Chapter 10

Singular Measures and $f(\alpha)$

10.1 Definition

Another approach to characterizing the complexity of chaotic attractors is through the singularities of the measure. Demonstration 1 illustrates the occurrence of singularities in the measure of the quadratic map. In fact for $a = 4$ we can calculate the measure analytically

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{10.1}$$

and we see that the measure shows $x^{-1/2}$ singularities at the endpoints. For other values of a, or other chaotic attractors, the distribution of singularities is more complicated. The measure can in fact be characterized in terms of intertwined fractals with different measure singularities.

Consider a covering of the attractor with m-box size l. Different regions of the attractor may lead to different singularities in the measure so for some i we have the measure associated with the box

$$p_i \sim l^{\alpha_i} \tag{10.2}$$

where α_i is the exponent given the singularity. In terms of the ideas of the previous chapter α_i is the pointwise dimension at the point on the attractor on which the boxes are centered. Points with the scaling α_i may occur on a fractal set, i.e. on a set of nonintegral dimension which we will call $f(\alpha_i)$ so that the number of boxes with the measure l^{α_i} will vary as

$$N(\alpha_i) \sim l^{-f(\alpha_i)}. \tag{10.3}$$

The function $f(\alpha)$ is used to characterize the attractor: $f(\alpha)$ is the dimension of the set of points with pointwise dimension α.

For the quadratic map with $a = 4$ we have the two endpoints (dimension 0) where the measure associated with a box l is $\int_0^l x^{-1/2}dx \sim l^{1/2}$ and the interval $0 < x < 1$ (dimension 1) where the measure associated with a box l is $\sim l^1$. In this case we have

$$\begin{array}{rcl} f\left(\frac{1}{2}\right) & = & 0 \\ f(1) & = & 1 \end{array} . \tag{10.4}$$

10.2 Relationship to D_q

In the box counting definition of the generalized dimension D_q (see chapter 9) the boxes are weighted with the factor p^q with p the measure associated with the box. The scaling of the measure with decreasing box size at each point is given by the pointwise dimension. Thus large *positive* q tends to weight those points with large pointwise dimension (often visited, sometimes known as the "hot spots" of the attractor), and large *negative* q weights the points with low pointwise dimension (rarely visited, or "cold spots"). Since $f(\alpha)$ is the dimension of points with pointwise dimension α, for attractors with smooth D_q and $f(\alpha)$ an approach called the "thermodynamic formalism" allows us to relate these two functions: the two functions contain the same information about the attractor.

We will assume, more typically than the special case of the quadratic map at $a = 4$, that there is a continuous range of exponents α with a weight $w(\alpha)$ so that for a box size l we can estimate

$$\sum_i p_i^q \sim \int d\alpha w(\alpha)\, l^{-f(\alpha)}\, l^{q\alpha} = \int d\alpha w(\alpha) e^{\log l[q\alpha - f(\alpha)]}. \tag{10.5}$$

where in the first integral l^α gives the scaling of the measure with the box size at some point on the attractor, $l^{-f(\alpha)}$ gives the dimension of the set of points scaling in this way and $w(\alpha)$ is a smooth weight function. Now we use the method of steepest descents to evaluate the integral: as $l \to 0$, $\log l \to -\infty$, and for smooth $w(\alpha)$ the integral will be dominated by the value of α maximizing the exponent, i.e.

$$q = f'(a) \tag{10.6}$$

(together with $f''(\alpha) < 0$). This tells us that each value of q for the generalized dimension picks out a particular measure singularity α given by this relationship. Using this maximum value now gives the estimate

$$\sum_i p_i^q \sim e^{\log l[q\alpha - f(\alpha)]}, \tag{10.7}$$

where we use the value of q given by equation 10.6, so that the dimension is given by

$$D_q = \frac{1}{q-1}\left[q\alpha(q) - f(\alpha(q))\right] \tag{10.8}$$

where $\alpha(q)$ is the solution to (10.6).

These relationships can be inverted to give $f(\alpha)$ knowing D_q: differentiating we find

$$\alpha = \frac{d}{dq}\left[(q-1)\, D_q\right] \tag{10.9}$$

and then

$$f(\alpha) = -(q-1)\, D_q + q\alpha. \tag{10.10}$$

A typical $f(\alpha)$ curve has the following properties:

- Convex: $f'' < 0$;
- A maximum given by $f'(\alpha) = 0$ corresponding to $q = 0$ and $f(\alpha) = D_0$;
- The value $q = 1$ gives $f(\alpha) = \alpha$ and $f'(\alpha) = 1$, corresponding to the result that the set of points on the attractor that make up most of the measure have dimension D_1 and the pointwise dimension at these points is also D_1;
- $|q| \to \infty$ picks out the regions where the measure is most concentrated ($q > 0$) or least concentrated ($q < 0$). Here $\left|f'(\alpha)\right| \to \infty$, $D_q = \alpha$ and usually $f(\alpha) \to 0$ corresponding to a single point, i.e. the intersections of the $f(\alpha)$ curve with the $f = 0$ axis gives $D_{\pm\infty}$.

This leads to the expectation for typical $f(\alpha)$ shown in figure 10.1.

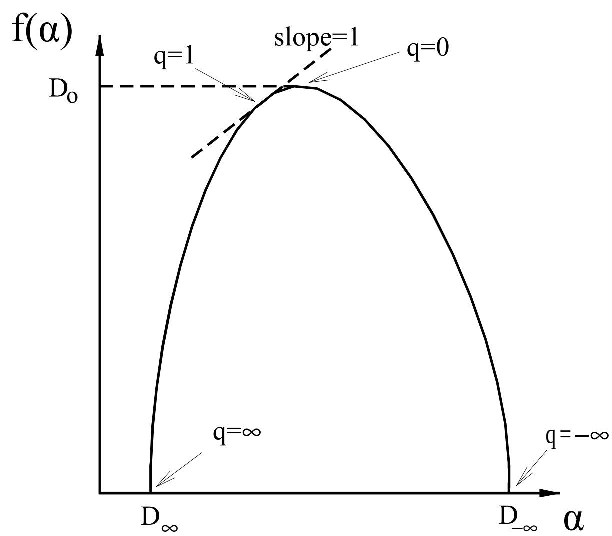

Figure 10.1: A typical $f(\alpha)$

There are formal analogies between the transformation $D_q \to f(\alpha)$ and the Legendre transformation between thermodynamic potentials e.g. $S(E) \to F(T)$, hence the description "thermodynamic formalism". For example we have

$$D_q = -\lim_{l\to 0} \frac{1}{q-1}\frac{1}{\log(1/l)} \log \sum_i \left(e^{-\alpha \log(1/l)}\right)^q , \quad \sum_i \to \int d\alpha\, e^{f(\alpha)\log(1/l)} \tag{10.11}$$

c.f. for the free energy as a function of inverse temperature in statistical physics

$$F_\beta = -\lim_{N\to\infty} \frac{1}{\beta}\frac{1}{N} \log \sum_i \left(e^{-E_i}\right)^\beta , \quad \sum_i \to \int dE\, e^{S(E)} \tag{10.12}$$

displaying the analogy through the translations $F(\beta) \leftrightarrow D_q$, $E/N \leftrightarrow \alpha$, $S(E)/N \leftrightarrow f(\alpha)$, $\beta \leftrightarrow q$, $N \leftrightarrow \log(q/l)$.

10.3 A simple multifractal

A simple example of a multifractal is given by the "two scale factor Cantor set", a generalization of the construction of the one-third Cantor set [1]. At first sight this seems an artificial construction. However sets generated by chaotic dynamics appear to show similar features, and looking at this set, although involving rather tedious algebra, provides useful insights into the significance of generalized dimensions and $f(\alpha)$. In addition the attractor for the bakers' map, which incorporates in a very simple way the "stretching and folding" that underlies chaotic dynamics, has exactly this structure. Thus studying the simple construction gives us many insights into what types of structure might occur in chaotic attractors.

Consider again successive divisions of the unit line, but now into unequal fractions l_1 and l_2. Also suppose a dynamics in which each segment is visited with the probability weights p_1 and p_2 (figure 10.2).

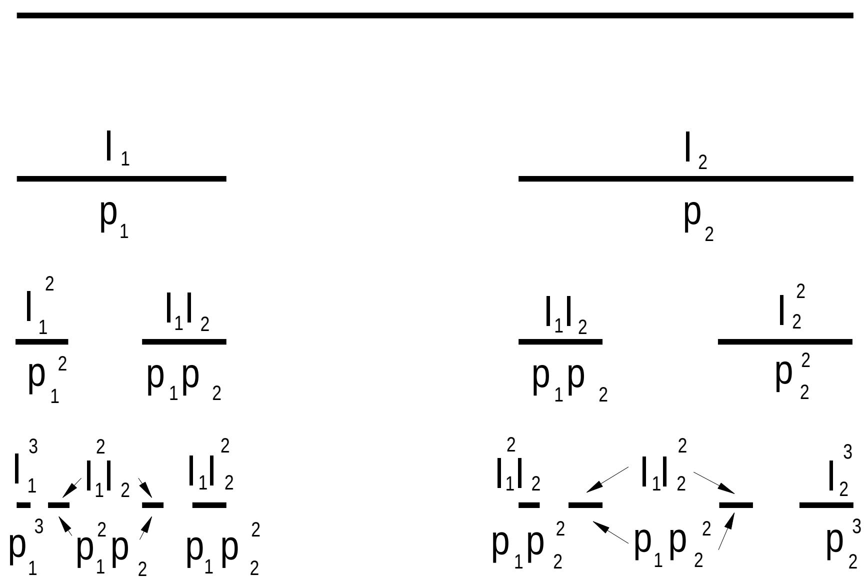

Figure 10.2: Construction of a multifractal

At the nth division level there are $C_m^n = \dfrac{n!}{m!(n-m)!}$ copies of the mth size $l_1^m l_2^{n-m}$ weighted with probability $p_1^m p_2^{n-m}$.

One feature of this set is that a chosen property is often completely dominated by a particular region of the set, identified by some fixed value of m/n as $n \to \infty$, and this is exploited in the analysis below. For example the measure associated with a particular length element $l_{mn} = l_1^m l_2^{n-m}$ is

$$W_{mn} = C_m^n p_1^m p_2^{n-m}. \tag{10.13}$$

Now take logs and use Stirling's formula for factorials $\log x! \simeq x \log x - x$ giving

$$\log C_m^n \simeq -n\left[\left(1-\frac{m}{n}\right)\log\left(1-\frac{m}{n}\right)+\frac{m}{n}\log\frac{m}{n}\right] \tag{10.14}$$

so that

$$\log W_{mn} \simeq n\left[-\left(1-\frac{m}{n}\right)\log\left(1-\frac{m}{n}\right)-\frac{m}{n}\log\frac{m}{n}+\frac{m}{n}\log p_1+\left(1-\frac{m}{n}\right)\log p_2\right]. \tag{10.15}$$

Expanding about the maximum at $m/n = p_1$ keeping terms to quadratic order gives

$$W_{mn} \sim \exp\left[-\frac{n\,(m/n-p_1)^2}{2p_1p_2}\right]. \tag{10.16}$$

Thus W_{mn} becomes a sharply peaked Gaussian function of m/n centered around p_1 with a width that goes to zero as $n \to \infty$, i.e. the measure is completely dominated by a narrow region of the set. (To get the correct normalization prefactor to the Gaussian we would need to take Stirling's formula to higher order.)

Calculation of $f(\alpha)$

At the nth level an interval of size $l = l_1^m l_2^{n-m}$ is weighted by a probability $p = p_1^m p_2^{n-m}$. The quantity α is defined by the singular dependence of the probability on interval size $p \sim l^\alpha$, so that taking logs

$$\alpha = \frac{\log p_1 + (\frac{n}{m}-1)\log p_2}{\log l_1 + (\frac{n}{m}-1)\log l_2} \tag{10.17}$$

The parameter α gives us the singularity in the measure as $l \to 0$. There is a spectrum of such singularities. The quantity f gives the dimension of the set experiencing the singularity α. Then the number of intervals with the singularity grows as l^{-f}:

$$C_m^n \sim \left(l_1^m l_2^{n-m}\right)^{-f} \tag{10.18}$$

to yield

$$f = \frac{(\frac{n}{m}-1)\log(\frac{n}{m}-1)-(\frac{n}{m})\log(\frac{n}{m})}{\log l_1 + (\frac{n}{m}-1)\log l_2} \tag{10.19}$$

Equations (10.17) and (10.19) give an implicit relationship $f(\alpha)$ (eliminate $\frac{m}{n}$). Notice again that a particular value of α characterizing the singularity of the measure picks out a region of the set labelled by m/n, and this region has a fractal dimension f given by (10.19).

Calculation of D_q

The calculation of D_q is a little more complicated, since we must use the partition function formalism. Form the "partition function" at the nth level

$$\Gamma^{(n)} = \Gamma(q, \tau, l^n) = \sum_{intervals} \frac{p_i^q}{l_i^\tau} = \sum_m \left\{ C_m^n p_1^{mq} p_2^{(n-m)q} l_1^{-m\tau} l_2^{-(n-m)\tau} \right\} \tag{10.20}$$

where l is the larger of l_1 and l_2 so that l^n is the largest interval at the nth level. Notice that from the binomial expansion $\Gamma^{(n)}$ is just $\left[\Gamma^{(1)}\right]^n$ so that the condition that $\Gamma(q, \tau, l^n)$ neither diverge nor go to zero as $l^n \to 0$ is simply $\Gamma^{(1)} = \Gamma(q, \tau, l) = 1$.

For large n we expect the largest term in the sum of (10.20) to dominate the sum. This can be calculated through $d\log\{\}/dm = 0$ where $\{\}$ is one of the terms in the sum, and again using Stirling's formula to evaluate C_m^n, to give an implicit equation for m/n for each (τ, q), which we write as

$$\tau = \frac{\log(\frac{n}{m} - 1) + q\log(\frac{p_1}{p_2})}{\log(\frac{l_1}{l_2})} \tag{10.21}$$

We then evaluate $\Gamma^{(n)}$ as its largest term, and $\tau = \tau(q)$ is given by requiring $\Gamma^{(n)} = \left(\Gamma^{(1)}\right)^n = 1$, i.e. $\log\{\} = 0$. This gives an expression that can be simplified to

$$q = \frac{\log(\frac{n}{m})\log(\frac{l_1}{l_2}) - \log(\frac{n}{m} - 1)\log l_1}{\log p_1 \log l_2 - \log p_2 \log l_1} \tag{10.22}$$

Equations (10.21) and (10.22) give an implicit equation for $\tau(q)$ (again by eliminating $\frac{n}{m}$) and hence $D_q = \tau(q)/(q-1)$. This calculation of D_q shows again how a particular value of m/n is singled out by the choice of q.

It is now straightforward to use n/m as an implicit parameter to plot $f(\alpha)$ and D_q. This is done for the example of $p_1 = 3/5$, $p_2 = 2/5$, $l_1 = 1/4$, $l_2 = 2/5$ in figure (10.3). For interest the values of m/n contributing to each q and α is shown in figure (10.4).

A direct expression for D_q is simply to use the result $\Gamma(q,\tau,l)=1$, i.e.

$$p_1^q l_1^{(1-q)D_q} + p_2^q l_2^{(1-q)D_q} = 1. \tag{10.23}$$

However this must be solved numerically.

10.3.1 Other dimensions

The case $l_1 = l_2 = l$ is particularly simple, since then at each level of construction all the line elements have the same length, and simple box counting arguments are easy to apply. This lets us easily study some of the other dimensions introduced in chapter 9. In particular using the expression (10.16) for the total measure associated with the C_m^n elements at level n of the construction labelled by the index m and each containing measure $p_1^m p_2^{n-m}$ we see that for large n essentially *all* measure is associated with values of m/n *very* close to p_1. Then the following results are easily proven:

1. The information density given by box-counting with boxes of size l^n is

$$D_1 = \lim_{n\to\infty} \sum_m W_{nm} \ln(p_1^m p_2^{n-m})/\ln l^n \tag{10.24}$$

   and since the width of W_{nm} is so narrow about $m/n = p_1$ we can take the second factor evaluated at this value out of the sum to give directly $D_1 = \ln(p_1^{p_1} p_2^{p_2})/\log l = (p_1 \ln p_1 + p_2 \ln p_2)/\ln l$.

2. The pointwise dimension at an element characterized by the index m/n is

$$D_P(m/n) = \lim_{n\to\infty} \ln p_1^m p_2^{n-m}/\ln l^n. \tag{10.25}$$

   Again, because almost all the measure is for $m/n = p_1$ this gives $D_P = D_1$ for almost all points in the set.

3. The number of boxes with index m/n is C_m^n given by (10.14). To make up a measure fraction $\theta \neq 1$ we need some small spread of m/n about p_1, but the spread goes to zero for large n, and this leads to terms that do not contribute to the capacity. Thus using (10.14)

$$D_C(\theta\neq 1) = n\left[\left(1-\frac{m}{n}\right)\log\left(1-\frac{m}{n}\right) + \frac{m}{n}\log\frac{m}{n}\right]/\ln l^n\Big|_{m/n=p_1} \tag{10.26}$$

which is again just D_1. (Note that near *its* maximum C_m^n can be written in Gaussian form analogous to (10.16) for W_{nm}

$$C_m^n \sim \exp\left[-2n\left(\frac{m}{n}-\frac{1}{2}\right)^2\right] \tag{10.27}$$

but for (10.26) we need to evaluate C_m^n in the tails far away from its maximum where the Gaussian expression is not valid.)

10.4 The Bakers' Map

From the construction (chapter 5) it is apparent that the bakers' map yields a set that is the full interval in the y direction and exactly given by the two scale factor Cantor set in the x direction, with $p_1 = \alpha$, $p_2 = 1-\alpha$, $l_1 = \lambda_a$ and $l_2 = \lambda_b$. Since the attractor is uniform in the y direction, the dimensions are given by $D_q = 1+\hat{D}_q$ with $\hat{D}_q$ the dimensions of the intersection of the set with a horizontal line. Using (10.23) gives the transcendental equation for the generalized dimensions

$$\alpha^q \lambda_a^{(1-q)\hat{D}_q} + \beta^q \lambda_b^{(1-q)\hat{D}_q} = 1. \tag{10.28}$$

This equation can be arrived at directly by considering the covering of the two portions of the set ($a \equiv 0 < x < \lambda_a$ and $b \equiv 1-\lambda_b < x < 1$) separately and the scaling given by iterating the map. Consider a covering of the intersection of the set with the x axis by line elements at scale ε, and the mapping of this coverage after one iteration. The mapping will give a covering of portion a with scale $\lambda_a\varepsilon$ with the measures p_i associated with each box multiplied by α, and portion b with scale $\lambda_b\varepsilon$ with measures multiplied by β. Define $S = \sum p_i^q$. Then

$$S(\varepsilon) = \alpha^q S_a(\lambda_a\varepsilon) + \beta^q S_b(\lambda_b\varepsilon). \tag{10.29}$$

But S scales as

$$S(\varepsilon) \sim \varepsilon^{(q-1)\hat{D}_q} \tag{10.30}$$

and substituting this into (10.29) then reproduces (10.28).

The capacity of the set $D_0 = 1+\hat{D}_0$ is given by

$$\lambda_a^{\hat{D}_0} + \lambda_b^{\hat{D}_0} = 1. \tag{10.31}$$

and by expanding in small $q-1$ the information density is

$$D_1 = 1 + \frac{\alpha \ln(1/\alpha) + \beta \ln(1/\beta)}{\alpha \ln(1/\lambda_a) + \beta \ln(1/\lambda_b)}. \tag{10.32}$$

For general λ_a and λ_b (10.28) has to be solved numerically. For $\lambda_a = \lambda_b$ the dimension is simply

$$D_q = 1 + \frac{1}{q-1} \frac{\log(\alpha^q + \beta^q)}{\log \lambda_a}. \tag{10.33}$$

The iteration of the bakers' map is performed in demonstration 2.

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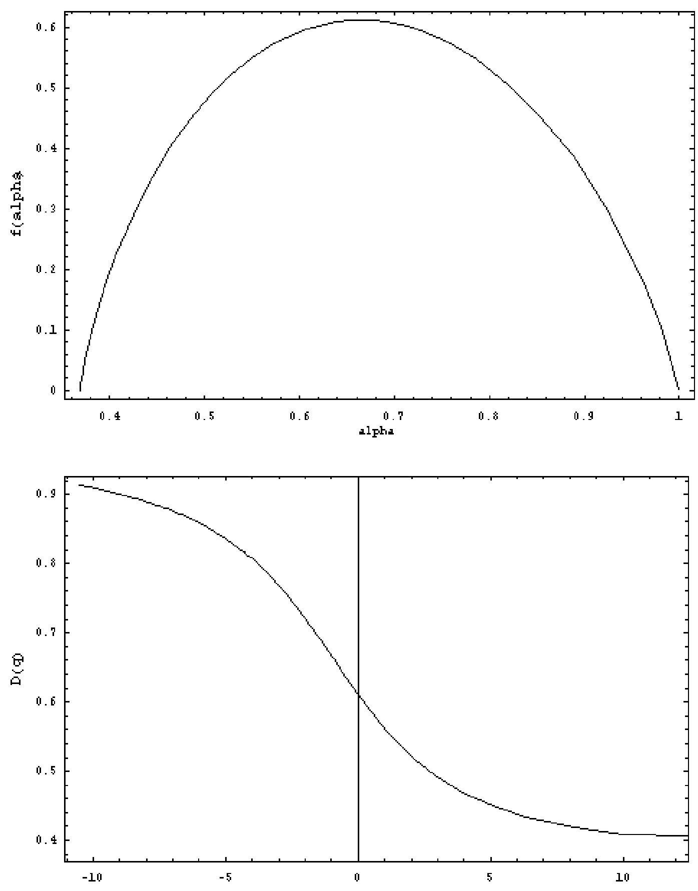

Figure 10.3: Plot of $f(\alpha)$ and D_q for $p_1 = 3/5,\ p_2 = 2/5$, $l_1 = 1/4$, $l_2 = 2/5$.

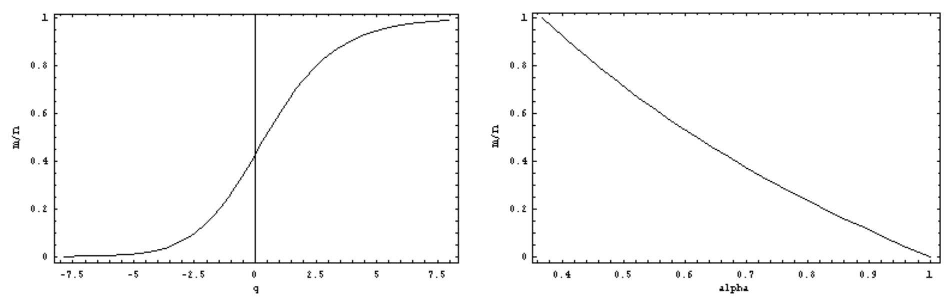

Figure 10.4: Value of m/n contributing for each q or α for $p_1 = 3/5,\ p_2 = 2/5$, $l_1 = 1/4$, $l_2 = 2/5$.

Bibliography

[1] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman, Phys. Rev. **A33**, 1141 (1986); Erratum **A34**, 1601.