL Instability

stripes unstable

stripes (convection)

no pattern (conduction)

$R_c$
Spiral and Domain Chaos in Rayleigh-Bénard Convection

- Rotation Rate
- Rayleigh Number

- Spiral chaos
- Domain chaos
- No pattern (conduction)
- Stripes (convection)

$\gamma$

$R_C$
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
Summary of Results

- Amplitude equation theory predicts
  
  \[ \xi \sim \varepsilon^{-1/2} \]
  
  \[ \tau \sim \varepsilon^{-1} \]
  
  \[ v \sim \varepsilon^{1/2} \]

  with \( \varepsilon = (R - R_c(\Omega))/R_c(\Omega) \)

- Numerical Tests
  
  - generalized Swift-Hohenberg equations✓
  
  - full fluid dynamic simulations✓

- Experiment × (but now we understand why)
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   ■ What is it?
   ■ Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   ■ Amplitude equation theory
   ■ Generalized Swift-Hohenberg simulations
   ■ Experiment
   ■ Simulations of full fluid equations

4 Conclusions
Rayleigh's linear stability analysis gives

\[ u = u_0 e^{\gamma t} \cos(qx) \ldots = A(t) \cos(qx) \ldots \]

so that in the linear approximation and for \( R \) near \( R_c \)

\[ \frac{dA}{dt} = \gamma A \quad \text{with} \quad \gamma \propto \varepsilon = \frac{R - R_c}{R_c} \]
Rayleigh's linear stability analysis gives

\[ u = u_0 e^{\gamma t} \cos(qx) \ldots = A(t) \cos(qx) \ldots \]

so that in the linear approximation and for \( R \) near \( R_c \)

\[ \frac{dA}{dt} = \gamma A \quad \text{with} \quad \gamma \propto \varepsilon = \frac{R - R_c}{R_c} \]

Use \( \varepsilon \) as small parameter in expansion about threshold
Rayleigh's linear stability analysis gives

\[ u = u_0 e^{\gamma t} \cos(qx) \ldots = A(t) \cos(qx) \ldots \]

so that in the linear approximation and for \( R \) near \( R_c \)

\[ \frac{dA}{dt} = \gamma A \quad \text{with} \quad \gamma \propto \varepsilon = \frac{R - R_c}{R_c} \]

- Use \( \varepsilon \) as small parameter in expansion about threshold
- Nonlinear saturation

\[ \frac{dA}{dt} = (\varepsilon - A^2)A \]
Rayleigh's linear stability analysis gives

\[ u = u_0 e^{\gamma t} \cos(qx) \ldots = A(t) \cos(qx) \ldots \]

so that in the linear approximation and for \( R \) near \( R_c \)

\[ \frac{dA}{dt} = \gamma A \quad \text{with} \quad \gamma \propto \varepsilon = \frac{R - R_c}{R_c} \]

Use \( \varepsilon \) as small parameter in expansion about threshold

Nonlinear saturation

\[ \frac{dA}{dt} = (\varepsilon - A^2)A \]

Spatial variation

\[ \frac{\partial A}{\partial t} = \varepsilon A - A^3 + \frac{\partial^2 A}{\partial x^2} \]
Amplitude Equations for KL Instability
(Busse-Heikes, May-Leonard)

\[ \frac{dA_1}{dt} = \epsilon A_1 - A_1 (A_1^2 + g_+ A_2^2 + g_- A_3^2) \]
\[ \frac{dA_2}{dt} = \epsilon A_2 - A_2 (A_2^2 + g_+ A_3^2 + g_- A_1^2) \]
\[ \frac{dA_3}{dt} = \epsilon A_3 - A_3 (A_3^2 + g_+ A_1^2 + g_- A_2^2) \]

give a heteroclinic cycle
Three Amplitudes + Rotation + Spatial Variation
(Tu and MCC, 1992)

\[ \frac{\partial A_1}{\partial t} = \varepsilon A_1 - A_1 (A^2_1 + g_+ A^2_2 + g_- A^2_3) + \frac{\partial^2 A_1}{\partial x^2_1} \]
\[ \frac{\partial A_2}{\partial t} = \varepsilon A_2 - A_2 (A^2_2 + g_+ A^2_3 + g_- A^2_1) + \frac{\partial^2 A_2}{\partial x^2_2} \]
\[ \frac{\partial A_3}{\partial t} = \varepsilon A_3 - A_3 (A^2_3 + g_+ A^2_1 + g_- A^2_2) + \frac{\partial^2 A_3}{\partial x^2_3} \]

gives chaos!
Grey: $A_1$ largest; White: $A_2$ largest; Black: $A_3$ largest
Scaling

Rescale $X = \varepsilon^{1/2} x$, $T = \varepsilon t$, $\tilde{A} = \varepsilon^{-1/2} A$

$$\partial_T \tilde{A}_1 = \tilde{A}_1 - \tilde{A}_1(\tilde{A}_1^2 + g_+ \tilde{A}_2^2 + g_- \tilde{A}_3^2) + \partial_{\tilde{x}_1}^2 \tilde{A}_1$$
$$\partial_T \tilde{A}_2 = \tilde{A}_2 - \tilde{A}_2(\tilde{A}_2^2 + g_+ \tilde{A}_3^2 + g_- \tilde{A}_1^2) + \partial_{\tilde{x}_2}^2 \tilde{A}_2$$
$$\partial_T \tilde{A}_3 = \tilde{A}_3 - \tilde{A}_3(\tilde{A}_3^2 + g_+ \tilde{A}_1^2 + g_- \tilde{A}_2^2) + \partial_{\tilde{x}_3}^2 \tilde{A}_3$$

Numerical simulations show chaotic dynamics with $O(1)$ length and time scales.
Therefore in unscaled (physical) units

- Length scale $\xi \sim \varepsilon^{-1/2}$
- Time scale $\tau \sim \varepsilon^{-1}$
- Velocity scale $v \sim \varepsilon^{1/2}$
Issues

Important Issues

- Validity of scaling results from truncated expansions
- Validity of “mean field” results in nonlinear fluctuating state

Other Approximations

- Restriction to 3 roll orientations
- Amplitudes assumed real
  - No wave number variation
  - No dislocations or phase grain boundaries
- No perpendicular derivative terms

\[
\left( \partial_{x_i} - \frac{i}{2q_c} \partial^2_{y_i} \right)^2 \rightarrow \partial^2_{x_i}
\]
Tests of the Theory

- Simulations of generalized Swift-Hohenberg equations in periodic geometries show results consistent with predictions [MCC, Meiron, and Tu (1994)]

- Experiments give results that are consistent either with finite values of $\xi$, $\tau$ at onset, or much smaller power laws $\xi \sim \varepsilon^{-0.2}$, $\tau \sim \varepsilon^{-0.6}$ [Hu et al. (1995) + many others]

- Simulations of generalized Swift-Hohenberg equations in circular geometries of radius $\Gamma$ gave results similar to experiment but also consistent with finite size scaling

$$\xi_M = \xi f(\Gamma/\xi) \quad \text{with} \quad \xi \sim \varepsilon^{-1/2}$$

[MCC, Louie, and Meiron (2001)]

- Fluid simulations …
Outline

1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
Real field of two spatial dimensions $\psi(x, y; t)$

\[
\frac{\partial \psi}{\partial t} = \varepsilon \psi + (\nabla^2 + 1)^2 \psi - \psi^3 \quad \text{gives stripes}
\]
Real field of two spatial dimensions $\psi(x, y; t)$

$$\frac{\partial \psi}{\partial t} = \varepsilon \psi + (\nabla^2 + 1)^2 \psi - \psi^3$$

$$+ g_2 \hat{z} \cdot \nabla \times [(\nabla \psi)^2 \nabla \psi] + g_3 \nabla \cdot [(\nabla \psi)^2 \nabla \psi]$$

gives domain chaos!

Stripes          Orientations          Domain Walls
Scaling of Correlation Length

MCC, Meiron, and Tu (1994)

![Graph showing the scaling of correlation length](image)

The graph illustrates the relationship between the inverse correlation length $\xi$ and the control parameter $\varepsilon$. The equation $y = ax^{1/2} + bx$, with $a = 0.059$ and $b = -0.016$, is used to model the data points. Another line with the equation $y = 0.059x^{1/2}$ is also plotted for comparison.
Outline

1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
Experiment and Diagnosis
Hu et al. (1995)
Experimental Results for Correlation Length
Hu et al. (1995)
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   - Amplitude equation theory
   - Generalized Swift-Hohenberg simulations
   - Experiment
   - Simulations of full fluid equations

4 Conclusions
Accurate simulation of long-time dynamics
Exponential convergence in space, third order in time
Efficient parallel algorithm, unstructured mesh
Arbitrary geometries, realistic boundary conditions
Simulations Complement Experiments

- Knowledge of full flow field and other diagnostics (e.g. total heat flow)
- No experimental/measurement noise (roundoff “noise” very small)
- Measure quantities inaccessible to experiment e.g. Lyapunov exponents and vectors
- Readily tune parameters
- Turn on and off particular features of the physics (e.g. centrifugal effects, realistic v. periodic boundary conditions)
Periodic Boundaries

Realistic Boundaries
Lyapunov Exponent
Jayaraman et al. (2005)

Temperature

Temperature Perturbation
Lyapunov Exponent
(Jayaraman et al., 2005)

Aspect ratio $\Gamma = 40$, Prandtl number $\sigma = 0.93$, rotation rate $\Omega = 40$

R=2275
Summary of results of full 3d fluid simulations:

- Simulations of Rayleigh-Bénard convection with Coriolis forces give $\tau \sim \varepsilon^{-1}$ for small enough $\varepsilon$. For larger $\varepsilon$ a slower growth is seen perhaps consistent with $\tau \sim \varepsilon^{-0.7}$ [Scheel and MCC (2005)]

- Scaling of largest Lyapunov exponent consistent with $\lambda \sim c + \varepsilon^{1}$ with $c$ comparable to the finite size shift in onset [Jayaraman et al. (2006)]

- Role of centrifugal force [Becker, Scheel, MCC, and Ahlers (2006)]
Slopes give frequency $\propto \varepsilon^{1.07}$ ($\Gamma = 40$ cylinder) and $\varepsilon^{1.04}$ (periodic)
Scaling of Lyapunov Exponent

Jayaraman et al. (2005)

\[ \Gamma = 40 \]
Importance of Centrifugal Force
Becker, Scheel, MCC, and Ahlers (2006)

Aspect ratio $\Gamma = 20$, $\varepsilon \approx 1.05$, $\Omega = 17.6$

Centrifugal force 0  Centrifugal force x4  Centrifugal force x10
Time Scaling
Becker, Scheel, MCC, and Ahlers (2006)

- simulations $\Gamma = 20$ with centrifugal force $\times 2$; $\square$ experiment $\Gamma = 40$
- simulations $\Gamma = 40$ no centrifugal force
Simulations of the full fluid equations near onset without centrifugal forces are consistent with predictions of scaling of times as $\varepsilon^{-1}$; not yet able to probe scaling of lengths.

Centrifugal forces are important in experiment, enhancing the finite size effects and limiting size of region of domain chaos.

Maximum centrifugal force cf. Coriolis force $\sim (\alpha \Delta T) \Omega \Gamma / u$. (Near threshold $\Omega_{KL} \sim 10^1$, $u \sim \varepsilon^{1/2}$).
1 Rayleigh-Bénard Convection

2 Spatiotemporal Chaos
   ■ What is it?
   ■ Spatiotemporal Chaos in Rayleigh-Bénard convection

3 Domain Chaos
   ■ Amplitude equation theory
   ■ Generalized Swift-Hohenberg simulations
   ■ Experiment
   ■ Simulations of full fluid equations

4 Conclusions
Conclusions

- Spatiotemporal chaos is a third paradigm of complex dynamics (cf. chaos, turbulence)
- Rotating convection shows spatiotemporal chaos in the weakly nonlinear regime near onset where there is hope for a quantitative understanding.
- Numerical simulations of realistic experimental geometries are now feasible
Conclusions

- Spatiotemporal chaos is a third paradigm of complex dynamics (cf. chaos, turbulence)
- Rotating convection shows spatiotemporal chaos in the weakly nonlinear regime near onset where there is hope for a quantitative understanding.
- Numerical simulations of realistic experimental geometries are now feasible
- Truncated amplitude equation model makes predictions for scaling of lengths $\propto \varepsilon^{-1/2}$ and times $\propto \varepsilon^{-1}$
- Scalings and features of dynamics predicted by truncated amplitude model confirmed by GSH simulations and full fluid simulations
- Disagreement between experiment and predictions resolved (finite size, centrifugal effects)
To Do

- More precise experimental tests of the (homogeneous) theory
- Understand theoretically how good the truncated amplitude equation model should be
- Relate to lattice systems of coupled heteroclinic oscillators
- Understand the origins of chaos in the system and models
To Do

- More precise experimental tests of the (homogeneous) theory
- Understand theoretically how good the truncated amplitude equation model should be
- Relate to lattice systems of coupled heteroclinic oscillators
- Understand the origins of chaos in the system and models

THE END