Physics 127b: Statistical Mechanics

Renormalization Group: 1d Ising Model

The ReNormalization Group (RNG) gives an understanding of *scaling* and *universality*, and provides various approximation schemes to calculate exponents etc. We will first motivate and illustrate the method using the 1d Ising ferromagnet, following Nelson and Fisher [Annals of Physics **91**, 226 (1975)].

The 1d Ising model is analytically soluble using various methods. We will be able to implement the RNG explicitly and without approximation. Usually, an explicit implementation requires approximations. The 1d Ising model (as is true for any 1d system with short range interactions) has a ordered phase only at zero temperature. We can think of this as a "zero temperature phase transition". This leads to an important difference from conventional, finite temperature phase transitions: rather than the *scaling variable* being $t = 1 - T/T_c$ (which makes no sense if T_c is zero) the analogous variable is $e^{-T_0/T}$ (which we will call x) with $k_B T_0$ an appropriately chosen excitation energy. Once this change is made, the results illustrate the general case very well.

Perturbation expansion

We consider N Ising spins $s_i = \pm 1$ with periodic boundary conditions (i.e. on a ring). The Hamiltonian in zero field is

$$H = -J \sum_i s_i s_{i+1}. \tag{1}$$

The ground state—the ordered state at $T = 0$—is the aligned state with spins all up or all down and energy $E = -NJ$. The minimum excitation energy to flip one spin is $4J$ (switch 2 bonds from $-J$ to J), and so the natural variable describing the finite temperature properties is

$$x = e^{-4J/k_B T}. \tag{2}$$

We will try to develop a perturbation expansion in powers of x to describe the low temperature behavior, and, since $T_c = 0$, the transition to the disordered phase.

We organize the expansion in terms of flipping an isolated single spin, an isolated pair of spins, an isolated triplet etc. There are N choices of an isolated spin to flip (energy cost $4J$). Two flip two isolated spins (energy cost $8J$) we can choose the first one in N ways, and the second one in $N - 3$ ways, since flipping the neighboring spins to the first one will not give isolated single spin flips, and will cost a different energy. Then we divide by 2 since the ordering of which spin is flipped first doesn't matter. In this we we have for the partition function

$$\begin{aligned} Q_N = e^{NJ/k_B T}\{2 & \\ & + Nx + \frac{1}{2}N(N-3)x^2 + \cdots \quad \text{isolated single spins} \\ & + Nx + \frac{1}{2}N(N-5)x^2 + \cdots \quad \text{isolated pairs of spins} \\ & \vdots \quad (\text{ total of } N-1 \text{ lines}) \\ \}. & \end{aligned} \tag{3}$$

Summing gives

$$Q_N = 2e^{NJ/k_B T}\{1 + \frac{1}{2}N(N-1)x + O(N^3 x^2)\cdots\} \tag{4}$$

and for the free energy per spin

$$f = -J + k_B T\{\frac{1}{2}(N-1)x + \cdots\}. \tag{5}$$

We see that in the thermodynamic limit the series *diverge* due to the possibility of thermal fluctuations over an infinite range of length scales (one spin flipped, a pair flipped,… half the spins flipped). For the $1d$ Ising model these fluctuations all have identical energy! This is indeed a special feature of the one dimensionality, but in higher dimensions fluctuations over all length scales indeed become important at the transition point.

Conclusion: $T = 0, B = 0$ is a *critical point* of the $1d$ Ising model. Taylor expansions about this point break down. In fact we will see that the expansion is nonanalytic, e.g.

$$\frac{f}{k_B T} = -\frac{J}{k_B T} + x^{1/2} + \cdots \tag{6}$$

with a *nontrivial exponent* $1/2$.

Renormalization Group

The key idea is not to try to treat all length scales in one shot. Instead use an iterative procedure to first treat short length scales, and study their effect ("renormalization") on the next larger scale etc. Furthermore, rather than studying how the free energy varies with parameters of a fixed Hamiltonian, the RNG studies how the Hamiltonian evolves to maintain a fixed free energy under the elimination of successive length scale fluctuations.

For the $1d$ Ising model we can proceed completely analytically. For convenience let as define a reduced Hamiltonian

$$\bar{H} = -\frac{H}{k_B T} = K\sum_i s_i s_{i+1} + h\sum_i s_i + CN \tag{7}$$

with $K = J/k_B T$, and $h = \mu B/k_B T$, and we have added a "zero of energy" constant CN for complete generality. A convenient definition of the corresponding partition function is

$$\bar{Q}_N = \prod_i \frac{1}{2}\sum_{s_i=\pm 1} e^{\bar{H}(\{s_i\})} = Tr_N(e^{\bar{H}}) \tag{8}$$

where the notation Tr_N is introduced to denote the trace over all configurations of the N spins. Note that we have added an extra factor of $1/2$ in the "trace", i.e. Tr_N is an average, rather than the mathematical trace. This corresponds to subtracting the entropy term $Nk_B T \ln 2$ from the free energy, so that

$$-k_B T \ln \bar{Q}_N = A - Nk_B T \ln 2. \tag{9}$$

The free energy we calculate from $\bar{Q}_N$ is therefore the deviation from the free spin result—precisely the quantity we are interested in.

Rather than doing the Tr_N all at once, we do the trace operation (i.e. average) over the states of *every other* spin (or in general every bth spin, leaving a fraction $(b-1)/b$ remaining—we are doing the case $b = 2$).

Consider first the effect of averaging over the states of a particular spin s with neighbors s_+ and s_-. Focus your attention on the terms in the product in Eq. (8) for Q_N involving the spin s

$$Tr_s = \frac{1}{2}\sum_{s=\pm 1} \ldots e^{Ks_- s + \frac{1}{2}h(s_- + s) + C} e^{Kss_+ + \frac{1}{2}h(s + s_+) + C} \ldots \tag{10a}$$

$$= \ldots e^{\frac{1}{2}h(s_- + s_+) + 2C} \cosh(Ks_- + Ks_+ + h) \ldots. \tag{10b}$$

(Notice that we consistently associate half the magnetic field term and the constant term with the "backward" bond, and half with the "forward bond"). Obviously the variable s no longer occurs in $\bar{Q}_N$—we have done the necessary average over its two possible configurations. No approximation has been made, and repeating this procedure correctly evaluates $\bar{Q}_N$.

Now we would like to automate this scheme by setting up an iterative procedure. We can try to do this by rewriting Eq. (10) in terms of a new *effective Hamiltonian* involving s_+ and s_- (the rest of the terms in the product for $\bar{Q}_N$ are, so far, unchanged). In the present case this new Hamiltonian takes the same *form*, but with changed (renormalized) parameters.

We look for K', h', C' so that

$$e^{\frac{1}{2}h(s_-+s_+)+2C}\cosh(Ks_- + Ks_+ + h) = e^{K's_-s_+ + \frac{1}{2}h'(s_-+s_+)+C'} \tag{11}$$

for *all* choices of $s_-, s_+ = \pm 1$. Since there are 4 states of s_- and s_+, and only 3 parameters, it is not immediately clear that this can be done. Indeed usually, i.e. more realistic systems or higher dimensions, it will not be possible, *and the Hamiltonian must be made more complicated as the iteration proceeds.* However in the present case the only quantities appearing, namely s_-s_+ and $s_- + s_+$, depend only on whether the spins are $\uparrow\uparrow$, $\downarrow\downarrow$, or ($\uparrow\downarrow$ or $\downarrow\uparrow$), i.e. only three different possibilities, so the three parameters are enough to satisfy Eq. (11). Explicitly

$$e^{h+2C}\cosh(2K + h) = e^{K'+h'+C'}, \tag{12a}$$

$$e^{2C}\cosh(h) = e^{-K'+C'}, \tag{12b}$$

$$e^{-h+2C}\cosh(-2K + h) = e^{K'-h'+C'}. \tag{12c}$$

The solutions are easily obtained by multiplying various combinations

$$e^{2h'} = e^{2h}\frac{\cosh(2K + h)}{\cosh(2K - h)} \tag{13a}$$

$$e^{4K'} = \frac{\cosh(2K + h)\cosh(2K - h)}{\cosh^2 h} \tag{13b}$$

$$e^{4C'} = e^{8C}\cosh(2K + h)\cosh(2K - h)\cosh^2 h \tag{13c}$$

(e.g. the first equation is given by dividing the first of Eq. (12a) by the third).

Recursion Relations

Clearly this procedure can be repeated for every other spin, and we end up with a system with the same free energy with $N/2$ spins, twice the lattice spacing, and described by the same Hamiltonian but with parameters K', h', C'. The final step of the renormalization group is to rescale lengths down by a factor of b (2 in our case), so that the lattice "looks the same".[1] This gives us the "scale factor $b = 2$" renormalization group

$$\bar{H}' = \mathcal{R}_b[\bar{H}] \tag{14}$$

defined by the *recursion relations* Eq. (13) which can be written

$$K' = \mathcal{R}_K(K, h), \tag{15a}$$

$$h' = \mathcal{R}_h(K, h), \tag{15b}$$

$$C' = bC + R_c(K, h). \tag{15c}$$

[1] The number of spins is $N/2$,down by a factor of b, but remember we are interested in the free energy density in the $N \to \infty$ limit.

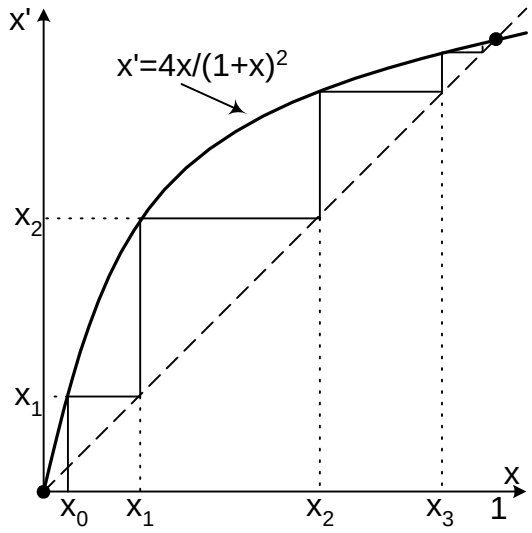

Figure 1: Recursion relation for the temperature variable x in the $1d$ Ising model. Fixed points are at $x = 0$ and $x = 1$. The "steps" yield successive values of x_l under the recursion.

Note that the constant term does not appear in the recursion relations for K, h, and only as a simple additive piece in the recursion relation for C. It keeps track of the contribution to the free energy coming from the variables averaged over. We will concentrate on the evolution of the interaction parameters K, h, and so will not consider the third equation further.

Remember that the partition function and so the free energy is *preserved* by the transformation, so we have for the free energy density

$$f[\bar{H}'] = b^d f[\bar{H}] = 2 f[\bar{H}] \tag{16}$$

corresponding to the thinning out of the spins (or the shrinking of the full lattice). Similarly the correlations between the surviving spins is preserved, so that for the correlation length

$$\xi[\bar{H}'] = b^{-1}\xi[\bar{H}] = \frac{1}{2}\xi[\bar{H}] \tag{17}$$

again corresponding to the trivial shrinking of the lattice. Equations (16),(17), although trivial, will later be important in understanding the content of the RNG.

We can now successively repeat the elimination i.e. solve the problem by iteration.

First consider $h = 0$. Equation (13a) becomes

$$e^{4K'} = \cosh^2(2K) = \frac{1}{4}(e^{2K} + e^{-2K})^2 \tag{18}$$

or for $x = e^{-4K}$

$$x' = \frac{4x}{(1+x)^2}. \tag{19}$$

The iteration of this equation can be understood graphically Fig. 1. The two *fixed points* defined by $x' = x$ play a key role. At a fixed point the renormalization does not change the Hamiltonian. Equation (17) then shows us that ξ must be zero or infinite at a fixed point.

The $x = 1$ (i.e. $K \to 0$) is the *high temperature fixed point*. The interactions renormalize to zero, and the behavior is simple. There are no large length scale phenomena, and the correlation length is zero. This

fixed point is *stable* or *attracting*—starting from any initial value of $x \neq 0$ eventually leads to this fixed point. Physically, any finite temperature state, inspected at long enough length scales, looks like the infinite temperature solution, i.e. is *disordered*.

The $x = x^* = 0$ fixed point is the nontrivial *critical* fixed point. The corresponding *fixed point Hamiltonian* $\bar{H}^* = \bar{H}(x^*, h^* = 0)$ satisfies

$$\mathcal{R}_b[\bar{H}^*] = \bar{H}^*. \tag{20}$$

The fixed point corresponds to $K \to \infty$, and so to nontrivial behavior. This state has an infinite correlation length $\xi \to \infty$. Note that this fixed point is *unstable* or *repelling*. This is a key result:

> The properties of the unstable, critical fixed point determine the physical behavior in the critical regime near the transition temperature.

The fixed points tell us particularly simple behavior. The recursion relations under the renormalization procedure allow us to relate the physical Hamiltonian (some given K or x and h) to another Hamiltonian. For example if we iterate many times, eventually the Hamiltonian for any nonzero temperature is related to the Hamiltonian at the high temperature fixed point, where the properties are easy to calculate. This gives a *tractable* scheme for calculating a *subset* of behavior, namely the long length scale behavior that is left after the elimination processes. Of particular interest is the behavior as $T \to 0$. This means the physical system corresponds to a value of x close to the unstable fixed point. We can understand properties of the system (beyond the simple fact that the state is disordered) by studying how x "flows away" from the unstable fixed point under the iteration procedure.

Critical Behavior

The critical behavior (i.e. x small) may be understood by *linearizing* about the unstable fixed point. Write the physical value of x as x_0, and assume

$$x_0 = x^* + \delta x_0 \tag{21}$$

with δx_0 small. Then under each iteration of the RNG we find by linearizing Eq. (19) that δx is simply multiplied by 4, so that after some chosen number l iterations

$$\delta x_l = (\Lambda_x)^l \delta x_0 \quad \text{with } \Lambda_x = 4 \tag{22}$$

where Λ_x is an *eigenvalue* of the linearization of the recursion relations about the fixed point $x = x^*$. This allows us to relate the physical behavior at x_0 to the behavior we calculate with the renormalized Hamiltonian given by $x_l = x^* + \delta x_l$. In particular we have for the free energy density

$$f(x_0) = \frac{1}{2^l} f(x_f = \Lambda_x^l x_0) \tag{23}$$

for x_0 small, where we have used the fact that x^* is zero. This result is good providing l is not so large that x_f is no longer small. A trivial-sounding statement, but another key point is:

> The trick of using the renormalization group at a critical point is to choose the number of iterations l so that something is learned (e.g. from the right hand side of Eq. (23)).

We often want to know the behavior as x is varied towards the critical point ($T \to T_c$). Let us choose l so that *as* x_0 *varies,* x_f *remains fixed* at some small value so that the linearization remains valid. This means that we choose

$$l = \frac{\ln(x_f/x_0)}{\ln \Lambda_x}. \tag{24}$$

Then[2]

$$f(x_0) = 2^{-\ln(x_f/x_0)/\ln\Lambda_x} f(x_f) \tag{25}$$

$$= x_0^{\ln 2/\ln\Lambda_x} [x_f^{-\ln 2/\ln\Lambda_x} f(x_f)]. \tag{26}$$

The first term tells us what we want to know—how does the free energy depend on the initial x (and so on temperature). The second term in the brackets does not depend on x_0, and is just a constant prefactor in the x_0 dependence.

Thus we have found for small x

$$f(x) = Ax^{1/\lambda_x} \quad \text{with} \quad \lambda_x = \frac{\ln\Lambda_x}{\ln 2} = 2. \tag{27}$$

or in conventional notation

$$f(x) \sim x^{2-\alpha_x} \quad \text{with} \quad \alpha_x = 2 - \frac{1}{\lambda_x}, \tag{28}$$

The nontrivial power law dependence on x (i.e. square root) corresponds to the power law dependence on t at a finite temperature phase transition. Similarly

$$\xi(x_0) = 2^l \xi(x_f) \tag{29}$$

so that following the same procedure

$$\xi \sim x^{-\nu_x} \quad \text{with} \quad \nu_x = \frac{1}{\lambda_x}. \tag{30}$$

Notice that the hyperscaling relation $2 - \alpha_x = d\nu_x$ is satisfied (remember $d = 1$ here): hyperscaling follows directly from the scaling of f with b^{-ld} and the correlation length with b^l.

Scaling of the field

Now we include the field h. The critical fixed point is $x^* = h^* = 0$. The recursion relation for h can be written

$$h' = h + \frac{1}{2}\ln\left[\frac{e^h + xe^{-h}}{e^{-h} + xe^h}\right]. \tag{31}$$

Linearizing about the fixed point gives

$$\delta h' = 2\delta h \tag{32}$$

and the equation for δx is unchanged. Thus, in general notation

$$\delta x' = \Lambda_x \delta x \quad \text{with} \quad \Lambda_x = b^{\lambda_x} \quad \text{and} \quad \lambda_x = 2, \tag{33a}$$

$$\delta h' = \Lambda_h \delta h \quad \text{with} \quad \Lambda_h = b^{\lambda_x} \quad \text{and} \quad \lambda_h = 1. \tag{33b}$$

(More generally we might expect a matrix equation

$$\begin{bmatrix} \delta x' \\ \delta h' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} \delta x \\ \delta h \end{bmatrix} \tag{34}$$

[2] We often have to manipulate exponent expressions such as $2^{-\ln(x_f/x_0)/\ln\Lambda_x}$, where we want to look at the dependence on x_f/x_0 behavior. To do this write the expression as $\exp[-\ln(x/x_0)\ln 2/\ln\Lambda_x]$ which can then be rewritten $(x/x_0)^{-\ln 2/\ln\Lambda_x}$.

and then we would have to diagonalize to find the two eigenvalues, and the different linear combinations of δx and δh that diverge exponentially from the fixed point.) So now iterating l times from the initial values $(x_0 = x, h_0 = h)$ near the fixed point (0, 0)

$$x_l = 2^{\lambda_x l} x \tag{35a}$$

$$h_l = 2^{\lambda_h l} h \tag{35b}$$

and

$$f(x, h) = 2^{-l} f(x_l, h_l). \tag{36}$$

Again choose l so that x_l is some small fixed value x_f as x varies

$$2^l = (x_f/x)^{1/\lambda_x} \tag{37}$$

so that

$$f(x, h) = \left(\frac{x}{x_f}\right)^{1/\lambda_x} f(x_f, \left(\frac{x_f}{x}\right)^{\lambda_h/\lambda_x} h), \tag{38}$$

which can be written in the form

$$f(x, h) = A\, x^{2-\alpha_x} Y(D\frac{h}{x^{\Delta_x}}) \tag{39}$$

and we have derived the *scaling form* with exponents

$$2 - \alpha_x = \frac{1}{\lambda_x} = \frac{1}{2}, \tag{40a}$$

$$\Delta_x = \frac{\lambda_h}{\lambda_x} = \frac{1}{2}. \tag{40b}$$