Lecture 2 Supplementary Notes: Derivation of the Phase Equation

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Derivation from Amplitude Equation

Near threshold the phase reduces to the phase of the complex amplitude, and the phase equation can be derived by "adiabatically eliminating" the relatively fast dynamics of the magnitude. The basic assumption is that we are looking at the dynamics driven by gradual spatial variations of the phase, i.e. that derivatives of θ are small. For simplicity we will also assume that we are looking at small deviations from a straight stripe pattern, so that the phase perturbations themselves may also be considered small. This leads to the *linear* phase diffusion equation first derived by Pomeau and Manneville (1979). We will consider the full nonlinear phase equation in the more general context away from threshold.

Consider the (scaled) amplitude equation

$$\partial_T \bar{A} = \bar{A} + \left(\partial_X - \frac{i}{2}\partial_Y^2\right)^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A}$$

Look at small perturbations about the state $\bar{A} = a_K e^{iKX}$ with $a_K^2 = 1 - K^2$,

$$\bar{A} = a e^{iKX} e^{i\theta}, \qquad a = a_K + \delta a$$

Expand in

- small phase perturbations θ and amplitude perturbations δa
- small derivatives of θ (up to second order)

Then using

$$e^{-iKX} e^{-i\theta} \partial_T A = \partial_T a + i a \partial_T \theta,$$

the real part of the equation gives the dynamical equation for a, and the imaginary part of the equation gives the dynamical equation for θ. The real part gives

$$\partial_T \delta a = -2a_K^2 \delta a - 2K a_K \partial_X \theta + \partial_X^2 \delta a$$

For time variations on a T-scale much longer than unity, the term on the left hand side is negligible, and δa is said to adiabatically follow the phase perturbations. The term in $\partial_X^2 \delta a$ will lead to phase derivatives that are higher than second order, and so can be ignored. Hence

$$a_K \delta a \simeq -K \partial_X \theta.$$

The imaginary part gives

$$a_K \partial_T \theta \simeq 2K \partial_X \delta a + a_K \partial_X^2 \theta + a_K K \partial_Y^2 \theta.$$

Eliminating δa and using $a_K^2 = 1 - K^2$ gives

$$\partial_T \theta = \left[\frac{1 - 3K^2}{1 - K^2}\right] \partial_X^2 \theta + K \partial_Y^2 \theta.$$

the phase *diffusion equation* in scaled units.

Returning to the unscaled units we get the phase diffusion equation

$$\partial_t \theta = D_\parallel \partial_x^2 \theta + D_\perp \partial_y^2 \theta$$

with diffusion constants for the state with wave number $q = q_c + k$ (with k related to K by $k = \xi_0^{-1}\varepsilon^{1/2}K$)

$$D_\parallel = (\xi_0^2 \tau_0^{-1}) \frac{\varepsilon - 3\xi_0^2 k^2}{\varepsilon - \xi_0^2 k^2}$$
$$D_\perp = (\xi_0^2 \tau_0^{-1}) \frac{k}{q_c}.$$

A negative diffusion constant leads to exponentially growing solutions, i.e. the state with wave number $q_c + k$ is *unstable* to long wavelength phase perturbations for

$\lvert\xi_0 k\rvert > \varepsilon^{1/2}/\sqrt{3}$	$D_\parallel < 0$: longitudinal (Eckhaus)
$k < 0$	$D_\perp < 0$: transverse (ZigZag)

General Method

Away from threshold the internal degrees of freedom as well as the overall magnitude again relax rapidly compared to the phase variable for gradual spatial variations of the phase. The method of multiple scales can again be used to derive the equation for the phase equation. This application of the method is a little different from the derivation of the amplitude equation in that the slow scale is not determined by an independent parameter such as ε, but itself *defines* the small parameter. The small parameter is essentially the reciprocal of the length scale of the spatial variation (in units of the periodicity of the pattern).

The starting point for the derivation of the phase equation is the definition of the phase variable in terms of the wave vector field

$$\nabla\theta(\mathbf{x}_\perp, t) = \mathbf{q}(\mathbf{X}, T), \tag{1}$$

or

$$\theta = \int \mathbf{q}(\mathbf{X}, \mathbf{T}) \cdot d\mathbf{x}_\perp. \tag{2}$$

In these equations a slow space variable $\mathbf{X}$ has been introduced. It is defined as

$$\mathbf{X} = \eta \mathbf{x}_\perp, \tag{3}$$

where η is the small parameter such that the slow spatial variations of interest in the pattern occur over a length scale of order unity in the $\mathbf{X}$ variable. Similarly a slow time scale T is introduced

$$T = \eta^2 t, \tag{4}$$

where the scaling with η^2 anticipates the diffusive nature of the dynamics. The wave vector defines the orientation and local periodicity of the pattern: this variable therefore varies on the long length scale, and is a function of the slow variable $\mathbf{X}$, but not of the fast variable $\mathbf{x}$. In turn this slow spatial variation will induce dynamics on the slow time scale. Note that Eq. (1) applies in regions of smooth variation of the pattern, away from defects and disordered regions.

The expressions Eqs. (1), (2) are not easy to work with, because they mix the fast and slow coordinates $\mathbf{x}_\perp$, $\mathbf{X}$ in an inconvenient way To develop the systematic perturbation expansion it is useful to introduce a *scaled* phase variable $\Theta(\mathbf{X}, T)$, through

$$\Theta = \eta\theta, \tag{5}$$

so that the derivatives of Θ with respect to $\mathbf{X}$ are $O(1)$ (the first derivative is just the wave vector). In terms of the scaled phase we have

$$\mathbf{q}(\mathbf{X}) = \nabla_{\mathbf{X}}\Theta(X), \quad \Theta(\mathbf{X}) = \int \mathbf{q}(\mathbf{X}) \cdot d\mathbf{X}. \tag{6}$$

This clever trick allows the inclusion in the same formal expansion scheme of both the first derivative of θ, which is $O(1)$ and gives the local wave vector, and higher derivatives of θ, which are $O(\eta)$, and give the slow spatial variation.

With the definitions Eq. (6), the derivation of the phase dynamics follows quite closely the multiple scales derivation of the amplitude equation. In the present case, we expand the evolution equations for the fields $\mathbf{u}(\mathbf{x}, t)$ in powers of η, corresponding to the slow spatial variation of $\mathbf{q}$.

The zeroth order solution for $\mathbf{u}$ (i.e. no effect of the spatial variation of $\mathbf{q}$) is the fully nonlinear, spatially periodic solution $\mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z)$, which corresponds to the ideal stripe state with wave vector $\mathbf{q}$. Since $\mathbf{u}_{\mathbf{q}}$ is periodic in $\mathbf{x}_\perp$ with period $2\pi q^{-1}$ in the $\hat{\mathbf{q}}$ direction, we redefine the spatially periodic function in terms of the phase

$$\mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z) = \bar{\mathbf{u}}_{\mathbf{q}}(\theta, z), \quad \theta = \mathbf{q} \cdot \mathbf{x}_\perp. \tag{7}$$

The expansion in powers of η is then

$$\mathbf{u}(\mathbf{x}_\perp, z, t) = \mathbf{u}^{(0)}(\theta, z; \mathbf{X}, T) + \eta \mathbf{u}^{(1)} + \cdots, \tag{8}$$

where the dependence of $\mathbf{u}^{(i)}$ on the slow variables $\mathbf{X}$, T arises through the implicit dependence on $\mathbf{q}(\mathbf{X}, T)$. In particular we have for the zeroth order term

$$u^{(0)}(\theta, z; \mathbf{X}, T) = \bar{\mathbf{u}}_{\mathbf{q}(\mathbf{X},T)}(\theta, z). \tag{9}$$

Equation (8)is substituted into the evolution equations for the system, and terms at each order in η are collected. To derive the lowest order phase equation, we need only go up to terms that are first order in η. These terms arise from slow spatial derivatives, slow time dependence, and also the term $\eta\mathbf{u}^{(1)}$ in Eq. (8). For example, a spatial derivative acting on $\mathbf{u}^{(i)}$ gives

$$\nabla \mathbf{u}^{(i)} \to \mathbf{q}\partial_\theta \mathbf{u}^{(i)} + \eta \nabla_{\mathbf{X}} \mathbf{u}^{(i)}. \tag{10}$$

Higher order derivatives may also be needed, for example

$$\nabla^2 \mathbf{u}^{(i)} \to q^2 \partial_\theta^2 \mathbf{u}^{(i)} + \eta D \partial_\theta \mathbf{u}^{(i)} + O(\eta^2), \tag{11}$$

with the operator D defined by

$$D = 2\mathbf{q} \cdot \nabla_{\mathbf{X}} + (\nabla_{\mathbf{X}} \cdot \mathbf{q}). \tag{12}$$

Also, the time derivative gives

$$\partial_t \mathbf{u}^{(i)}(\theta, z; \mathbf{X}, T) = \eta^2 \partial_T \theta \, \partial_\theta \mathbf{u}^{(i)} + \eta^2 \partial_T \mathbf{u}^{(i)} = \eta \partial_T \Theta \, \partial_\theta \mathbf{u}^{(i)} + O(\eta^2). \tag{13}$$

At $O(\eta)$ there are also terms $\eta L \mathbf{u}^{(1)}$, with L the linear operator given by linearizing the equations of motion about $\mathbf{u}^{(0)}$. We know from physical arguments that L has an eigenvector with zero eigenvalue, and so the phase equation appears as the solvability condition that the equation for $\mathbf{u}^{(1)}$ has a finite solution. Here we see the close similarity with the derivation of the amplitude equation. The zero mode in the present case corresponds to a translation of the solution, and so takes the from $\nabla \mathbf{u}^{(0)}$.

This procedure is illustrated for the simple example of the Swift-Hohenberg equation in the following section.

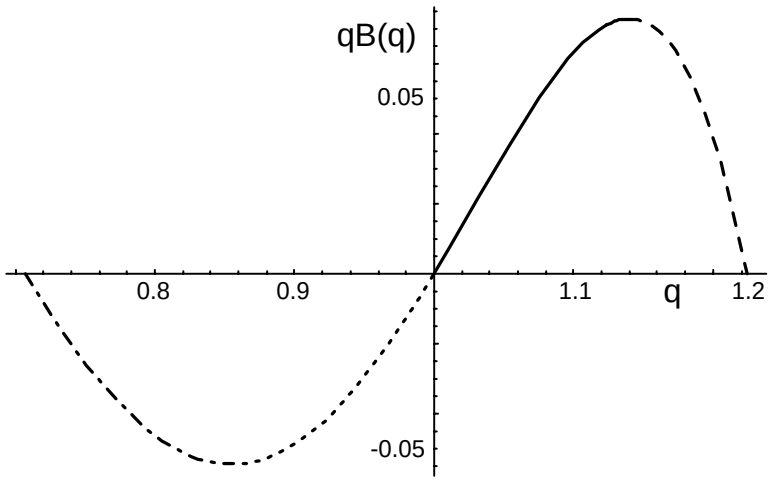

Figure 1: Plot of $qB(q)$ in the phase equation for the Swift-Hohenberg model. A control parameter value of $r = 0.25$ was used in constructing the plot. The solid line is where B and $(qB)'$ (with the prime denoting the q derivative) are positive, the dashed line is where $(qB)'$ is negative, the dotted line where B is negative, and the dashed-dotted line where both B and $(qB)'$ are negative. The range of q for the solid curve is the wave number band for which stripes are linearly stable against long length scale perturbations.

Phase Equation for the Swift-Hohenberg Equation

The Swift-Hohenberg equation is the equation for a real scalar field u. Here we will use the equation in two space dimensions, when it can be written in the form

$$\partial_t u(\mathbf{x}, t) = ru - (\nabla^2 + 1)^2 u - u^3, \tag{14}$$

with $\mathbf{x} = (x, y)$, and $\nabla^2 = \partial_x^2 + \partial_y^2$. As in Eq. (8), we expand u as an expansion in powers of η, to give

$$u(\mathbf{x}, t) = u^{(0)}(\theta, z; \mathbf{X}, T) + \eta u^{(1)} + \text{h.o.t.}, \tag{15}$$

with $\mathbf{X}$, T the slow space and time variables, as in Eqs. (3,4), and $u^{(0)}$ the zeroth order solution

$$u^{(0)}(\theta, z; \mathbf{X}, T) = \bar{u}_{q(\mathbf{X},T)}(\theta), \tag{16}$$

where $\bar{u}_q(\theta = qx)$ is the nonlinear, spatially periodic, time independent solution for straight stripes which satisfies

$$r\bar{u}_q(\theta) - (q^2\partial_\theta^2 + 1)^2 \bar{u}_q(\theta) - \bar{u}_q^3(\theta) = 0. \tag{17}$$

The h.o.t. in Eq. (15) denotes terms that are second order and higher in η.

We now substitute Eq. (15) into the evolution equation, Eq. (14). We will need the rather complicated operator involving up to fourth order derivatives

$$(\nabla^2 + 1)^2 \to \left[(q^2\partial_\theta^2 + 1) + \eta D\partial_\theta\right]\left[(q^2\partial_\theta^2 - 1) + \eta D\partial_\theta\right] + \text{h.o.t.} \tag{18}$$

$$= (q^2\partial_\theta^2 + 1)^2 + \eta\left\{2\partial_\theta(q^2\partial_\theta^2 + 1)D + [2\mathbf{q}\cdot\nabla_X(q^2)]\partial_\theta^3\right\} + \text{h.o.t.}. \tag{19}$$

The other terms in Eq. (14) are easy to evaluate up to first order in η

$$\partial_t u(\mathbf{x}, t) \to \eta(\partial_T\Theta)\partial_\theta\bar{u}_q(\theta) + \text{h.o.t.}, \tag{20}$$

$$ru - u^3 \to r\bar{u}_q - \bar{u}_q^3 + \eta\left[r - 3\bar{u}_q^2\right]u^{(1)} + \text{h.o.t.}. \tag{21}$$

Now collecting terms at $O(\eta)$ we find the equation

$$\left[r-(q^2\partial_\theta^2+1)^2-3\bar{u}_q^2\right]u^{(1)}=(\partial_T\Theta)\partial_\theta\bar{u}_q(\theta)+\left\{2\partial_\theta(q^2\partial_\theta^2+1)D+[2\mathbf{q}\cdot\nabla_X(q^2)]\partial_\theta^3\right\}\bar{u}_q(\theta). \tag{22}$$

It is straightforward to check that $\partial_\theta\bar{u}_q$ is a zero-eigenvalue eigenvector of the operator on the left hand side

$$\left[r-(q^2\partial_\theta^2-1)^2-3\bar{u}_q^2\right]\partial_\theta\bar{u}_q=0, \tag{23}$$

as is expected from the translational symmetry. The operator acting on $u^{(1)}$ in Eq. (22) is self adjoint, and so the solvability condition, that the right hand side have no component along this eigenvector, reduces to the orthogonality condition for the right hand side with $\partial_\theta\bar{u}_q$:

$$(\partial_T\Theta)\int_0^{2\pi}d\theta(\partial_\theta\bar{u}_q)^2+\int_0^{2\pi}d\theta(\partial_\theta\bar{u}_q)\left\{2\partial_\theta(q^2\partial_\theta^2+1)D+[2\mathbf{q}\cdot\nabla_X(q^2)]\partial_\theta^3\right\}\bar{u}_q=0. \tag{24}$$

After integrating by parts with respect to θ some terms in the second integral, and rearranging, this reduces to

$$(\partial_T\Theta)\int_0^{2\pi}d\theta(\partial_\theta\bar{u}_q)^2=\nabla_X\cdot\left\{\mathbf{q}\int_0^{2\pi}d\theta\left[q^2(\partial_\theta^2\bar{u}_q)^2-(\partial_\theta\bar{u}_q)^2\right]\right\}. \tag{25}$$

Eq. (25) is in the form introduced in the lecture (returning to unscaled variables)

$$\tau(q)\partial_t\theta=\nabla\cdot[qB(q)] \tag{26}$$

with

$$\tau(q)=\frac{1}{\pi}\int_0^{2\pi}d\theta(\partial_\theta\bar{u}_q)^2, \tag{27a}$$

$$B(q)=\frac{1}{\pi}\int_0^{2\pi}d\theta\left[q^2(\partial_\theta^2\bar{u}_q)^2-(\partial_\theta\bar{u}_q)^2\right]. \tag{27b}$$

(Since we can multiply τ and B by the same arbitrary constant without changing the equation, I have included a normalization constant $1/\pi$ in these expressions for convenience.)

These integral expressions depend on knowing the full nonlinear, but spatially periodic stripe solutions to some satisfactory level of approximation. A simple lowest order mode truncation gives $\bar{u}_q\simeq a_q\cos\theta$, $\theta=qx$, with

$$a_q^2=\frac{4}{3}\left[r-(q^2-1)^2\right]. \tag{28}$$

Then we find

$$\tau(q)=a_q^2, \tag{29a}$$

$$B(q)=(q^2-1)a_q^2. \tag{29b}$$

The function a_q^2 is positive everywhere between the neutral stability curve of the uniform state, and goes to zero on the neutral stability curve. The function $B(q)$ changes sign at $q=1$. It is useful to plot $qB(q)$, since the slope of this curve is needed to calculate the parallel diffusion constant, and the signs of B and $(qB)'$ with the prime denoting the derivative with respect to q, are important in determining the stability of the stripe state against long wavelength perturbations. The dependence of qB on the wave number q for the Swift-Hohenberg model at $r=0.25$ is shown in the figure.